Lösningar TMA205/225 K2, Bt2, Kf2, 4 poäng, 03-10-25

1. Betrakta begynnelsevärdesproblemet

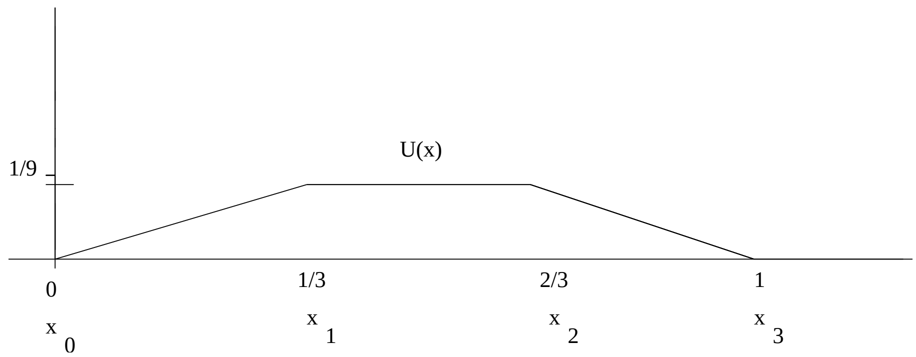
$$-u'' = 1, \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

(a) Lös ekvationen analytiskt.
(b) Dela in intervallet $[0,1]$ i tre delintervall och beräkna för hand den styckvis linjära finita element lösningen U.

Lösning: (a) Vi har att

$$\begin{aligned} &-u''(x) = 1 \Longrightarrow u'(x) = -x + C \Longrightarrow u(x) = -\tfrac{1}{2}x^2 + Cx. \\ &u(1) = 0 \Longrightarrow -\tfrac{1}{2} + C = 0 \Longrightarrow C = \tfrac{1}{2} \Longrightarrow \underline{u(x) = \tfrac{1}{2}x(1-x)}. \end{aligned}$$

(b) Med dem givna homogena randdata och partitionen $x_0 = 0$, $x_1 = 1/3$, $x_2 = 2/3$, $x_3 = 1$, har vi



med endast två bas funktioner:

$$\varphi_1(x) = \begin{cases} 3x, & 0 \le x \le 1/3, \\ 3(2/3 - x), & 1/3 \le x \le 2/3, \end{cases}$$

och

$$\varphi_2(x) = \begin{cases} 3(x - 1/3), & 1/3 \le x \le 2/3, \\ 3(1 - x), & 2/3 \le x \le 1. \end{cases}$$

För styckvis linjära finit element lösningen gäller

$$\begin{aligned} U(x) &= U_1\varphi_1(x) + U_2\varphi_2(x) = u(x_1)\varphi_1(x) + u(x_2)\varphi_2(x) = \\ &= \frac{1}{9}\Big(\varphi_1(x) + \varphi_2(x)\Big). \end{aligned}$$

Insättningen av $\varphi_i, \ i = 1, 2$ ger

$$U(x) = \begin{cases} x/3, & 0 \le x \le 1/3, \\ (2/3 - x)/3 + (x - 1/3)/3 = 1/9, & 1/3 \le x \le 2/3, \\ (1 - x)/3, & 2/3 < x < 1. \end{cases}$$

2. Visa en a priori feluppskattning för en styckvis linjär FEM för problemet

$$-u'' + u = f, \quad x \in (0,1), \quad u(0) = u(1) = 0.$$

Lösning: Varitionsformuleringen blir (efter partiell integration)

$$(VF) \qquad \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \qquad \forall v \in H_0^1.$$

Låt nu

$$V_h^0 =: \{v:\ v \text{ linjär, styckvis kontinuerlig och } v(0) = v(1) = 0\} \subset H_0^1.$$

Finit element formulering: Finn $U \in V_h^0$ så att

$$(FEM) \qquad \int_0^1 U'v'\,dx + \int_0^1 Uv\,dx = \int_0^1 fv\,dx, \qquad \forall v \in V_h^0.$$

Nu (VF)-(FEM) ger Galerkin ortogonalitet:

$$(G\perp) \qquad \int_0^1 (u-U)'v'\,dx + \int_0^1 (u-U)v\,dx = 0, \qquad \forall v \in V_h^0.$$

Relevant norm för uppskattning av felet $e = u - U$, blir då energinormen:

$$\|e\|_E := \Big(\int_0^1 (e')^2\,dx + \int_0^1 e^2\,dx\Big)^{1/2}.$$

Då har vi med $v \in V_h^0$ att

$$\begin{aligned}
\|e\|_E^2 =& \|(u-U)'\|^2 + \|u-U\|^2\\
=& \int_0^1 (u-U)'(u-v+v-U)'\,dx + \int_0^1 (u-U)(u-v+v-U)\,dx\\
=& \int_0^1 (u-U)'(u-v)'\,dx + \int_0^1 (u-U)(u-v)\,dx\\
&+ \int_0^1 (u-U)'(v-U)'\,dx + \int_0^1 (u-U)(v-U)\,dx \{\text{denna rad } =0 \text{ pga } G\perp\}\\
=& \int_0^1 (u-U)'(u-v)'\,dx + \int_0^1 (u-U)(u-v)\,dx\\
\le& \{\text{Cauchy-Schwartz}\} \le \|(u-U)'\|\|(u-v)'\| + \|u-U\|\|u-v\|\\
\le& \frac{1}{2}\|(u-U)'\|^2 + \frac{1}{2}\|(u-v)'\|^2 + \frac{1}{2}\|u-U\|^2 + \frac{1}{2}\|u-v\|^2.
\end{aligned}$$

Alltså

$$\|e\|_E^2 = \|(u-U)'\|^2 + \|u-U\|^2 \le \|(u-v)'\|^2 + \|u-v\|^2, \quad \forall v \in V_h^0.$$

Välj nu $v = \pi_h u$. $\pi_h u$ är den linjära interpolanten av u. Genom att använda feluppskattningar för interpolanten får vi

$$\begin{aligned}
\|e\|_E^2 &\le \|(u-\pi_h u)'\|^2 + \|u-\pi_h u\|^2\\
&\le C_i^2\|hu''\|^2 + C_i^2\|hu'\|^2 \le \Big(C_i\|hu''\| + C_i\|hu'\|\Big)^2.
\end{aligned}$$

Alltså vi har följande a priori feluppskattning:

$$\|e\|_E \le C_i\Big(\|hu''\| + \|hu'\|\Big) = \mathcal{O}(h).$$

3. Consider the boundary value problem

$$\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d,\ d = 2, 3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}$$

Prove the L_2 stability estimate

$$||\nabla u||^2_{L_2(\Omega)} + \frac{1}{2}||u||^2_{L_2(\Gamma)} \le \frac{1}{2}||g||^2_{L_2(\Gamma)}.$$

Solution: Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u \cdot \nabla u = -\int_\Omega (\Delta u)u + \int_{\partial\Omega} \frac{\partial u}{\partial n} u = \int_{\partial\Omega} (g - u)u.$$

In other words

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Gamma)} = \int_{\partial\Omega} gu \le ||g||^2_{L_2(\Gamma)} ||u||^2_{L_2(\Gamma)} \le \frac{1}{2}||g||^2_{L_2(\Gamma)} + \frac{1}{2}||u||^2_{L_2(\Gamma)},$$

which gives the desired estimate.

4. Let $\|\cdot\|$ denote the $L_2(0,1)$-norm. Consider the one-dimensional problem:

$$\begin{cases} \dot{u} - u'' = 0, & 0 < x < 1, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1, & \\ u(0,t) = u_x(1,t) = 0, & & t > 0. \end{cases}$$

a) Verify that $\|u(\cdot,t)\|$ and $\|u_x(\cdot,t)\|$ are nonincreasing functions in t.
b) Show that $\|u_x(\cdot,t)\| \to 0$ as $t \to \infty$. c) Interprete a) and b) physically.

Solution: a) Multiply the equation by u, integrate over $(0,1)$. Integrating by parts and using the boundary conditions we have

$$\begin{aligned} 0 &= \int_0^1 \dot{u}u\,dx - \int_0^1 u''u\,dx \\ &= \frac{1}{2}\frac{d}{dt}\int_0^1 u^2\,dx + \int_0^1 \Big(u'\Big)^2\,dx - u'(1,t)u(1,t) + u'(0,t)u(0,t) \\ &= \frac{1}{2}\frac{d}{dt}||u||^2 + ||u'||^2 \end{aligned}$$

This implies that

$$\frac{1}{2}\frac{d}{dt}||u||^2 = -||u'||^2 \le 0,$$

with equality only if $u_0 = 0$. Thus u is decreasing.

Multiplying the equation by u'' would yiels

$$0 = \int_0^1 \dot{u}u''\,dx - \int_0^1 (u'')^2\,dx = [\dot{u}u']_0^1 - \frac{1}{2}\int_0^1 \frac{\partial}{\partial t}(u')^2 + 2(u'')^2\,dx.$$

Now since $u(0,t) = 0$ we get $\dot{u}(0,t) = \frac{\partial}{\partial t}\Big(u(0,t)\Big) = 0$ which together with $u'(1,t) = 0$ gives that

$$\frac{1}{2}\frac{d}{dt}||u'||^2 - ||u''||^2 \le 0,$$

and hence u' is decreasing as well.
b) Time integration of the first equality in a) yields:

$$||u(\cdot,T)||^2 + 2\int_0^T ||u'||^2\,dt = ||u(\cdot,0)||^2 = C.$$

To get a convergent integral as $T \to \infty$ it is necessary to have $||u'||^2 \to 0$.
c) In the absence of a heat source, the temperature and heat flux are decreasing (non-increasing) in time, in particular the heat flux tends to 0 as $T \to \infty$.

5. Antag att $f \in \mathcal{C}^2(a,b)$. Visa att det finns en interpolationskonstant C_i, oberoende av f och intervallet (a,b), så att för linjär interpolation $\Pi_1 f$ har vi följande feluppskattning

$$||\Pi_1 f - f||_{L_\infty(a,b)} \leq C_i (b-a)^2 ||f''||_{L_\infty(a,b)}.$$

Lösning: See PDE Lecture Notes Chapter 5

6. Betrakta Dirichlets problem för Poissonekvationen, $-\Delta u = f$, i Ω, $u = 0$, på $\partial\Omega$, där Ω är ett polygonområde i $\mathbb{R}^2$. Visa stabilitetsuppskattningen

$$\|\nabla u\| \leq C_\Omega \|f\|.$$

Lösning: Se PDE Lecture Notes Chapter 15.

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