Applied Mathematics/ Partial Differential Equations, part A

Solutions to Problems III

September 11, 2003

Problem 1. Let u be the solution to

$$-(au')' + cu = f \quad \text{in } (0,1), \tag{1}$$
$$u(0) = u(1) = 0, \tag{2}$$

where a, c, and f are given functions.
(a) Show that u satisfies the variational equation

$$\int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx, \tag{3}$$

for all sufficiently smooth v with $v(0) = v(1) = 0$.
(b) Introduce a partition of $(0,1)$ and the corresponding space of continuous piecewise linear functions V_{h0} which are zero for $x = 0$ and $x = 1$. Formulate a finite element method based on the variational equation in (a).
(c) Let $|||u||| = \left(\int_0^1 (au'u' + cuu)\,dx\right)^{1/2}$. Verify that $|||\cdot|||$ is a norm if $a(x) > 0$ and $c(x) \geq 0$ for all $x \in (0,1)$.
(d) Prove the a priori error estimate

$$|||u - U||| \leq |||u - v|||, \tag{4}$$

for all $v \in V_{h0}$.
(e) Assume that there are constants C_a and C_c such that $\|a\|_{L_\infty(0,1)} \leq C_a$ and $\|c\|_{L_\infty(0,1)} \leq C_c$, and that $\|u''\|_{L^2(0,1)}$ is bounded. Show that $|||u - U|||$ converges to zero as the meshsize tends to zero.
Solution:
(a) Multiply both sides of the differential equation by $v(x)$, such that $v(0) = v(1) = 0$, and integrate from $x = 0$ to $x = 1$ to get the following equality:

$$\int_0^1 (\,-(au')'v + cuv\,)\,dx = \int_0^1 fv\,dx.$$

Integrate by parts in the first term on the left-hand side, and use the fact that $v(0) = v(1) = 0$ to see that the boundary terms vanish:

$$-[au'v]_{x=0}^{x=1} + \int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx;$$
$$\int_0^1 (au'v' + cuv)\,dx = \int_0^1 fv\,dx.$$

(b) Let $0 = x_0 < x_1 < \cdots < x_N < x_{N+1} = 1$ be a partition of (0, 1) and let $\{\varphi_i\}_{i=1}^N$ be the "hat-functions" on this partition that are equal to one in an *internal* node. Define $V_{h0} = \text{span}\{\varphi_1, \varphi_2, \ldots, \varphi_N\}$, i.e., V_{h0} is the vector space of continuous, piece-wise linear functions $v(x)$ that are zero at $x = 0$ and $x = 1$. The Finite Element Method now reads: Find $U \in V_{h0}$ such that

$$\int_0^1 (aU'v' + cUv)\,dx = \int_0^1 fv\,dx \quad \text{for all } v \in V_{h0}.$$

(c) To prove that $|||\cdot|||$ is a norm we must verify that:

(i) $|||u+v||| \le |||u||| + |||v|||$ for all u and $v \in V_0$,

(ii) $|||\alpha u||| = |\alpha|\,|||u|||$ if $u \in V_0$ and $\alpha \in \boldsymbol{R}$,

(iii) $|||u||| = 0$ for $u \in V_0$ implies $u = \mathbf{0}$,

where V_0 denotes the vector space of functions that are zero at the boundary, and that are smooth enough for the integrals in the definition of $|||u|||$ to exist.

Since

$$|||u||| = (u,\, u)_E^{1/2},$$

where

$$(u,\, v)_E = \int_0^1 (\, a(x)u'(x)v'(x) + c(x)u(x)v(x)\,)\,dx,$$

is a *scalar product* between functions in V_0, property (i) follows from *the Cauchy-Schwarz inequality*:

$$\begin{aligned} |||u+v|||^2 &= (u+v,\, u+v)_E = (u,\, u)_E + 2(u,\, v)_E + (v,\, v)_E \\ &\le |||u|||^2 + 2|||u|||\cdot|||v||| + |||v|||^2 = (|||u||| + |||v|||)^2. \end{aligned}$$

Property (ii) follows since

$$\int_0^1 (\, a(x)(\alpha u'(x))^2 + c(x)(\alpha u(x))^2\,)\,dx = \alpha^2 \int_0^1 (\, a(x)u'(x)^2 + c(x)u(x)^2\,)\,dx.$$

To prove property (iii) we notice that $a(x)u'(x)^2 \ge 0$ and $c(x)u(x)^2 \ge 0$. This means that $\int_0^1 a(x)u'(x)^2\,dx \ge 0$ and $\int_0^1 c(x)u(x)^2\,dx \ge 0$. If $0 = |||u|||^2 = \int_0^1 a(x)u'(x)^2\,dx + \int_0^1 c(x)u(x)^2\,dx$, both these integrals must therefore be equal to zero. Since $a(x) > 0$ this implies $u'(x) \equiv 0$, which means that $u(x) \equiv K$ where K is a constant. But since $u(0) = u(1) = 0$ we must have $K = 0$.

Remark. If $c(x) > 0$ is (also) *strictly positive* then $\int_0^1 c(x)u(x)^2\,dx = 0$ immediately implies that $u(x) \equiv 0$ and we don't need to use the boundary conditions.

(d) Observe that, by using the definition of $(u,\, v)_E$ in (c), the variational equation in (a) can be written

$$(u,\, v)_E = \int_0^1 f v\, dx \quad \text{for all } v \in V_0,$$

and the Finite Element Method in (b) can be written

$$(U,\, v)_E = \int_0^1 f v\, dx \quad \text{for all } v \in V_{h0}.$$

Since $V_{h0} \subset V_0$ we get by subtracting:

$$(u - U,\, v)_E = 0 \quad \text{for all } v \in V_{h0}.$$

The last equation expresses *the Galerkin orthogonality*. This shows that the Finite Element approximation $U(x)$ of $u(x)$ is *the orthogonal projection* of u onto V_{h0} with respect to the scalar product $(\cdot,\, \cdot)_E$. This orthogonality, and the Cauchy-Schwarz inequality, implies that for an *arbitrary* function $v(x) \in V_{h0}$:

$$|||u - U|||^2 = (u - U,\, u - U)_E = (u - U,\, u - U + (U - v))_E$$

$$= (u - U,\, u - v)_E \leq |||u - U||| \cdot |||u - v|||,$$

since $U - v \in V_{h0}$. Dividing both sides by $|||u - U|||$ now completes the proof.

Remark. Observe the complete analogy between this proof and the corresponding proof for the L^2-projection.

(e) Assume for simplicity that the partition is uniform, i.e., that the mesh function $h(x) \equiv h$ is a constant function. Choosing v in (d) to be the nodal interpolant $\pi_h u(x) \in V_{h0}$ of u, we get:

$$|||u - U|||^2 \leq |||u - \pi_h u|||^2$$

$$= \int_0^1 (\, a(x)(u - \pi_h u)'(x)^2 + c(x)(u - \pi_h u)(x)^2 \,)\, dx$$

$$\leq C_a \int_0^1 (u - \pi_h u)'(x)^2\, dx + C_c \int_0^1 (u - \pi_h u)(x)^2\, dx$$

$$= C_a ||(u - \pi_h u)'||^2_{L^2(0,1)} + C_c ||u - \pi_h u||^2_{L^2(0,1)}$$

$$\leq C_a C_i^2 ||h u''||^2_{L^2(0,1)} + C_c C_i^2 ||h^2 u''||^2_{L^2(0,1)}$$

$$= C_a C_i^2 h^2 ||u''||^2_{L^2(0,1)} + C_c C_i^2 h^4 ||u''||^2_{L^2(0,1)},$$

which tends to zero as h tends to zero. (C_i denotes interpolation constants.) □

Problem 2. Let u be the solution to

$$-u''(x) = 1 \quad \text{in } (0,1), \tag{5}$$

$$u(0) = u(1) = 0. \tag{6}$$

(a) Solve the problem analytically.
(b) Let $I = (0,\,1)$ be divided into a uniform mesh with $h = 1/N$. Calculate (by hand) the finite element approximation U for $N = 2, 3$.
(c) Plot your solutions in a figure. Compare your results.
Solution:
(a) Integrating the differential equation twice gives:

$$u''(x) = -1 \quad \Rightarrow \quad u'(x) = -x + C_1 \quad \Rightarrow \quad u(x) = -x^2/2 + C_1 x + C_2.$$

The boundary condition $u(0) = 0$ then gives $C_2 = 0$, and $u(1) = 0$ gives $-1/2 + C_1 + C_2 = 0$, i.e., $C_1 = 1/2; \quad C_2 = 0$. Therefore:

$$u(x) = -\frac{x^2}{2} + \frac{x}{2} = \frac{x(1-x)}{2}.$$

(b) The finite element approximation $U(x) = \sum_{j=1}^{M} \xi_j \varphi_j(x)$ can be computed by solving the linear system of equations (see *Applied Mathematics: B&S*, Part D, equation 54.4, with $a = 1$):

$$\sum_{j=1}^{M} \xi_j \int_0^1 \varphi_j' \varphi_i' \, dx = \int_0^1 f \varphi_i \, dx \quad i = 1, \ldots, M,$$

which determines the unknown coefficients $\xi_1, \ldots, \xi_M$. Here M is the number of *internal* nodes, since we have *homogeneous Dirichlet boundary conditions.*

If the number of subintervals is $N = 2$, then there is only one internal node, $M = 1$, and the equation above simplifies to:

$$\xi_1 \int_0^1 \varphi_1' \varphi_1' \, dx = \int_0^1 f \varphi_1 \, dx.$$

Since $f(x) = 1$, $\varphi_1' = 2$ on $[0,\, \frac{1}{2}]$ and $\varphi_1' = -2$ on $[\frac{1}{2},\, 1]$, we get

$$\xi_1 \Big(\int_0^{0.5} 2^2 \, dx + \int_{0.5}^1 (-2)^2 \, dx \Big) = 4\xi_1 = \int_0^1 \varphi_1 \, dx = \frac{1}{2},$$

which gives that $\xi_1 = \frac{1}{8}$. That is: $U(x) = \frac{1}{8}\, \varphi_1(x)$.

Remark. The integral $\int_0^1 \varphi_1\,dx$ is geometrically the *area* under φ_1, i.e., the area of a triangle.

If the number of subintervals is $N = 3$, then there are two internal nodes, $M = 2$, and we get the following linear system of equations:

$$\xi_1 \int_0^1 \varphi_1'\varphi_1'\,dx + \xi_2 \int_0^1 \varphi_2'\varphi_1'\,dx = \int_0^1 f\varphi_1\,dx,$$
$$\xi_1 \int_0^1 \varphi_1'\varphi_2'\,dx + \xi_2 \int_0^1 \varphi_2'\varphi_2'\,dx = \int_0^1 f\varphi_2\,dx.$$

Since $f(x) = 1$ and

$$\varphi_1'(x) = \begin{cases} 0, & x \notin [0,\,\frac{2}{3}], \\ 3, & x \in [0,\,\frac{1}{3}], \\ -3, & x \in [\frac{1}{3},\,\frac{2}{3}], \end{cases} \qquad \varphi_2'(x) = \begin{cases} 0, & x \notin [\frac{1}{3},\,1], \\ 3, & x \in [\frac{1}{3},\,\frac{2}{3}], \\ -3, & x \in [\frac{2}{3},\,1], \end{cases}$$

we get:

$$\xi_1 \left(\int_0^{\frac{1}{3}} 3^2\,dx + \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)^2\,dx \right) + \xi_2 \int_{\frac{1}{3}}^{\frac{2}{3}} 3(-3)\,dx = 6\xi_1 - 3\xi_2 = \int_0^1 \varphi_1\,dx = \frac{1}{3},$$
$$\xi_1 \int_{\frac{1}{3}}^{\frac{2}{3}} (-3)3\,dx + \xi_2 \left(\int_{\frac{1}{3}}^{\frac{2}{3}} 3^2\,dx + \int_{\frac{2}{3}}^{1} (-3)^2\,dx \right) = -3\xi_1 + 6\xi_2 = \int_0^1 \varphi_2\,dx = \frac{1}{3},$$

with solution $\xi_1 = \xi_2 = \frac{1}{9}$. That is: $U(x) = \frac{1}{9}\,\varphi_1(x) + \frac{1}{9}\,\varphi_2(x)$.
(c) See Figure 1. □

Problem 3*.
(a) Show that the finite element approximations U that you have computed in *Problem 2 (Week 3)* actually are exactly equal to u *at the nodes*, by simply evaluating u and U at the nodes.
(b) Prove this result. *Hint:* Show that the error $e = u - U$ can be written

$$e(z) = \int_0^1 g_z'(x)e'(x)\,dx, \quad 0 \le z \le 1,$$

where

$$g_z(x) = \begin{cases} (1-z)x, & 0 \le x \le z, \\ z(1-x), & z \le x \le 1, \end{cases}$$

and then use the fact the $g_{x_j} \in V_{h0}$.
(c) Does the result in (b) extend to variable $a = a(x)$?
Solution:
(a) From *Problem 2 (Week 3)* with $N = 2$ we get

$$u(1/2) = \frac{1}{2}(1 - \frac{1}{2})/2 = 1/8,$$

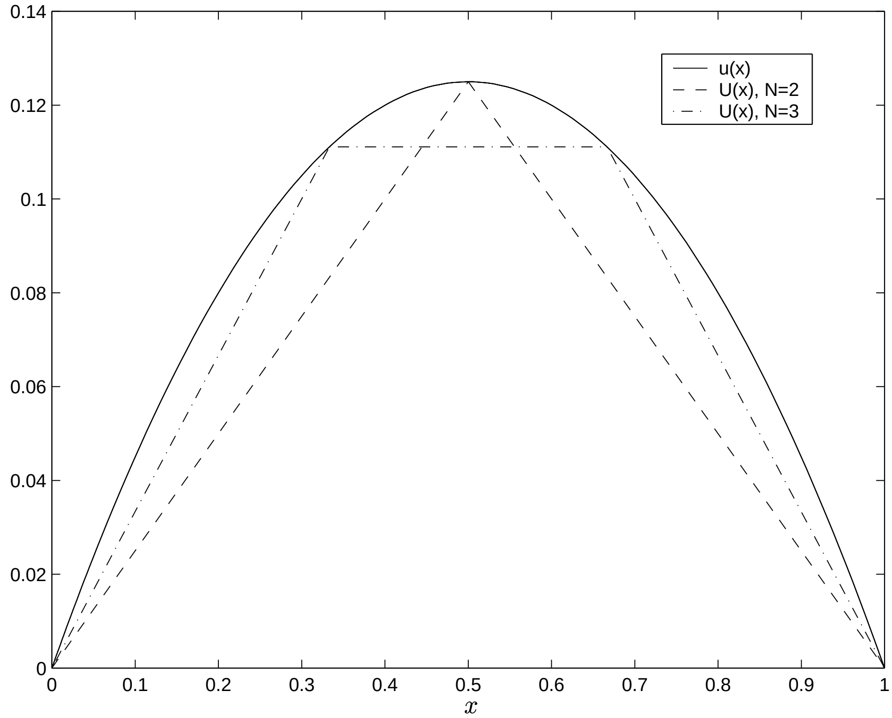

Figure 1: Problem 2 (Week 3). Plots of $u(x)$ and $U(x)$ for $N = 2, 3$.

and

$$U(1/2) = \frac{1}{8}\,\varphi_1(1/2) = 1/8.$$

Hence, $u(1/2) = U(1/2)$.

Using $N = 3$ we have for the first inner node

$$u(1/3) = \frac{1}{3}(1 - \frac{1}{3})/2 = 1/9,$$

and

$$U(1/3) = \frac{1}{9}\,\varphi_1(1/3) + \frac{1}{9}\,\varphi_2(1/3) = \frac{1}{9} \cdot 1 + 0 = \frac{1}{9}.$$

For the second inner node:

$$u(2/3) = \frac{2}{3}(1 - \frac{2}{3})/2 = 1/9,$$

and

$$U(2/3) = \frac{1}{9}\,\varphi_1(2/3) + \frac{1}{9}\,\varphi_2(2/3) = 0 + \frac{1}{9} \cdot 1 = \frac{1}{9}.$$

Hence, $u(1/3) = U(1/3)$ and $u(2/3) = U(2/3)$.

(b) To check the given formula for $e(z)$ we must compute the integral. Before we can do that, we must calculate the derivative of $g_z(x)$:

$$g_z'(x) = \frac{dg_z(x)}{dx} = \begin{cases} 1-z, & 0 \le x < z, \\ -z, & z < x \le 1. \end{cases}$$

Thus, we have:

$$\begin{aligned} \int_0^1 g_z'(x)e'(x)\,dx &= \int_0^z (1-z)e'(x)\,dx + \int_z^1 -ze'(x)\,dx \\ &= (1-z)(e(z)-e(0)) - z(e(1)-e(z)) \\ &= e(z) - \underbrace{e(0) + ze(0) - ze(1)}_{=0} \\ &= e(z), \end{aligned}$$

since the error $e = u - U$ is equal to zero at the boundary points $x = 0$ and $x = 1$. This follows from the boundary conditions, $u(0) = U(0) = 0$ and $u(1) = U(1) = 0$.

To show that the error is zero also at all internal nodal points x_j, we only need to show that $g_{x_j} \in V_{h0}$. The result then follows from *the Galerkin orthogonality* (cf. *Problem 1(d) (Week 3)* with $a = 1$ and $c = 0$), $\int_0^1 e'v'\,dx = (e, v)_E = 0$ for all $v \in V_{h0}$, by taking $v = g_{x_j}$. But from Figure 2 we see that g_{x_j} can be written as

$$g_{x_j}(x) = \sum c_i\,\varphi_i(x)$$

with weights $c_i = g_{x_j}(x_i)$. Hence, $g_{x_j} \in V_{h0}$. Also note that $g_z(x) \notin V_{h0}$ if $z \neq x_j$, which can be seen from Figure 3.
(c) No. As a counter-example, consider the case $a(x) = 1 + x$:

$$\begin{aligned} -((1+x)u')' = 1, \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{aligned}$$

The solution is $u(x) = \frac{\log(1+x)}{\log(2)} - x$. Computing the Finite Element approximation $U(x)$ for $N = 2$ in the same way as in *Problem 2(b) (Week 3)* gives $U(x) = \frac{1}{12}\,\varphi_1(x)$. We thus have that $U(1/2) = \frac{1}{12} \neq \frac{\log(3/2)}{\log(2)} - \frac{1}{2} = u(1/2)$. □

Problem 4. Consider the system of ODE:

$$M\dot{\xi}(t) + A\xi(t) = b \quad \text{in } (0, T), \tag{7}$$

$$\xi(0) = \xi^0. \tag{8}$$

Assume that

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 4 & 14 \\ 4 & 8 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \xi^0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \tag{9}$$

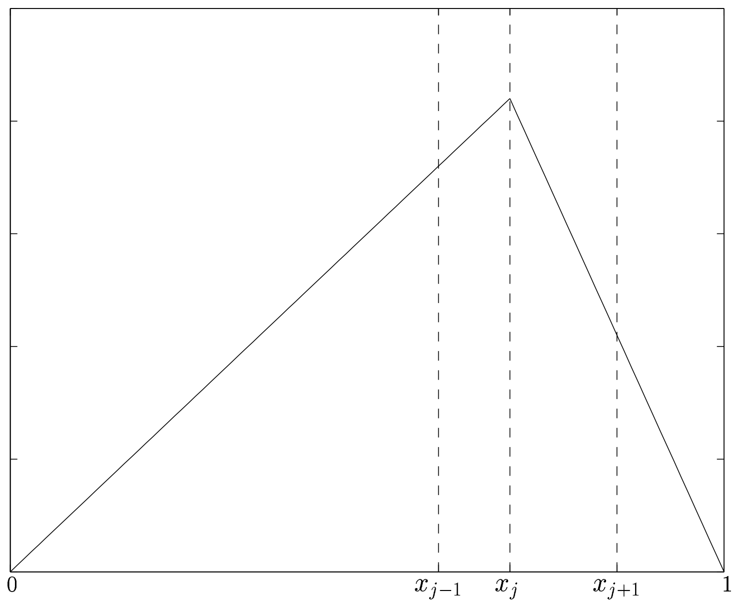

Figure 2: Problem 3 (Week 3). $g_z(x)$ when $z = x_j$.

Make a uniform partition of the time interval $(0, 1)$ into two sub-intervals and compute an approximation of $\xi(1)$ with the *backward Euler* method.
Solution: We divide the time interval: $0 = t_0 < t_1 < t_2 = 1$, with $t_1 = 0.5$, i.e., into two subintervals with length $\triangle t = 0.5$. The Euler backward method approximates the time derivative with a difference quotient in the following manner:

$$M\frac{\xi^n - \xi^{n-1}}{\triangle t} + A\xi^n = b, \quad n = 1,\ 2,$$
$$\xi^0 = \xi(0).$$

So to compute $\xi^2 \approx \xi(t_2)$ we have to solve, in order, the equations:

$$M\frac{\xi^1 - \xi^0}{\triangle t} + A\xi^1 = b,$$

$$M\frac{\xi^2 - \xi^1}{\triangle t} + A\xi^2 = b.$$

Rearrangement of the first of these equations yields:

$$M\xi^1 + \triangle t\, A\xi^1 = M\xi^0 + \triangle t\, b;$$

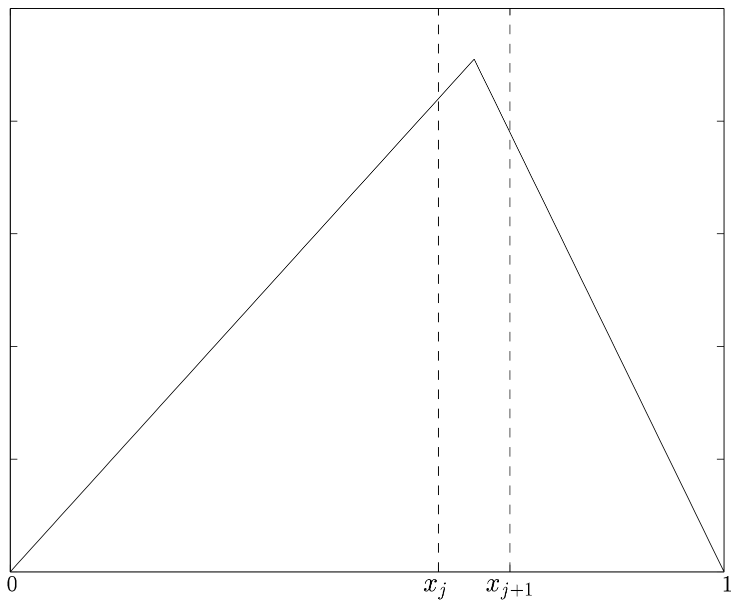

Figure 3: Problem 3 (Week 3). $g_z(x)$ when $z \neq x_j$.

$$\begin{aligned}
(M + \triangle t\, A)\xi^1 =& M\xi^0 + \triangle t\, b; \\
\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}4 & 14\\4 & 8\end{bmatrix}\right)\xi^1 =& \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}; \\
\begin{bmatrix}3 & 7\\2 & 5\end{bmatrix}\xi^1 =& \begin{bmatrix}1\\1\end{bmatrix}; \\
\xi^1 =& \begin{bmatrix}-2\\1\end{bmatrix},
\end{aligned}$$

where the linear system of equations is solved by *Gaussian elimination.* Similarly, we get for the second equation:

$$\begin{aligned}
M\xi^2 + \triangle t\, A\xi^2 =& M\xi^1 + \triangle t\, b; \\
(M + \triangle t\, A)\xi^2 =& M\xi^1 + \triangle t\, b; \\
\left(\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}4 & 14\\4 & 8\end{bmatrix}\right)\xi^2 =& \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}-2\\1\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix}0\\0\end{bmatrix}; \\
\begin{bmatrix}3 & 7\\2 & 5\end{bmatrix}\xi^2 =& \begin{bmatrix}-2\\1\end{bmatrix}; \\
\xi^2 =& \begin{bmatrix}-17\\7\end{bmatrix}.
\end{aligned}$$

The vector $\xi^2 = \begin{bmatrix}-17\\7\end{bmatrix}$ is thus an approximation of the solution $\xi(t)$ at time $t = 1$ (and

$\xi^1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ at time $t = 0.5$). □

Problem 5. Show that, for the time dependent reaction-diffusion problem with Robin boundary conditions,

$$\begin{aligned}
\dot{u} - (au')' + cu &= f(x,t), \quad x_{\min} < x < x_{\max}, \quad 0 < t < T, \\
a(x_{\min})u'(x_{\min},t) &= \gamma(x_{\min})(u(x_{\min},t) - g_D(x_{\min})) + g_N(x_{\min}), \quad 0 < t < T, \\
-a(x_{\max})u'(x_{\max},t) &= \gamma(x_{\max})(u(x_{\max},t) - g_D(x_{\max})) + g_N(x_{\max}), \quad 0 < t < T, \\
u(x,0) &= u_0(x), \quad x_{\min} < x < x_{\max},
\end{aligned}$$

semi-discretization in space leads to the following system of ODE:

$$M\,\dot{\xi}(t) \,+\, (A \,+\, M_c \,+\, R)\,\xi(t) \,=\, b(t) + rv, \quad 0 < t < T.$$

Solution: *Hint:* To derive the variational formulation, first multiply both sides of the differential equation by a function $v = v(x)$. Then integrate both sides from $x = x_{\min}$ to $x = x_{\max}$. Integrate by parts in "the diffusive term" $\int_{x_{\min}}^{x_{\max}} (\,-(au')'v\,)\,dx$. Finally use the boundary conditions to *replace* au' in the boundary terms at $x = x_{\min}, x_{\max}$. This gives *the variational formulation*:

Find $u(x,t)$ such that for every fixed t: $u(x,t) \in V$, and

$$\int_{x_{\min}}^{x_{\max}} \dot{u}v\,dx \,+\, \gamma uv|_{x=x_{\max}} \,+\, \gamma uv|_{x=x_{\min}} \,+\, \int_{x_{\min}}^{x_{\max}} au'v'\,dx \,+\, \int_{x_{\min}}^{x_{\max}} cuv\,dx \,=$$

$$(\gamma g_D - g_N)v|_{x=x_{\max}} \,+\, (\gamma g_D - g_N)v|_{x=x_{\min}} \,+\, \int_{x_{\min}}^{x_{\max}} fv\,dx, \quad 0 < t < T, \quad \text{for all } v \in V,$$

where V is the vector space of functions $v = v(x)$ that are smooth enough for the integrals in the variational formulation to exist.

The corresponding *Finite Element Method* reads:

Find $U(x,t)$ such that for every fixed t: $U(x,t) \in V_h$, and

$$\int_{x_1}^{x_N} \dot{U}v\,dx \,+\, \gamma Uv|_{x=x_N} \,+\, \gamma Uv|_{x=x_1} \,+\, \int_{x_1}^{x_N} aU'v'\,dx \,+\, \int_{x_1}^{x_N} cUv\,dx \,=$$

$$(\gamma g_D - g_N)v|_{x=x_N} \,+\, (\gamma g_D - g_N)v|_{x=x_1} \,+\, \int_{x_1}^{x_N} fv\,dx, \quad 0 < t < T, \quad \text{for all } v \in V_h,$$

where V_h is the vector space of functions $v = v(x)$ that are continuous and piecewise linear on a partition $x_{\min} = x_1 < x_2 < \ldots < x_N = x_{\max}$ of $[x_{\min},\, x_{\max}]$.

Finally, insert the *Ansatz*

$$U(x,t) = \sum_{j=1}^{N} \xi_j(t)\varphi_j(x),$$

into the Finite Element formulation and choose $v = \varphi_i$ for $i = 1, \ldots, N$.