

Applied Mathematics/Differential Equations and
Scientific Computing,
part A

Solutions to Problems Week 1

August 29, 2003

Week 1:

Problem 1. Use the expressions $\lambda_a(x) = \frac{b-x}{b-a}$ and $\lambda_b(x) = \frac{x-a}{b-a}$ to show that

$$\lambda_a(x) + \lambda_b(x) = 1; \quad a \lambda_a(x) + b \lambda_b(x) = x.$$

Give a geometrical interpretation by plotting $\lambda_a(x)$, $\lambda_b(x)$, $\lambda_a(x) + \lambda_b(x)$, $a \lambda_a(x)$, $b \lambda_b(x)$, $a \lambda_a(x) + b \lambda_b(x)$ in the same figure.

Solution: Direct calculation gives $\lambda_a(x) + \lambda_b(x) = \frac{b-x}{b-a} + \frac{x-a}{b-a} = 1$ and $a \lambda_a(x) + b \lambda_b(x) = a \frac{b-x}{b-a} + b \frac{x-a}{b-a} = x$. The functions for the case $a = 2$ and $b = 3$ are plotted in Figure 1. \square

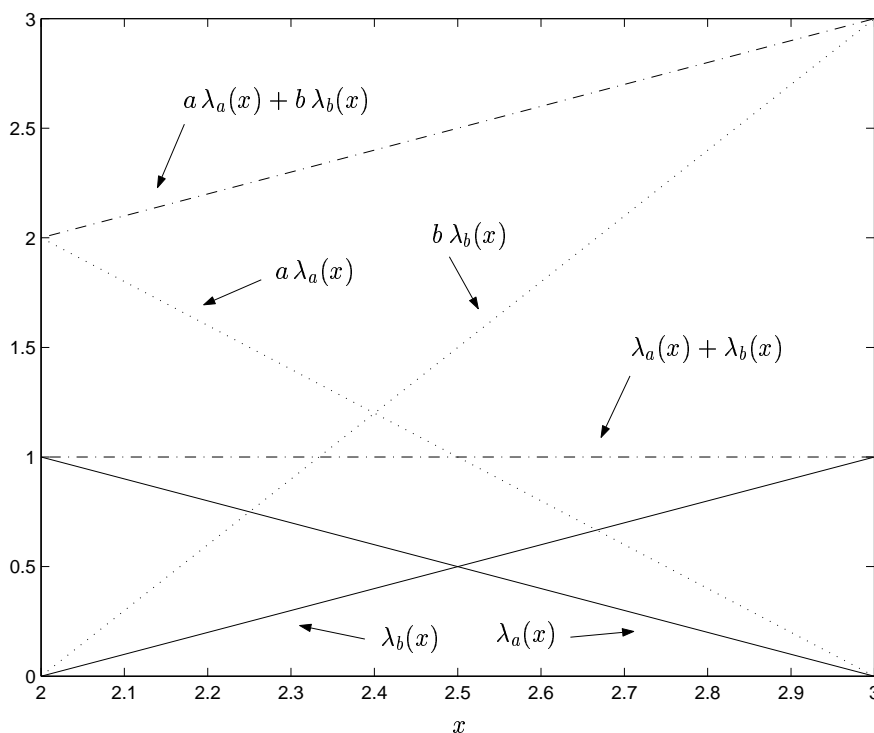


Figure 1: Problem 1 (Week 1). A plot of the functions.

Problem 2. Let $0 = x_0 < x_1 < x_2 < x_3 = 1$, where $x_1 = 1/6$ and $x_2 = 1/2$, be a partition of the interval $[0, 1]$ into three subintervals.

(a) Determine analytical expressions for the “hat-functions” $\varphi_0, \varphi_1, \varphi_2, \varphi_3$ in V_h (the space of continuous piecewise linear functions on this partition). Draw a figure.

(b) Which is the dimension of V_h ?

(c) Plot the mesh function $h(x)$.

Solution:

(a) The “hat-functions” are given by the formula (with obvious modifications for φ_0 and φ_3):

$$\varphi_i(x) = \begin{cases} 0, & x \notin [x_{i-1}, x_{i+1}] \\ \frac{x-x_{i-1}}{x_i-x_{i-1}}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1}-x}{x_{i+1}-x_i}, & x \in [x_i, x_{i+1}] \end{cases}$$

This gives

$$\varphi_0(x) = \begin{cases} 0, & x \notin [x_0, x_1] \\ 1-6x, & x \in [x_0, x_1] \end{cases}, \quad \varphi_1(x) = \begin{cases} 0, & x \notin [x_0, x_2] \\ 6x, & x \in [x_0, x_1] \\ \frac{3-6x}{2}, & x \in [x_1, x_2] \end{cases}$$

and

$$\varphi_2(x) = \begin{cases} 0, & x \notin [x_1, x_3] \\ \frac{6x-1}{2}, & x \in [x_1, x_2] \\ 2-2x, & x \in [x_2, x_3] \end{cases}, \quad \varphi_3(x) = \begin{cases} 0, & x \notin [x_2, x_3] \\ 2x-1, & x \in [x_2, x_3] \end{cases},$$

where $x_0 = 0$, $x_1 = 1/6$, $x_2 = 1/2$ and $x_3 = 1$. See Figure 2.

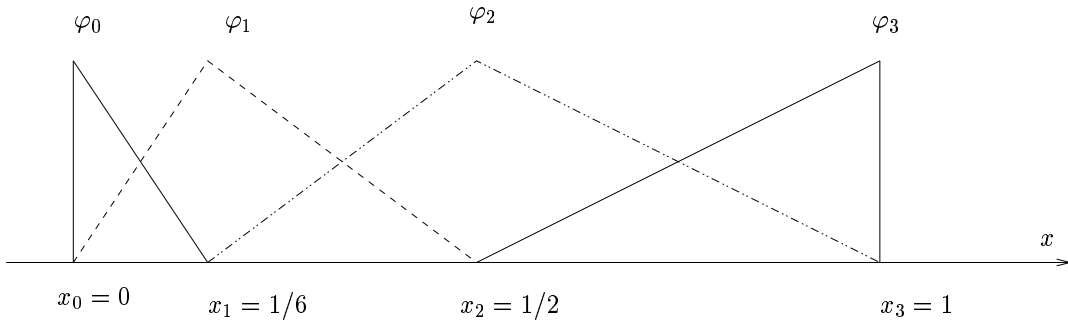


Figure 2: Problem 2(a) (Week 1). A plot of the “hat-functions”.

(b) The dimension of V_h is equal to the number of basis functions which in this case is 4.

(c) See Figure 3. □

Problem 3. Let $f : [0, 1] \rightarrow \mathbf{R}$ be a Lipschitz continuous function. Determine the linear interpolant $\pi f \in \mathcal{P}(0, 1)$ and plot f and πf in the same figure, when

(a) $f(x) = x^2$,

(b) $f(x) = \sin(\pi x)$.

Solution: In general, the linear (nodal) interpolant $\pi f \in \mathcal{P}(x_0, x_1)$ can be written as

$$\pi f(x) = f(x_0) \varphi_0(x) + f(x_1) \varphi_1(x),$$

where $\varphi_i(x)$ form a basis of the space $\mathcal{P}(x_0, x_1)$ of linear polynomials on $I = [x_0, x_1]$. The x_i 's are *nodes* where the interpolant's value is the same as the function's value. The basis functions we use are the “hat functions” $\varphi_i(x)$, $i = 0, 1$. Remember that $\varphi_0(x) = 1 - x$ and $\varphi_1(x) = x$, on $I = [0, 1]$.

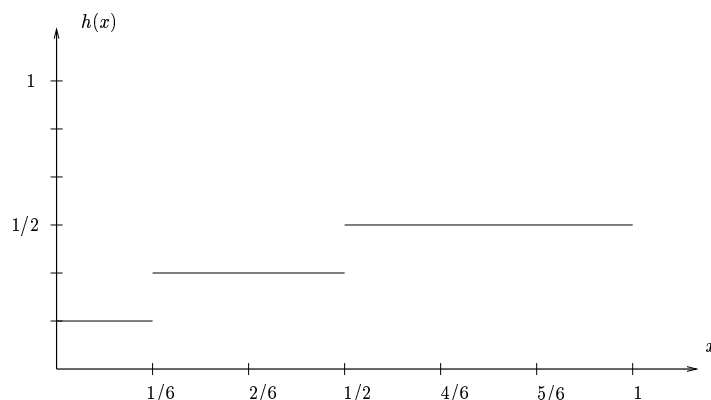


Figure 3: Problem 2(c) (Week 1). A plot of the mesh function.

(a) Therefore, the linear (nodal) interpolant of $f(x) = x^2$ on I can be written as

$$\begin{aligned}
 \pi f(x) &= \sum_{i=1}^2 f(x_i) \varphi_i(x) \\
 &= f(0) \varphi_0(x) + f(1) \varphi_1(x) \\
 &= 0 \cdot (1-x) + 1 \cdot x \\
 &= x
 \end{aligned}$$

(b) With $f(x) = \sin(\pi x)$, we analogously get

$$\begin{aligned}
 \pi f(x) &= \sum_{i=1}^2 f(x_i) \varphi_i(x) \\
 &= f(0) \varphi_0(x) + f(1) \varphi_1(x) \\
 &= \sin(0) (1-x) + \sin(\pi) x \\
 &= 0
 \end{aligned}$$

Obviously, the interpolant we've computed here is a poor approximation of $f(x) = \sin(\pi x)$. \square

Problem 4. Let $f : [0, 1] \rightarrow \mathbf{R}$ be a Lipschitz continuous function. Determine the continuous piecewise linear interpolant $\pi_h f \in V_h$, with $h(x)$ and V_h as in *Problem 2 (Week 1)*, and plot f and $\pi_h f$ in the same figure, when

- (a) $f(x) = x^2$,
- (b) $f(x) = \sin(\pi x)$.

Have we chosen a proper partition to approximate these functions? Can you think of a better one in case (a) and (b) if we are restricted to three subintervals?

Solution: The interval is partitioned according to $0 = x_0 < x_1 < x_2 < x_3 = 1$, with $x_1 = 1/6$ and $x_2 = 1/2$. On each subinterval we want the approximation, $\pi_h f(x)$, of $f(x)$

to be a straight line, $\alpha x + \beta$, and we want $\pi_h f(x)$ to *interpolate* $f(x)$ at the nodes $\{x_i\}_{i=0}^3$, i.e., $\pi_h f(x_i) = f(x_i)$, $i = 0, \dots, 3$. (Note that this makes $\pi_h f(x)$ continuous also at the node-points.) This is accomplished by defining

$$\pi_h f(x) = f(x_0) \varphi_0(x) + f(x_1) \varphi_1(x) + f(x_2) \varphi_2(x) + f(x_3) \varphi_3(x)$$

where the “hat-functions” $\{\varphi_i\}_{i=0}^3$ have been computed in *Problem 2 (Week 1)*.

(a) With $f(x) = x^2$ we get

$$\begin{aligned} \pi_h f(x) &= f(x_0) \varphi_0(x) + f(x_1) \varphi_1(x) + f(x_2) \varphi_2(x) + f(x_3) \varphi_3(x) \\ &= 0 \cdot \varphi_0(x) + \frac{1}{36} \cdot \varphi_1(x) + \frac{1}{4} \cdot \varphi_2(x) + 1 \cdot \varphi_3(x) \\ &= \begin{pmatrix} \begin{cases} 0 \cdot (1 - 6x) + \frac{1}{36} \cdot 6x, & x \in [0, 1/6] \\ \frac{1}{36} \cdot (\frac{3}{2} - 3x) + \frac{1}{4} \cdot (3x - \frac{1}{2}), & x \in [1/6, 1/2] \\ \frac{1}{4} \cdot (2 - 2x) + 1 \cdot (2x - 1), & x \in [1/2, 1] \end{cases} \\ \begin{cases} x/6, & x \in [0, 1/6] \\ -1/12 + 2x/3, & x \in [1/6, 1/2] \\ -1/2 + 3x/2, & x \in [1/2, 1] \end{cases} \end{pmatrix} \end{aligned}$$

Remark. We are usually content with writing a function $v \in V_h$ in the form $v(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) + c_2 \varphi_2(x) + c_3 \varphi_3(x)$ which tells us that the nodal values are (c_0, c_1, c_2, c_3) and that v is linear in between. Compare with plotting v in Matlab: `>> plot([x0 x1 x2 x3], [c0 c1 c2 c3])` connects the four points (x_i, c_i) with straight lines. If we for some reason need to know the analytical expressions on each subinterval, they may of course be computed as above.

(b) For $f(x) = \sin(\pi x)$ we similarly get

$$\begin{aligned} \pi_h f(x) &= f(x_0) \varphi_0(x) + f(x_1) \varphi_1(x) + f(x_2) \varphi_2(x) + f(x_3) \varphi_3(x) \\ &= 0 \cdot \varphi_0(x) + \frac{1}{2} \cdot \varphi_1(x) + 1 \cdot \varphi_2(x) + 0 \cdot \varphi_3(x) \\ &= \begin{pmatrix} \begin{cases} 0 \cdot (1 - 6x) + \frac{1}{2} \cdot 6x, & x \in [0, 1/6] \\ \frac{1}{2} \cdot (\frac{3}{2} - 3x) + 1 \cdot (3x - \frac{1}{2}), & x \in [1/6, 1/2] \\ 1 \cdot (2 - 2x) + 0 \cdot (2x - 1), & x \in [1/2, 1] \end{cases} \\ \begin{cases} 3x, & x \in [0, 1/6] \\ 3x/2 + 1/4, & x \in [1/6, 1/2] \\ 2 - 2x, & x \in [1/2, 1] \end{cases} \end{pmatrix} \end{aligned}$$

□

Problem 5. Let $h(x)$ be the mesh function for the partition defined in *Problem 2 (Week 1)*. Compute $\|f - \pi_h f\|_{L_\infty(0,1)}$ and $\frac{1}{8} \|h^2 f''\|_{L_\infty(0,1)}$, when

- (a) $f(x) = x^2$,
 (b) $f(x) = \sin(\pi x)$.

Hint: To compute $\|f - \pi_h f\|_{L^\infty(0,1)}$ you need to maximize the function $g(x) = |f(x) - \pi_h f(x)|$ on each subinterval and choose the largest of these three maxima. You can control your answers by also doing the computations with *Piecewise Polynomial Lab*.

If you think you have found better partitions in the end of *Problem 4 (Week 1)*, repeat the computations for these. Utilize *Piecewise Polynomial Lab*!

Solution: We have the mesh function

$$h(x) = \begin{cases} 1/6, & 0 < x < 1/6 \\ 1/3, & 1/6 < x < 1/2, \\ 1/2, & 1/2 < x < 1, \end{cases}$$

and want to compute $\|f - \pi_h f\|_{L^\infty(0,1)} = \max_{x \in [0,1]} |f(x) - \pi_h f(x)|$ and $\frac{1}{8} \|h^2 f''\|_{L^\infty(0,1)} = \frac{1}{8} \max_{x \in [0,1]} |h(x)^2 f''(x)|$.

(a) From *Problem 4(a) (Week 1)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |x^2 - x/6|, & 0 \leq x \leq 1/6, \\ |x^2 - (2x/3 - 1/12)|, & 1/6 \leq x \leq 1/2, \\ |x^2 - (3x/2 - 1/2)|, & 1/2 \leq x \leq 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = 2 > 0$, $f(x)$ is a *convex* function and the interpolant $\pi_h f(x)$ will therefore always be greater than $f(x)$. Further, since $f(x) - \pi_h f(x) = 0$ at the nodes, the local maxima will occur in the interior of the subintervals.)

$0 \leq x \leq 1/6$:

$$g(x) = \underbrace{|x^2 - \frac{x}{6}|}_{\leq 0, x \in [0, 1/6]} = \frac{x}{6} - x^2;$$

$$g'(x) = \frac{1}{6} - 2x = 0 \Rightarrow x = \frac{1}{12}; \quad g''(x) = -2 \Rightarrow \text{Maximum: } g\left(\frac{1}{12}\right) = \frac{1}{144}.$$

$1/6 \leq x \leq 1/2$:

$$g(x) = \underbrace{|x^2 - \left(\frac{2x}{3} - \frac{1}{12}\right)|}_{\leq 0, x \in [1/6, 1/2]} = \frac{2x}{3} - \frac{1}{12} - x^2;$$

$$g'(x) = \frac{2}{3} - 2x = 0 \Rightarrow x = \frac{1}{3}; \quad g''(x) = -2 \Rightarrow \text{Maximum: } g\left(\frac{1}{3}\right) = \frac{1}{36}.$$

$1/2 \leq x \leq 1$:

$$g(x) = \underbrace{|x^2 - \left(\frac{3x}{2} - \frac{1}{2}\right)|}_{\leq 0, x \in [1/2, 1]} = \frac{3x}{2} - \frac{1}{2} - x^2;$$

$$g'(x) = \frac{3}{2} - 2x = 0 \Rightarrow x = \frac{3}{4}; \quad g''(x) = -2 \Rightarrow \text{Maximum: } g\left(\frac{3}{4}\right) = \frac{1}{16}.$$

By comparing the three local maxima we get:

$$\|f - \pi_h f\|_{L_\infty(0,1)} = \frac{1}{16}.$$

Since

$$\frac{1}{8}|h(x)^2 f''(x)| = \begin{cases} \frac{2}{8 \cdot 36} = \frac{1}{144}, & 0 < x < 1/6, \\ \frac{2}{8 \cdot 9} = \frac{1}{36}, & 1/6 < x < 1/2, \\ \frac{2}{8 \cdot 4} = \frac{1}{16}, & 1/2 < x < 1, \end{cases}$$

we also get

$$\frac{1}{8}\|h^2 f''\|_{L_\infty(0,1)} = \frac{1}{16}.$$

Remark. The reason we have equality in this case is that $f''(x) = 2$ is *constant*.

(b) From *Problem 4(b) (Week 1)*, we have

$$|f(x) - \pi_h f(x)| = \begin{cases} |\sin(\pi x) - 3x|, & 0 \leq x \leq 1/6, \\ |\sin(\pi x) - (3x/2 + 1/4)|, & 1/6 \leq x \leq 1/2, \\ |\sin(\pi x) - (2 - 2x)|, & 1/2 \leq x \leq 1. \end{cases}$$

Find maxima for each subinterval: (*Note:* Since $f''(x) = -\pi^2 \sin(\pi x) < 0$, for $x \in (0, 1)$, $f(x)$ is *concave* on this interval and the interpolant $\pi_h f(x)$ will therefore be lesser than $f(x)$.)

$0 \leq x \leq 1/6$:

$$g(x) = \underbrace{|\sin(\pi x) - 3x|}_{\geq 0, x \in [0, 1/6]} = \sin(\pi x) - 3x;$$

$$g'(x) = \pi \cos(\pi x) - 3 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi} \arccos\left(\frac{3}{\pi}\right) \approx 0.096 \in [0, 1/6];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \text{Maximum: } g\left(\frac{1}{\pi} \arccos\left(\frac{3}{\pi}\right)\right) \approx 0.009.$$

$1/6 \leq x \leq 1/2$:

$$g(x) = \underbrace{\left| \sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right) \right|}_{\geq 0, x \in [1/6, 1/2]} = \sin(\pi x) - \left(\frac{3x}{2} + \frac{1}{4}\right);$$

$$g'(x) = \pi \cos(\pi x) - \frac{3}{2} = 0 \quad \Rightarrow \quad x = \frac{1}{\pi} \arccos\left(\frac{3}{2\pi}\right) \approx 0.342 \in [1/6, 1/2];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \text{Maximum: } g\left(\frac{1}{\pi} \arccos\left(\frac{3}{2\pi}\right)\right) \approx 0.116.$$

$1/2 \leq x \leq 1$:

$$g(x) = \underbrace{|\sin(\pi x) - (2 - 2x)|}_{\geq 0, x \in [1/2, 1]} = \sin(\pi x) - (2 - 2x);$$

$$g'(x) = \pi \cos(\pi x) + 2 = 0 \quad \Rightarrow \quad x = \frac{1}{\pi} \arccos\left(-\frac{2}{\pi}\right) \approx 0.720 \in [1/2, 1];$$

$$g''(x) = -\pi^2 \sin(\pi x) < 0 \quad \Rightarrow \quad \text{Maximum: } g\left(\frac{1}{\pi} \arccos\left(-\frac{2}{\pi}\right)\right) \approx 0.211.$$

By comparing the three local maxima we get:

$$\|f - \pi_h f\|_{L_\infty(0,1)} = \sin\left(\arccos\left(-\frac{2}{\pi}\right)\right) - 2\left(1 - \frac{1}{\pi} \arccos\left(-\frac{2}{\pi}\right)\right) \approx 0.211.$$

Since

$$\frac{1}{8} |h(x)^2 f''(x)| = \begin{cases} \frac{\pi^2 \sin(\pi x)}{8 \cdot 36} \leq \frac{\pi^2 \sin(\frac{\pi}{6})}{8 \cdot 36} = \frac{\pi^2}{576}, & 0 < x < 1/6, \\ \frac{\pi^2 \sin(\pi x)}{8 \cdot 9} \leq \frac{\pi^2 \sin(\frac{\pi}{2})}{8 \cdot 9} = \frac{\pi^2}{72}, & 1/6 < x < 1/2, \\ \frac{\pi^2 \sin(\pi x)}{8 \cdot 4} \leq \frac{\pi^2 \sin(\frac{\pi}{2})}{8 \cdot 4} = \frac{\pi^2}{32}, & 1/2 < x < 1, \end{cases}$$

we get

$$\frac{1}{8} \|h^2 f''\|_{L_\infty(0,1)} = \frac{\pi^2}{32} \approx 0.308.$$

□