1 Separation of Variables

Use separation of variables to solve the following problems:

Problem 1.1 *Solve the following initial boundary value problem*

$$\begin{cases} u_{xx} = u_t, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = 1, & 0 \le x \le 1. \end{cases}$$

Problem 1.2 *Solve the following problem with isolated boudaries*

$$\begin{cases} u_{xx} = u_t, & 0 < x < 1, \quad t > 0, \\ u_x(0,t) = u_x(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = x, & 0 \le x \le 1. \end{cases}$$

Problem 1.3 *Solve the following problem with isolated boudaries*

$$\begin{cases} u_t = 7u_{xx}, & 0 < x < \pi, \quad t > 0, \\ u(0,t) = u(\pi,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = 3\sin(2x) - 6\sin(5x), & 0 \le x \le \pi. \end{cases}$$

Problem 1.4 *Solve the homogeneous heat equation*

$$\begin{cases} u_{xx} = 4u_t, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = \min(x, 1-x), & 0 \le x \le 1. \end{cases}$$

Problem 1.5 *Solve the homogeneous heat equation*

$$\begin{cases} u_{xx} = u_t + u, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u(x,0) = \sin(\pi x), & 0 \le x \le 1. \end{cases}$$

Problem 1.6 *Solve the homogeneous heat equation*

$$\begin{cases} u_{xx} = u_t + u, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u(1,t) = 0, & \qquad\qquad\quad t > 0, \\ u_t(x,0) = 1, & 0 < x < 1. \end{cases}$$

Problem 1.7 *Solve the following heat equation having inhomogeneuos boundary condition*

$$\begin{cases} u_{xx} = u_t, & 0 < x < 1, & t > 0, \\ u_x(0,t) = 0, & u(1,t) = e^t, & t > 0, \\ u(x,0) = 1, & 0 < x < 1. & \end{cases}$$

Problem 1.8 *Solve the following inhomogeneuos heat equation*

$$\begin{cases} u_t = u_{xx} - \sin\frac{2\pi x}{L}, & 0 < x < L, & t > 0, \\ u(0,t) = u(L,t) = 0, & & t > 0, \\ u(x,0) = 0, & 0 < x < L. & \end{cases}$$

Problem 1.9 *Solve the following initial boundary value problem*

$$\begin{cases} u_{tt} = 4u_{xx}, & 0 < x < \pi, & t > 0, \\ u(0,t) = u(\pi,t) = 0, & & t \geq 0, \\ u(x,0) = \sin(3x), & u_t(x,0) = 2\sin(4x), & 0 \leq x \leq \pi. \end{cases}$$

Problem 1.10 *Solve the following initial boundary value problem*

$$\begin{cases} u_{tt} = u_{xx}, & 0 < x < 1, & t > 0, \\ u(0,t) = u_x(1,t) = 0, & & t > 0, \\ u_t(x,0) = 0, & u(x,0) = x^2 - 2x, & 0 < x < 1. \end{cases}$$

Problem 1.11 *Solve the following wave equation*

$$\begin{cases} u_{xx} = u_{tt} + u, & 0 < x < \pi, & t > 0, \\ u(0,t) = u_x(\pi,t) = 0, & & t > 0, \\ u(x,0) = x, & u_t(x,0) = 0, & 0 < x < \pi. \end{cases}$$

Problem 1.12 *Solve the following homogeneous wave equation*

$$\begin{cases} u_{tt} = u_{xx} - u, & 0 < x < 1, & t > 0, \\ u(0,t) = u(1,t) = 0, & & t > 0, \\ u_t(x,0) = 0, & u(x,0) = x\sin(\pi x), & 0 < x < 1. \end{cases}$$

Problem 1.13 *Solve the following inhomogeneous wave equation*

$$\begin{cases} u_{xx} = u_{tt} - 1, & 0 < x < 1, & t > 0, \\ u(0,t) = u(1,t) = 0, & & t > 0, \\ u(x,0) = u_t(x,0) = 0, & & 0 < x < 1. \end{cases}$$

Problem 1.14 *Solve the following inhomogeneous wave equation*

$$\begin{cases} u_{xx} = u_{tt} - 1, & 0 < x < 1, & t > 0, \\ u(0,t) = 0, & u(1,t) = 1, & t > 0, \\ u(x,0) = x, & u_t(x,0) = 0, & 0 < x < 1. \end{cases}$$

Problem 1.15 *Solve the Laplace equation in the rectangular domain given by* $\Omega = [0,\pi] \times [0,\pi]$*:*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < \pi, & 0 < y < \pi, \\ u(0,y) = 0, & u(\pi,y) = \pi, & 0 < y < \pi, \\ u(x,0) = 0, & u(x,\pi) = x, & 0 < x < \pi. \end{cases}$$

Problem 1.16 *a) Solve the Laplace equation in the rectangular domain given by* $\Omega = [0,a] \times [0,1]$*:*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < a, & 0 < y < 1, \\ u_x(0,y) = u_x(a,y) = 0, & & 0 < y < 1, \\ u(x,0) = 0, & u(x,1) = f(x), & 0 < x < a. \end{cases}$$

b) Show that

$$\int_0^a u(x,y)\,dx = y \int_0^a f(x)\,dx.$$

Problem 1.17 *Solve the homogeneous Laplace equation*

$$\begin{cases} u_{xx} + u_{yy} = 0, & 0 < x < 1, & y > 0, \\ u(0,y) = e^{-y}, & u(1,y) = 0, & y > 0, \\ u(x,0) = 1 - x, & |\lim_{y\to\infty} u(x,y)| < \infty, & 0 < x < 1. \end{cases}$$

Problem 1.18 *Solve the inhomogeneous Laplace equation*

$$\begin{cases} u_{xx} + u_{yy} = -2, & 0 < x < 1, & y > 0, \\ u(0,y) = u_x(1,y) = 0, & & y > 0, \\ u(x,0) = 0, & |\lim_{y\to\infty} u(x,y)| < \infty, & 0 < x < 1. \end{cases}$$

Answer 1.1

$$u(x,t) = \frac{4}{\pi}\sum_{k=0}^{\infty}\frac{1}{2k+1}e^{-(2k+1)^2\pi^2\,t}\sin(2k+1)\pi x$$

Answer 1.2

$$u(x,t) = \frac{1}{2} - \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}e^{-(n\pi)^2\,t}\cos(n\pi x).$$

Answer 1.3

$$u(x,t) = 3e^{-28\,t}\sin(2x) - 6e^{-175\,t}\sin(5x).$$

Answer 1.4

$$u(x,t) = \frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}e^{-\frac{(2n+1)^2\pi^2}{4}\times t}\sin(2n+1)\pi x$$

Answer 1.5

$$u(x,t) = e^{-(\pi^2+1)\,t}\sin(\pi x).$$

Answer 1.6

$$u(x,t) = -\frac{4}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)\Big(1+\pi^2(2n+1)^2\Big)}e^{-\Big(1+\pi^2(2n+1)^2\Big)\times t}\sin(2n+1)\pi x$$

Answer 1.7

$$u(x,t) = \frac{\cosh x}{\cosh 1}e^t + 2\sum_{n=0}^{\infty}\frac{(-1)^n}{\beta_n(1+\beta_n^2)}e^{-\beta_n^2\,t}\cos\beta_n x, \qquad \beta_n = (2n+1)\frac{\pi}{2}.$$

Answer 1.8

$$u(x,t) = \frac{L^2}{4\pi^2}\Big(e^{-\frac{4\pi^2}{L^2}\,t} - 1\Big)\sin\frac{2\pi x}{L}.$$

Answer 1.9

$$u(x,t) = \cos(6t)\sin(3x) + \frac{1}{4}\sin(8t)\sin(4x).$$

Answer 1.10

$$u(x,t) = -\frac{16}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\Big[\sin\frac{(2n+1)\pi}{2}(x+t) + \sin\frac{(2n+1)\pi}{2}(x-t)\Big].$$

Answer 1.11

$$u(x,t) = \frac{8}{\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)^2}\cos[\sqrt{(n+1/2)^2+1}\ t]\sin[(n+1/2)\pi x].$$

Answer 1.12

$$u(x,t) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)\cos[\sqrt{1+\pi^2 n^2}\ t], \qquad A_n = \begin{cases} -\frac{8n}{(n^2-1)\pi^2}, & n \text{ even} \\ \frac{1}{2}, & n=1, \\ 0, & \text{else} \end{cases}$$

Answer 1.13

$$u(x,t) = \frac{1}{2}(x-x^2) - \frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos[(2n+1)\pi t]\sin[(2n+1)\pi x].$$

Answer 1.14

$$u(x,t) = \frac{1}{2}(3x-x^2) - \frac{4}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\cos[(2n+1)\pi t]\sin[(2n+1)\pi x].$$

Note that the solution to problem 14: $u_{14} = u_{13} + x$, where u_9 is just the solution of problem 13. Why?

Answer 1.15

$$u(x,y) = x + \sum_{n=1}^{\infty}\frac{2(-1)^n}{n}\Big[\frac{e^{ny}}{1-e^{2n\pi}} + \frac{e^{-ny}}{1-e^{-2n\pi}}\Big]\sin(nx).$$

Answer 1.16 a)

$$u(x,y) = C_0 y + \sum_{n=1}^{\infty} C_n \cos\frac{n\pi x}{a}\sinh\frac{n\pi y}{a}, \qquad \text{where}$$

$$C_0 = \frac{1}{a}\int_0^a f(x)\,dx, \qquad \text{and } C_n = \frac{2}{a\sinh(n\pi/a)}\int_0^a f(x)\cos\frac{n\pi x}{a}\,dx.$$

b) Integrating $u(x,y)$ in x-direction we have $\int_0^a \cos\frac{n\pi x}{a}\,dx = 0$. Thus

$$\int_0^a u(x,y)\,dx = \int_0^a C_0 y\,dx = aC_0 y = y\int_0^a f(x)\,dx.$$

Answer 1.17

$$u(x,y) = (1-x)e^{-y} + \sum_{n=1}^{\infty} \frac{2}{n\pi(n^2\pi^2-1)} \left(e^{-y} - e^{-n\pi y}\right) \sin(n\pi x).$$

Answer 1.18

$$u(x,y) = x(2-x) - \frac{4}{\pi^3} \sum_{n=0}^{\infty} \frac{1}{(n+\frac{1}{2})^3} e^{-(n+\frac{1}{2})^2\pi^2 y} \sin(n+\frac{1}{2})\pi x.$$