Applied Mathematics/ Partial Differential Equations part A

Solutions to Problems IV

September 12, 2003

Problem 1. Consider the triangulation of the unit square $\Omega = [0, 1] \times [0, 1]$ into 8 triangles drawn in Figure 1.

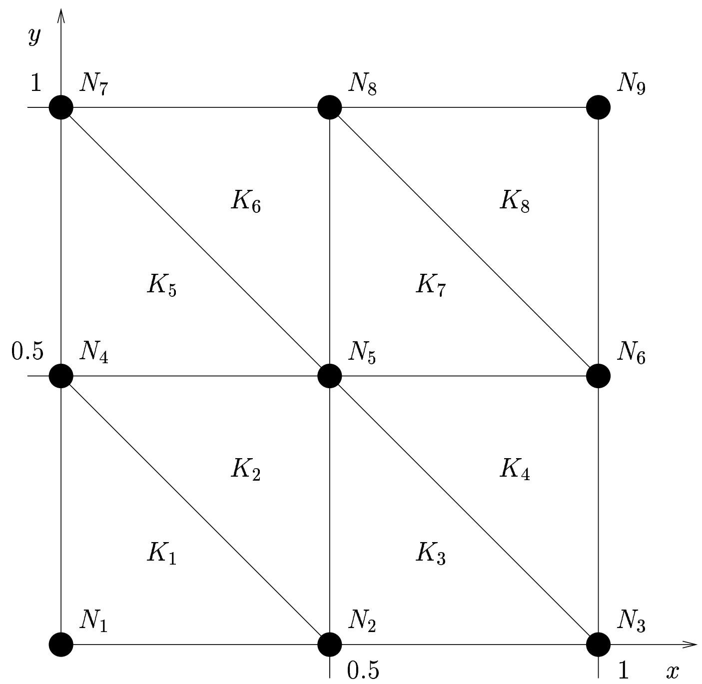

Figure 1: Problem 1 (Week 5). The triangulation of Ω.

(a) Compute the length of the largest side h_{K_j}, and the smallest angle τ_{K_j} of the triangles.
(b) Determine the *point matrix* `p` that describes this triangulation in Matlab. *Hint:* Since node 1 is located at the origin, the first column in `p` is `[0; 0]`.
(c) Determine the *triangle matrix* `t` that describes this triangulation in Matlab. *Hint:* Since triangle 1 has corners in node number 1, 2 and 4, the first column in `t` can e.g. be `[1; 2; 4]`. It is not important which node comes first, but they must be listed in a *counter-clockwise* order.
(d) Verify your results by creating `p` and `t` in Matlab:

```
>> p(:, 1) = [0; 0]
>> p(:, 2) = ...
...
>> p(:, 9) = ...
>> t(:, 1) = [1; 2; 4]
>> t(:, 2) = ...
...
>> t(:, 8) = ...
```

and plot the triangulation by the Matlab-command:

```
>> pdemesh(p, [], t)
```

Solution:
(a) Pythagoras' theorem and simple trigonometry gives that $h_{K_j} = 1/\sqrt{2}$ and $\tau_{K_j} = \pi/4$ for all triangles K_j.
(b)

$$p = \begin{bmatrix} 0 & 0.5 & 1 & 0 & 0.5 & 1 & 0 & 0.5 & 1 \\ 0 & 0 & 0 & 0.5 & 0.5 & 0.5 & 1 & 1 & 1 \end{bmatrix}$$

(c) For example:

$$t = \begin{bmatrix} 1 & 2 & 2 & 3 & 4 & 5 & 5 & 6 \\ 2 & 5 & 3 & 6 & 5 & 8 & 6 & 9 \\ 4 & 4 & 5 & 5 & 7 & 7 & 8 & 8 \end{bmatrix}$$

(d) - □

Problem 2. Consider the same triangulation as in *Problem 1 (Week 5)*.
(a) The continuous piecewise linear function $\varphi_2(x, y)$ is defined by:

$$\varphi_2(N_2) = 1; \quad \varphi_2(N_j) = 0 \text{ for } j \neq 2.$$

Compute the analytical expression for φ_2. *Hint:* The analytical expressions on K_1, K_2 and K_3 may be determined by solving linear systems of equations as you have seen in the lecture. On the other triangles, $\varphi_2 \equiv 0$. Why?
(b) Plot φ_2 in Matlab by giving the command:

```
>> pdesurf(p, t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

or

```
>> pdemesh(p, [], t, [0; 1; 0; 0; 0; 0; 0; 0; 0])
```

Try both! The argument `[0; 1; 0; 0; 0; 0; 0; 0; 0]` is a *column vector* containing the *nodal values* of φ_2. Try also to plot some other "tent functions" φ_j!
(c) Since an arbitrary continuous piecewise linear function v can be written as a linear combination of "tent functions":

$$v(x, y) = v(N_1)\,\varphi_1(x, y) + \ldots + v(N_9)\,\varphi_9(x, y)$$

the "tent functions" $\{\varphi_i\}_{i=1}^{9}$ form a *basis* for the vector space V_h of continuous piecewise linear functions on the triangulation in Figure 1. What is the *dimension* of V_h?
(d) Try plotting some different functions in V_h using the Matlab commands `pdesurf` and `pdemesh`. *Hint:* Cf. how you plotted φ_2.
Solution:
(a) The analytical expression for φ_2 is different on each triangle. Since φ_2 is equal to one

at node N_2 and zero at all other nodes, it is only non-zero on triangles K_1, K_2 and K_3, and therefore $\varphi_2(x, y) = 0$ on $K_4 \cup K_5 \cup K_6 \cup K_7 \cup K_8$.

On each triangle φ_2 is a linear function $\varphi_2(x, y) = c_0 + c_1 x + c_2 y$, where c_0, c_1, and c_2 are constants to be determined for the three triangles K_1, K_2 and K_3. These constants can be computed by solving the linear system of equations (see *Applied Mathematics: B&S*, Part D, page 1032):

$$\begin{pmatrix} 1 & a_1^1 & a_2^1 \\ 1 & a_1^2 & a_2^2 \\ 1 & a_1^3 & a_2^3 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \varphi_2(a_1^1, a_2^1) \\ \varphi_2(a_1^2, a_2^2) \\ \varphi_2(a_1^3, a_2^3) \end{pmatrix}$$

where (a_1^1, a_2^1), (a_1^2, a_2^2) and (a_1^3, a_2^3) are the node coordinates of the triangle. On triangle K_1, with nodes N_1, N_2 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 0.5 & 0 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 0$, $c_1 = 2$ and $c_2 = 0$. That is: on K_1, $\varphi_2(x, y) = c_0 + c_1 x + c_2 y = 2x$.

Similarly, on triangle K_2, with nodes N_2, N_5 and N_4 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 0.5 & 0.5 \\ 1 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 1$, $c_1 = 0$ and $c_2 = -2$. That is: on K_2, $\varphi_2(x, y) = 1 - 2y$.

Finally, on triangle K_3, with nodes N_2, N_3 and N_5 (in that order), we get:

$$\begin{pmatrix} 1 & 0.5 & 0 \\ 1 & 1 & 0 \\ 1 & 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

which has the solution $c_0 = 2$, $c_1 = -2$ and $c_2 = -2$. That is: on K_3, $\varphi_2(x, y) = c_0 + c_1 x + c_2 y = 2 - 2(x + y)$.

(b) -

(c) The dimension of V_h is 9, since there are 9 nodes and therefore 9 basis functions.

(d) To plot $v(x, y) = 2\,\varphi_1(x, y) + 3\,\varphi_8(x, y)$:

```
>> pdesurf(p, t, [2; 0; 0; 0; 0; 0; 0; 3; 0])
```

or

```
>> pdemesh(p, [], t, [2; 0; 0; 0; 0; 0; 0; 3; 0])
```

Comment: `[]` is an empty matrix. We don't need to use this argument but we still have to pass something to the function `pdemesh`, which expects an argument (actually the "edge matrix" `e`) between `p` and `t`. See `>> help pdemesh` ☐