Applied Mathematics/ Partial Differential Equations
part A

Solutions to Problems V

September 12, 2003

Problem 1. Calculate $\|f\|_{L_\infty(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1,\, x_2) = x_2^2\,(x_1 - 2/3)^3$. Hint: To compute $\max_{(x_1,\, x_2)\in\Omega} |f(x_1,\, x_2)|$, maximize the absolute value of each factor of f separately.
(b) $f(x_1,\, x_2) = 11/36 - x_1^2 + x_1 - x_2^2 + 8x_2/3$. Hint: Compute both $\max_{(x_1,\, x_2)\in\Omega} f(x_1,\, x_2)$ and $\min_{(x_1,\, x_2)\in\Omega} f(x_1,\, x_2)$.
Solution:
(a) Since $\|f\|_{L_\infty(\Omega)} = \max_{(x_1,x_2)\in\Omega} |f(x_1,x_2)|$ we want to find the maximum of the absolute value $|f(x_1,x_2)|$ of $f(x_1,x_2)$. From the hint we start by maximising the x_2-dependent factor over the interval $[0,1]$: The result is trivially 1 (for $x_2 = 1$). The maximum of the absolute value of the x_1-dependent factor is $8/27$ for $x_1 = 0$. This means that $\|f\|_{L_\infty(\Omega)} = 8/27$.
(b) We complete the squares to get:

$$f(x_1,x_2) = 11/36 - x_1^2 + x_1 - x_2^2 + 8x_2/3 = 7/3 - (x_1 - 1/2)^2 - (x_2 - 4/3)^2$$

We can now determine the maximum by minimising the two negative terms over Ω: Maximum of f thus occurs for $x_1 = 1/2$ and $x_2 = 1$ which gives us that $\max_{(x_1,x_2)\in\Omega} f(x_1,x_2) = 7/3 - 1/9 = 20/9$. In the same way minimum occurs when the last two terms are maximal, i.e., for $x_1 = 0$ or $x_1 = 1$ and $x_2 = 0$. Hence $\min_{(x_1,x_2)\in\Omega} f(x_1,x_2) = 7/3 - 1/4 - 16/9 = 11/36$. Since the minimum is positive, $f(x_1,\, x_2) = |f(x_1,\, x_2)|$ in Ω, and we conclude that $\|f\|_{L_\infty(\Omega)} = \max_{(x_1,\, x_2)\in\Omega} f(x_1,\, x_2) = 20/9$. $\square$

Problem 2. Calculate $\|f\|_{L^2(\Omega)}$ where $\Omega = [0,1] \times [0,1]$ and
(a) $f(x_1,\, x_2) = x_1\, x_2^2$.
(b) $f(x_1,\, x_2) = \sin(n\pi x_1)\sin(m\pi x_2)$ with n and m arbitrary integers.
Hint: $\sin^2 u = \frac{1-\cos(2u)}{2}$
Solution: The $L^2(\Omega)$-norm of f is defined by: $\|f\|_{L^2(\Omega)} = (\iint_\Omega f(x_1,x_2)^2\, dx_1\, dx_2)^{\frac{1}{2}}$.
(a)

$$\|f\|^2_{L^2(\Omega)} = \int_0^1\int_0^1 x_1^2 x_2^4\, dx_1\, dx_2 = \int_0^1 x_1^2\, dx_1 \int_0^1 x_2^4\, dx_2 = [x_1^3/3]_0^1 \cdot [x_2^5/5]_0^1 = \frac{1}{15}$$

so $\|f\|_{L^2(\Omega)} = \frac{1}{\sqrt{15}}$.
(b) If n and/or m is equal to zero then f is identically equal to zero implying that $\|f\|_{L^2(\Omega)} = 0$. Otherwise we get:

$$\begin{aligned}
\|f\|^2_{L^2(\Omega)} &= \int_0^1\int_0^1 \sin^2(n\pi x_1)\sin^2(m\pi x_2)\, dx_1\, dx_2 \\
&= \int_0^1 \frac{1-\cos(2n\pi x_1)}{2}\, dx_1 \cdot \int_0^1 \frac{1-\cos(2m\pi x_2)}{2}\, dx_2 \\
&= \left[x_1/2 - \frac{\sin(2n\pi x_1)}{4n\pi}\right]_0^1 \cdot \left[x_2/2 - \frac{\sin(2m\pi x_2)}{4m\pi}\right]_0^1 \\
&= \left(1/2 - \frac{\sin(2n\pi)}{4n\pi}\right) \cdot \left(1/2 - \frac{\sin(2m\pi)}{4m\pi}\right) = 1/4,
\end{aligned}$$

and thus $\|f\|_{L^2(\Omega)} = 1/2$ if $n \neq 0$ and $m \neq 0$.

Problem 3. Let $\mathcal{P}(K) = \{v(x) = c_0 + c_1x_1 + c_2x_2,\ c_i \in \mathbf{R},\ i = 1, 2, 3;\ x = (x_1, x_2) \in K\}$ be the space of linear polynomials defined on a triangle K with corners a^1, a^2, and a^3. Derive explicit expressions (in terms of the corner coordinates $a^1 = (a^1_1,\ a^1_2)$, $a^2 = (a^2_1,\ a^2_2)$, and $a^3 = (a^3_1,\ a^3_2)$) for the basis functions $\lambda_1,\ \lambda_2,\ \lambda_3 \in \mathcal{P}(K)$ defined by

$$\lambda_i(a^j) = \begin{cases} 1 & i = j, \\ 0 & i \neq j, \end{cases} \tag{1}$$

with $i, j = 1, 2, 3$. Hint: set up the linear system of equations which relates c_0, c_1, and c_2 to the values at the corners $v(a^1)$, $v(a^2)$, and $v(a^3)$ of a function $v \in \mathcal{P}(K)$. Solve for the coefficients corresponding to corner values of the basis functions.

Solution: Look at the basis function λ_1 first. Since λ_1 is *linear* on K we make the Ansatz $\lambda_1(x_1,\ x_2) = c_0 + c_1x_1 + c_2x_2$. According to the definition λ_1 has the value one in a^1 and zero in a^2 and a^3. (See Figure 1.) Hence, we have in these corners respectively:

$$\begin{cases} 1 = c_0 + c_1a^1_1 + c_2a^1_2 \\ 0 = c_0 + c_1a^2_1 + c_2a^2_2 \\ 0 = c_0 + c_1a^3_1 + c_2a^3_2 \end{cases}$$

Or in matrix form:

$$\underbrace{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}}_{b} = \underbrace{\begin{pmatrix} 1 & a^1_1 & a^1_2 \\ 1 & a^2_1 & a^2_2 \\ 1 & a^3_1 & a^3_2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} c_0 \\ c_1 \\ c_2 \end{pmatrix}}_{c}$$

We have three equations and three unknowns (c_0, c_1 and c_2). We can solve the linear system of equations above by Gaussian elimination. The result is

$$\begin{aligned} c_0 &= \frac{a^2_1a^3_2 - a^3_1a^2_2}{\det A} \\ c_1 &= \frac{a^2_2 - a^3_2}{\det A} \\ c_2 &= \frac{a^3_1 - a^2_1}{\det A} \end{aligned}$$

where $\det A = a^3_1a^1_2 + a^2_1a^3_2 - a^2_1a^1_2 - a^3_1a^2_2 - a^1_1a^3_2 + a^1_1a^2_2$.

For the basis function λ_2 we get the same matrix A as above, but here $b = (0,\ 1,\ 0)^T$ (since λ_2 is one in the node a^2 and zero in the other two nodes). Solving the system of equations gives

$$c_0 = \frac{a^3_1a^1_2 - a^1_1a^3_2}{\det A}$$

$$
\begin{aligned}
c_1 &= \frac{a_2^3 - a_2^1}{\det A} \\
c_2 &= \frac{a_1^1 - a_1^3}{\det A}
\end{aligned}
$$

And similarly for λ_3 with $b = (0,\, 0,\, 1)^T$ gives the coefficients

$$
\begin{aligned}
c_0 &= \frac{a_1^1 a_2^2 - a_1^2 a_2^1}{\det A} \\
c_1 &= \frac{a_2^1 - a_2^2}{\det A} \\
c_2 &= \frac{a_1^2 - a_1^1}{\det A}
\end{aligned}
$$

Remark. Note that $\det A$ equals $2\,\mu(K)$ where $\mu(K)$ is the area of K. See *Problem 4 (Week 6)*. Note further that it might not be necessary to actually compute λ_2 and λ_3. Given the expression for λ_1 it is possible to make a permutation of the node indices.

□

Problem 4. Derive an expression for the area of the triangle K in *Problem 3 (Week 6)* in terms of the corner coordinates $a^1 = (a_1^1,\, a_2^1)$, $a^2 = (a_1^2,\, a_2^2)$ and $a^3 = (a_1^3,\, a_2^3)$.
Solution:

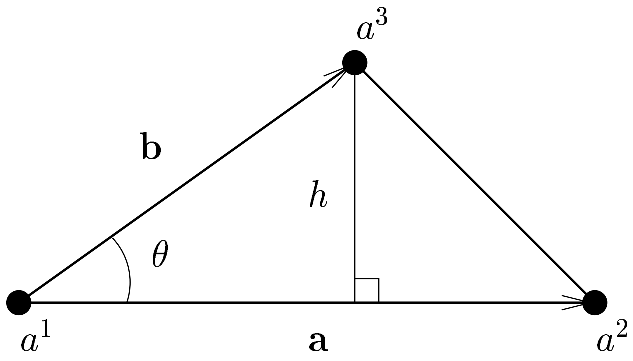

Figure 1: Problem 3 and Problem 4 (Week 6).

From Figure 1 we calculate the area $\mu(K)$ as follows.

$$\mu(K) = \tfrac{1}{2}\,|\mathbf{a}|h = \tfrac{1}{2}\,|\mathbf{a}||\mathbf{b}|\sin\theta = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| \tag{2}$$

Now, clearly the vectors $\mathbf{a}$ and $\mathbf{b}$ are given by

$$\mathbf{a} = a^2 - a^1 = (a_1^2 - a_1^1,\, a_2^2 - a_2^1), \tag{3}$$

$$\mathbf{b} = a^3 - a^1 = (a_1^3 - a_1^1,\, a_2^3 - a_2^1). \tag{4}$$

Explicitly the area is thus given by

$$\mu(K) = \tfrac{1}{2}|\mathbf{a}\times\mathbf{b}| = |\begin{vmatrix} a_1^2 - a_1^1 & a_2^2 - a_2^1 \\ a_1^3 - a_1^1 & a_2^3 - a_2^1 \end{vmatrix}| \tag{5}$$

$$= \tfrac{1}{2}|(a_1^2 - a_1^1)(a_2^3 - a_2^1) - (a_2^2 - a_2^1)(a_1^3 - a_1^1)|. \tag{6}$$

Note that the cross-product between vectors in two dimensions is a number.

Remark. With $\mathbf{a}$ and $\mathbf{b}$ oriented as in Figure 1 the cross-product $\mathbf{a}\times\mathbf{b}$ is positive and thus $\mu(K) = \frac{1}{2}(\mathbf{a}\times\mathbf{b})$.

□

Problem 5. Consider the triangulation of $\Omega = [0,\, 2]\times[0,\, 1]$ into 3 triangles drawn in Figure 2.

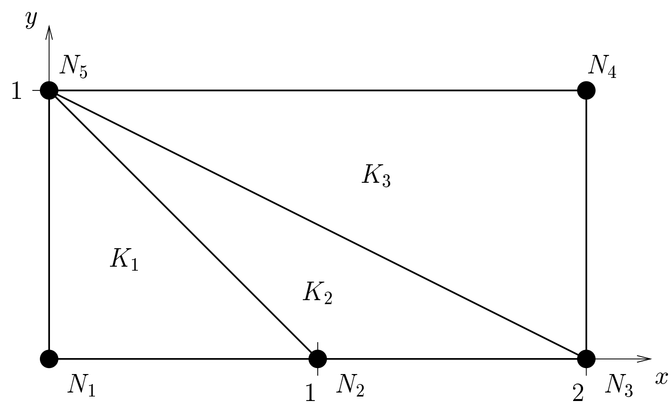

Figure 2: Problem 5 (Week 6). The triangulation of Ω.

(a) Compute the mass matrix M with elements $m_{ij} = \iint_\Omega \varphi_j(x,\, y)\, \varphi_i(x,\, y)\, dx\, dy,\ i, j = 1, \ldots, 5$.

Hint: The easiest way is to use the quadrature formula based on the value of the integrand, $\varphi_j(x,\, y)\, \varphi_i(x,\, y)$, at the mid-points on the triangle sides, since this formula is exact for polynomials of degree 2. It is also possible to write down explicit analytical expressions for the "tent-functions" on each triangle (cf. *Problem 3 (Week 6)*) and integrate the products analytically. This, however, is a much harder way. Observe that, using quadrature, we don't need to know the analytical expressions, only *the values at some given points* which are much easier to compute.

(b) Compute the "lumped" mass matrix $\hat{M}$, which is the diagonal matrix with the diagonal element in each row being the sum of the elements in the corresponding row of M.

(c*) Prove that, using nodal quadrature, the approximate mass matrix you get is actually the "lumped" mass matrix.

Hint: $\sum_{j=1}^{5} \varphi_j(x,\, y) \equiv 1$

Solution:

(a) We start to compute the area $\mu(K_i)$ of the triangles, $i = 1, 2, 3$:

$$\mu(K_1) = \frac{1 \cdot 1}{2} = \frac{1}{2},$$
$$\mu(K_2) = \frac{1 \cdot 1}{2} = \frac{1}{2},$$
$$\mu(K_3) = \frac{2 \cdot 1}{2} = 1.$$

Then, we compute a few elements of M: m_{11}, m_{12}, m_{13}, and m_{22}. Note that the integrands $\varphi_1\, \varphi_1$ and $\varphi_2\, \varphi_1$ are non-zero only over K_1, and $\varphi_2\, \varphi_2$ is non-zero over K_1 and K_2. On the other hand $\varphi_3\, \varphi_1$ is nowhere non-zero and therefore $m_{13} = 0$.

$$\begin{aligned} m_{11} &= \iint_\Omega \varphi_1\, \varphi_1\, dxdy = \frac{(\varphi_1(\frac{1}{2}, 0))^2 + (\varphi_1(0, \frac{1}{2}))^2 + (\varphi_1(\frac{1}{2}, \frac{1}{2}))^2}{3}\, \mu(K_1) \\ &= \frac{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 0 \cdot 0}{3}\, \mu(K_1) = \frac{1}{6}\, \mu(K_1) = \frac{1}{12}, \\ m_{12} &= (M \text{ symmetric!}) = m_{21} = \frac{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot 0 + 0 \cdot \frac{1}{2}}{3}\, \mu(K_1) = \frac{1}{12}\, \mu(K_1) = \frac{1}{24}, \\ m_{22} &= \frac{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 0}{3}\, \mu(K_1) + \frac{\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2} + 0}{3}\, \mu(K_2) = \frac{1}{6}\, (\mu(K_1) + \mu(K_2)) = \frac{1}{6}. \end{aligned}$$

Continuing analogously gives:

$$M = \begin{bmatrix} \frac{1}{12} & \frac{1}{24} & 0 & 0 & \frac{1}{24} \\ \frac{1}{24} & \frac{1}{6} & \frac{1}{24} & 0 & \frac{1}{12} \\ 0 & \frac{1}{24} & \frac{1}{4} & \frac{1}{12} & \frac{1}{8} \\ 0 & 0 & \frac{1}{12} & \frac{1}{6} & \frac{1}{12} \\ \frac{1}{24} & \frac{1}{12} & \frac{1}{8} & \frac{1}{12} & \frac{1}{3} \end{bmatrix}$$

(b) From $\hat{m}_{ii} = \sum_{j=1}^{5} m_{ij}, \quad i = 1, \ldots, 5$, we compute:

$$\hat{m}_{11} = \frac{1}{12} + \frac{1}{24} + 0 + 0 + \frac{1}{24} = \frac{1}{6}.$$

Analogously:

$$\hat{m}_{22} = \frac{1}{3}; \quad \hat{m}_{33} = \frac{1}{2}; \quad \hat{m}_{44} = \frac{1}{3}; \quad \hat{m}_{55} = \frac{2}{3}.$$

Thus:

$$\hat{M} = \begin{bmatrix} \frac{1}{6} & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 & \frac{2}{3} \end{bmatrix}$$

(c*) *Hint:* Adding the elements in row number i gives:

$$\hat{m}_{ii} = \iint_\Omega \left(\sum_{j=1}^{5} \varphi_j(x,\, y) \right) \varphi_i(x,\, y)\, dx\, dy = \iint_\Omega \varphi_i(x,\, y)\, dx\, dy.$$

Now use the formula for the volume of a pyramid, and compare the result to what you get when using nodal quadrature. $\square$