Applied Mathematics/Differential Equations and Scientific Computing, part A

Problems Week 2

Quadrature

1. Let $I = (0,1)$ and $f(x) = x^2$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on the single interval $(0,1)$.
(c) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on the single interval $(0,1)$.
(d) Compute the errors in (b) and (c). Compare with theory.
(e) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on each subinterval.
(f) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on each subinterval.
(g) Compute the errors in (e) and (f), and compare with the errors in (b) and (c) respectively. By what factor has the error decreased?

2. Let $I = (0,1)$ and $f(x) = x^4$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on the single interval $(0,1)$.
(c) Compute the error in (b). Compare with theory.
(d) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on each subinterval.
(e) Compute the error in (d), and compare with the error in (b). By what factor has the error decreased?

L^2-projection

3. Let $I = (0,1)$ and $f(x) = x^2$ for $x \in I$.
(a) Let V_h be the space of linear functions on I and calculate the L^2-projection $P_h f \in V_h$ of f.
(b) Divide I into two subintervals of equal length and let V_h be the corresponding space of piecewise linear functions. Calculate the L^2-projection $P_h f \in V_h$ of f.
(c) Illustrate your results in figures and compare with the nodal interpolant $\pi_h f$.

4. Let $I = (0,1)$ and $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of I into subintervals $I_j = (x_{j-1}, x_j)$ of length h_j.
(a) Assume $h_j = 1/N$ for all j. Calculate the mass matrix M.

(b) Calculate the mass matrix M in the general case.

5. Recall that $(f, g) = \int_I fg\, dx$ and $\|f\|^2_{L^2(I)} = (f, f)$ are the L^2-scalar product and norm, respectively. Let $I = (0, \pi)$, $f = \sin x$, $g = \cos x$ for $x \in I$.
(a) Calculate (f, g).
(b) Calculate $\|f\|_{L^2(I)}$ and $\|g\|_{L^2(I)}$.

6. Show that $(f - P_h f, v) = 0, \quad \forall v \in V_h$, if and only if $(f - P_h f, \varphi_i) = 0, \quad i = 0, \dots, N$; where $\{\varphi_i\}_{i=0}^N \subset V_h$ is the basis of hat-functions.

7. Let V be a linear subspace of $\mathbf{R}^n$ with basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ with $m < n$. Let $P\boldsymbol{x} \in V$ be the orthogonal projection of $\boldsymbol{x} \in \mathbf{R}^n$ onto the subspace V. Derive a linear system of equations that determines $P\boldsymbol{x}$. Note that your results are analogous to the L^2-projection when the usual scalar product in $\mathbf{R}^n$ is replaced by the scalar product in $L^2(I)$. Compare this method of computing the projection $P\boldsymbol{x}$ to the method used for computing the projection of a three dimensional vector onto a two dimensional subspace. What happens if the basis $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is orthogonal?