Applied Mathematics/ Partial Differential Equations, part A

Problems V

Norms

1. Calculate $\|f\|_{L_\infty(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1,\, x_2) = x_2^2\,(x_1 - 2/3)^3$. Hint: To compute $\max_{(x_1,\,x_2)\in\Omega} |f(x_1,\, x_2)|$, maximize the absolute value of each factor of f separately.
(b) $f(x_1,\, x_2) = 11/36 - x_1^2 + x_1 - x_2^2 + 8x_2/3$. Hint: Compute both $\max_{(x_1,\,x_2)\in\Omega} f(x_1,\, x_2)$ and $\min_{(x_1,\,x_2)\in\Omega} f(x_1,\, x_2)$.
2. Calculate $\|f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1,\, x_2) = x_1\, x_2^2$.
(b) $f(x_1,\, x_2) = \sin(n\pi x_1)\sin(m\pi x_2)$ with n and m arbitrary integers.
Hint: $\sin^2 u = \frac{1-\cos(2u)}{2}$

Linear basis functions on a general triangle

3. Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1 x_1 + c_2 x_2,\ c_i \in \mathbf{R},\ i = 1,2,3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle K with corners a^1, a^2, and a^3. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a_1^1,\, a_2^1)$, $a^2 = (a_1^2,\, a_2^2)$, and $a^3 = (a_1^3,\, a_2^3)$) for the basis functions $\lambda_1,\, \lambda_2,\, \lambda_3 \in \mathcal{P}(K)$ defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{1}$$

with $i, j = 1, 2, 3$. Hint: set up the linear system of equations which relates c_0, c_1, and c_2 to the values at the corners $v(a^1)$, $v(a^2)$, and $v(a^3)$ of a function $v \in \mathcal{P}(K)$. Solve for the coefficients corresponding to corner values of the basis functions.

Quadrature

4. Derive an expression for the area of the triangle K in *Problem 3* in terms of the corner coordinates $a^1 = (a_1^1,\, a_2^1)$, $a^2 = (a_1^2,\, a_2^2)$ and $a^3 = (a_1^3,\, a_2^3)$.

L^2-projection

5. Consider the triangulation of $\Omega = [0,\, 2] \times [0,\, 1]$ into 3 triangles drawn in Figure 1.
(a) Compute the mass matrix M with elements $m_{ij} = \iint_\Omega \varphi_j(x,\, y)\, \varphi_i(x,\, y)\, dx\, dy$, $i, j = 1, \ldots, 5$.

Hint: The easiest way is to use the quadrature formula based on the value of the integrand, $\varphi_j(x,\, y)\, \varphi_i(x,\, y)$, at the mid-points on the triangle sides, since this formula is exact for polynomials of degree 2. It is also possible to write down explicit analytical expressions

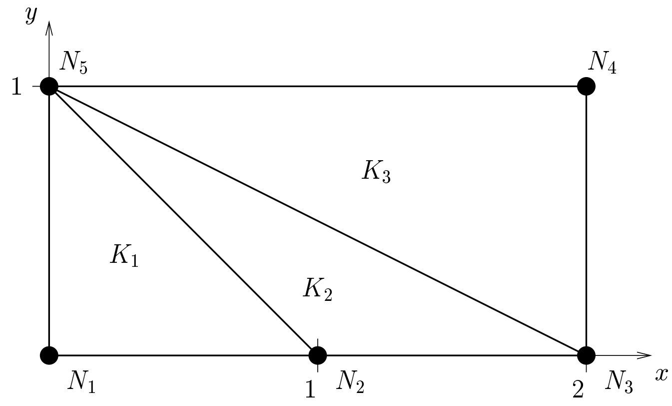

Figure 1: The triangulation in Problem 5.

for the "tent-functions" on each triangle (cf. *Problem 3*) and integrate the products analytically. This, however, is a much harder way. Observe that, using quadrature, we don't need to know the analytical expressions, only *the values at some given points* which are much easier to compute.
(b) Compute the "lumped" mass matrix $\hat{M}$, which is the diagonal matrix with the diagonal element in each row being the sum of the elements in the corresponding row of M.
(c*) Prove that, using nodal quadrature, the approximate mass matrix you get is actually the "lumped" mass matrix.

Hint: $\sum_{j=1}^{5} \varphi_j(x,\, y) \equiv 1$