**Applied Mathematics/Differential Equations and Scientific Computing, part A**

# Problems Week 3

## The Finite Element Method: Stationary Problems.

**1.** Let $u$ be the solution to

$$-(au')' + cu = f \quad \text{in } (0,1), \tag{1}$$
$$u(0) = u(1) = 0, \tag{2}$$

where $a, c$, and $f$ are given functions.
(a) Show that $u$ satisfies the variational equation

$$\int_0^1 (au'v' + cuv)\, dx = \int_0^1 fv\, dx, \tag{3}$$

for all sufficiently smooth $v$ with $v(0) = v(1) = 0$.
(b) Introduce a partition of $(0,1)$ and the corresponding space of continuous piecewise linear functions $V_{h0}$ which are zero for $x = 0$ and $x = 1$. Formulate a finite element method based on the variational equation in (a).
(c) Let $|||u||| = \left(\int_0^1 (au'u' + cuu)\, dx\right)^{1/2}$. Verify that $|||\cdot|||$ is a norm if $a(x) > 0$ and $c(x) \geq 0$ for all $x \in (0,1)$.
(d) Prove the a priori error estimate

$$|||u - U||| \leq |||u - v|||, \tag{4}$$

for all $v \in V_{h0}$.
(e) Assume that there are constants $C_a$ and $C_c$ such that $\|a\|_{L_\infty(0,1)} \leq C_a$ and $\|c\|_{L_\infty(0,1)} \leq C_c$, and that $\|u''\|_{L^2(0,1)}$ is bounded. Show that $|||u - U|||$ converges to zero as the meshsize tends to zero.
**2.** Let $u$ be the solution to

$$-u'' = 1 \quad \text{in } (0,1), \tag{5}$$
$$u(0) = u(1) = 0. \tag{6}$$

(a) Solve the problem analytically.
(b) Let $I = (0,1)$ be divided into a uniform mesh with $h = 1/N$. Calculate (by hand) the finite element approximation $U$ for $N = 2, 3$.
(c) Plot your solutions in a figure. Compare your results.
**3*.**
(a) Show that the finite element approximations $U$ that you have computed in Problem 2

actually are exactly equal to $u$ *at the nodes*, by simply evaluating $u$ and $U$ at the nodes.
(b) Prove this result. *Hint:* Show that the error $e = u - U$ can be written

$$e(z) = \int_0^1 g_z'(x) e'(x)\, dx, \quad 0 \le z \le 1,$$

where

$$g_z(x) = \begin{cases} (1-z)x, & 0 \le x \le z, \\ z(1-x), & z \le x \le 1, \end{cases}$$

and then use the fact the $g_{x_j} \in V_{h0}$.
(c) Does the result in (b) extend to variable $a = a(x)$?