Differential Equations and Scientific Computing, part A

Solutions to Problems Week 2

September 8, 2003

Week 2:

Problem 1. Let $I = (0,1)$ and $f(x) = x^2$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on the single interval $(0,1)$.
(c) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on the single interval $(0,1)$.
(d) Compute the errors in (b) and (c). Compare with theory.
(e) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using the *trapezoidal rule* on each subinterval.
(f) Compute an approximation of $\int_I f(x)\,dx$ by using the *mid-point rule* on each subinterval.
(g) Compute the errors in (e) and (f), and compare with the errors in (b) and (c) respectively. By what factor has the error decreased?
Solution:
(a)

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

(b)

$$\int_0^1 x^2\,dx \approx \frac{0^2+1^2}{2} = \frac{1}{2}$$

(c)

$$\int_0^1 x^2\,dx \approx (\frac{0+1}{2})^2 = \frac{1}{4}$$

(d) The error for the trapezoidal rule is $|\frac{1}{3}-\frac{1}{2}| = \frac{1}{6}$ and the error for the mid-point rule is $|\frac{1}{3}-\frac{1}{4}| = \frac{1}{12}$. Both agree with the bounds for the error on a single interval of length h: $\frac{h^3}{12}\max_{y\in[0,1]}|f''(y)| = \frac{1}{6}$ and $\frac{h^3}{24}\max_{y\in[0,1]}|f''(y)| = \frac{1}{12}$ in *Quadrature (1D)*.

Remark. The reason that we have *equality* between the error and the error bound in this case is that $f''(y) = 2$ is *constant*.

(e)

$$\int_0^1 x^2\,dx \approx \frac{0^2+(\frac{1}{2})^2}{4} + \frac{(\frac{1}{2})^2+1^2}{4} = \frac{3}{8}$$

(f)

$$\int_0^1 x^2\,dx \approx \frac{(\frac{1}{4})^2}{2} + \frac{(\frac{3}{4})^2}{2} = \frac{5}{16}$$

(g) The trapezoidal rule gives $|\frac{1}{3}-\frac{3}{8}| = \frac{1}{24}$ which means that the error decreases by a factor 4 when the mesh size decreases by a factor 2. This agrees with the *global* error bound $\frac{b-a}{12}\max_{y\in[0,1]}|h^2(y)f''(y)|$ in *Quadrature (1D)*. For the mid-point rule we get the error $|\frac{1}{3}-\frac{5}{16}| = \frac{1}{48}$ which shows a similar behaviour. $\square$

Problem 2. Let $I = (0, 1)$ and $f(x) = x^4$ for $x \in I$.
(a) Compute (analytically) $\int_I f(x)\,dx$.
(b) Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on the single interval $(0, 1)$.
(c) Compute the error in (b). Compare with theory.
(d) Divide I into two subintervals of equal length. Compute an approximation of $\int_I f(x)\,dx$ by using *Simpson's rule* on each subinterval.
(e) Compute the error in (d), and compare with the error in (b). By what factor has the error decreased?
Solution:
(a)

$$\int_I f(x)\,dx = \int_0^1 x^4\,dx = \frac{1}{5}$$

(b)

$$\int_I f(x)\,dx \approx \frac{f(0) + 4f(\frac{0+1}{2}) + f(1)}{6} = \frac{0 + 4(\frac{1}{2})^4 + 1}{6} = \frac{5}{24}$$

(c) $Error_1 = |\frac{1}{5} - \frac{5}{24}| = |\frac{24}{120} - \frac{25}{120}| = \frac{1}{120}$. From the theory we know that the error using *Simpson's rule* on a single interval of length h must be less than or equal to

$$\frac{h^5}{2880} \max_{y\in[0,1]} |f^{(4)}(y)| = \frac{24}{2880} = \frac{1}{120}$$

Remark. The reason that we have *equality* between the error and the error bound in this case is that $f^{(4)}(y) = 24$ is *constant.*
(d)

$$\begin{aligned}
\int_I f(x)\,dx &= \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx \\
&\approx \frac{f(0) + 4f(\frac{0+1/2}{2}) + f(\frac{1}{2})}{6} \cdot \frac{1}{2} + \frac{f(\frac{1}{2}) + 4f(\frac{1/2+1}{2}) + f(1)}{6} \cdot \frac{1}{2} \\
&= \frac{0 + 4(\frac{1}{4})^4 + (\frac{1}{2})^4}{12} + \frac{(\frac{1}{2})^4 + 4(\frac{3}{4})^4 + 1^4}{12} = \frac{77}{384}
\end{aligned}$$

(e) $Error_2 = |\frac{1}{5} - \frac{77}{384}| = |\frac{384-5\cdot 77}{1920}| = \frac{1}{1920}$. If we compare this error to the one computed above in exercise (c):

$$\frac{Error_1}{Error_2} = \frac{\frac{1}{120}}{\frac{1}{1920}} = \frac{1920}{120} = 16,$$

we see that the error has decreased by a factor 16 when the mesh size has decreased by a factor 2! This agrees with the *global* error bound $\frac{b-a}{2880} \max_{y\in[0,1]} |h^4(y) f^{(4)}(y)|$. □

Problem 3. Let $I = (0, 1)$ and $f(x) = x^2$ for $x \in I$.
(a) Let V_h be the space of linear functions on I and calculate the L^2-projection $P_h f \in V_h$ of f.

Remark. In this case $h(y) \equiv 1$ and $V_h = \mathcal{P}(0,1)$.

(b) Divide I into two subintervals of equal length and let V_h be the corresponding space of continuous piecewise linear functions. Calculate the L^2-projection $P_h f \in V_h$ of f.
(c) Illustrate your results in figures and compare with the nodal interpolant $\pi_h f$.

Solution:

(a) The L^2-projection $P_h f \in V_h$ of f is the *orthogonal projection* of f onto V_h. Therefore $f - P_h f$ must be orthogonal to all $v \in V_h$, that is

$$\int_I (f - P_h f) v \, dx = 0, \quad \forall v \in V_h,$$

but from *Problem 6 (Week 2)* this is equivalent to

$$\begin{cases} \int_I (f - P_h f)\varphi_0 \, dx = 0 \\ \int_I (f - P_h f)\varphi_1 \, dx = 0 \end{cases}$$

since the "hat functions" $\varphi_0 = 1 - x$ and $\varphi_1 = x$ are a basis for V_h.

Since $P_h f \in V_h$, we make the *Ansatz*

$$P_h f = \sum_{j=0}^{1} c_j \, \varphi_j(x),$$

and inserting this Ansatz into the orthogonality relation gives

$$\sum_{j=0}^{1} c_j \int_I \varphi_j \, \varphi_i \, dx = \int_I f \, \varphi_i \, dx, \quad i = 0, 1,$$

which is a linear system with two equations and two unknowns: c_0 and c_1. It is therefore natural to state the system in matrix form, $Mc = b$, with the mass matrix $M = (m_{ij})$, $m_{ij} = \int_I \varphi_j \varphi_i \, dx$, $c = (c_0, c_1)^t$ and $b = (b_0, b_1)^t$ where $b_i = \int_I f \varphi_i \, dx$. Now, we only have to compute these integrals and solve for c. Note that $m_{ij} = m_{ji}$ (the mass matrix is *symmetric*).

$$\begin{aligned} m_{00} &= \int_I \varphi_0 \varphi_0 \, dx \\ &= \int_0^1 (1-x)^2 \, dx \\ &= 1/3 \\ m_{10} &= \int_I \varphi_0 \varphi_1 \, dx \\ &= \int_0^1 (1-x) x \, dx \\ &= 1/6 \end{aligned}$$

$$
\begin{aligned}
m_{11} &= \int_I \varphi_1 \varphi_1 \, dx \\
&= \int_0^1 x^2 \, dx \\
&= 1/3 \\
b_0 &= \int_I f \varphi_0 \, dx \\
&= \int_0^1 x^2 (1-x) \, dx \\
&= 1/12 \\
b_1 &= \int_I f \varphi_1 \, dx \\
&= \int_0^1 x^2 \cdot x \, dx \\
&= 1/4
\end{aligned}
$$

The system of equations we have to solve is then

$$
\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \begin{bmatrix} 1/12 \\ 1/4 \end{bmatrix}.
$$

Hence, $c_0 = -1/6$ and $c_1 = 5/6$, which gives $P_h f(x) = c_0 \varphi_0(x) + c_1 \varphi_1(x) = -1/6 \, \varphi_0(x) + 5/6 \, \varphi_1(x) = -1/6 \cdot (1-x) + 5/6 \cdot x = -1/6 + x$.

Remark. We could in principle use any set (pair, in this case) of basis functions, for instance $\{1, x\} \subset V_h$. This choice would lead to the orthogonality relation

$$
\begin{cases} \int_I (f - P_h f) \cdot 1 \, dx = 0 \\ \int_I (f - P_h f) \cdot x \, dx = 0 \end{cases}
$$

and the Ansatz

$$
P_h f(x) = a \cdot 1 + b \cdot x = a + bx,
$$

from which a $(= -1/6)$ and b $(= 1)$ can be computed.

(b) We now divide I into the two subintervals $(0, \frac{1}{2})$ and $(\frac{1}{2}, 1)$. As in (a), we choose the "hat functions" as basis functions:

$$
\varphi_0 = \begin{cases} 1 - 2x, & x \in (0, \frac{1}{2}) \\ 0, & x \in (\frac{1}{2}, 1) \end{cases}
$$

$$
\varphi_1 = \begin{cases} 2x, & x \in (0, \frac{1}{2}) \\ 2 - 2x, & x \in (\frac{1}{2}, 1) \end{cases}
$$

$$
\varphi_2 = \begin{cases} 0, & x \in (0, \frac{1}{2}) \\ 2x - 1, & x \in (\frac{1}{2}, 1) \end{cases}
$$

Using the same technique as in (a), we obtain a 3×3 linear system of equations (since the number of nodes is 3 when the number of intervals is 2). The elements of the mass matrix are

$$
\begin{aligned}
m_{00} &= \int_I \varphi_0 \varphi_0 \, dx \\
&= \int_0^{1/2} (1-2x)^2 \, dx \\
&= 1/6 \\
m_{10} &= \int_I \varphi_0 \varphi_1 \, dx \\
&= \int_0^{1/2} (1-2x) 2x \, dx \\
&= 1/12 \\
m_{20} &= \int_I \varphi_0 \varphi_2 \, dx \\
&= 0 \\
m_{11} &= \int_I \varphi_1 \varphi_1 \, dx \\
&= \int_0^{1/2} (2x)^2 \, dx + \int_{1/2}^1 (2-2x)^2 \, dx \\
&= 1/3 \\
m_{12} &= \int_I \varphi_2 \varphi_1 \, dx \\
&= \int_{1/2}^1 (2x-1)(2-2x) \, dx \\
&= 1/12 \\
m_{22} &= \int_I \varphi_2 \varphi_2 \, dx \\
&= \int_{1/2}^1 (2x-1)^2 \, dx \\
&= 1/6
\end{aligned}
$$

Similarly, we get for the right hand side

$$
\begin{aligned}
b_0 &= \int_I f \varphi_0 \, dx \\
&= \int_0^{1/2} x^2 (1-2x) \, dx \\
&= 1/96
\end{aligned}
$$

$$
\begin{aligned}
b_1 &= \int_I f\varphi_1\,dx \\
&= \int_0^{1/2} x^2 2x\,dx + \int_{1/2}^1 x^2(2-2x)\,dx \\
&= 7/48 \\
b_2 &= \int_I f\varphi_2\,dx \\
&= \int_{1/2}^1 x^2(2x-1)\,dx \\
&= 17/96
\end{aligned}
$$

The system we have to solve is

$$
\begin{bmatrix} 1/6 & 1/12 & 0 \\ 1/12 & 1/3 & 1/12 \\ 0 & 1/12 & 1/6 \end{bmatrix}
\begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}
=
\begin{bmatrix} 1/96 \\ 7/48 \\ 17/96 \end{bmatrix}
$$

with the solution $c_0 = -1/24, c_1 = 5/24$ and $c_2 = 23/24$. Hence,

$$
\begin{aligned}
P_h f(x) &= c_0\,\varphi_0(x) + c_1\,\varphi_1(x) + c_2\,\varphi_2(x) \\
&= -1/24\,\varphi_0(x) + 5/24\,\varphi_1(x) + 23/24\,\varphi_2(x) \\
&\left(\begin{aligned}
&= \begin{cases} -1/24\cdot(1-2x) + 5/24\cdot 2x, & x \in (0,1/2) \\ 5/24\cdot(2-2x) + 23/24\cdot(2x-1), & x\in(1/2,1) \end{cases} \\
&= \begin{cases} -1/24 + x/2, & x\in(0,1/2) \\ -13/24 + 3x/2, & x \in (1/2,1) \end{cases}
\end{aligned}\right)
\end{aligned}
$$

Remark. Cf. the Remark at the end of *Problem 4(a) (Week 1).*

Remark. Also in this case one might try the Ansatz

$$
P_h f(x) = \begin{cases} a + bx, & x \in (0, \frac{1}{2}) \\ c + dx, & x \in (\frac{1}{2}, 1) \end{cases}
$$

using $\{1, x\}$ as *local* basis functions on each subinterval. In addition to the orthogonality requirement (against three *global* basis functions, for instance $\{\varphi_i\}_{i=0}^2$) we will in this case need to enforce continuity at the point $x = 1/2$, and will therefore end up with 4 equations instead of 3, from which a $(= -1/24)$, b $(= 1/2)$, c $(= -13/24)$, d $(= 3/2)$, can be computed. This, however, is disadvantageous since we have to solve a linear system of four equations instead of three.

(c) See Figure 1 and Figure 2. □

Problem 4. Let $I = (0,1)$ and $0 = x_0 < x_1 < \cdots < x_N = 1$ be a partition of I into subintervals $I_j = (x_{j-1}, x_j)$ of length h_j.

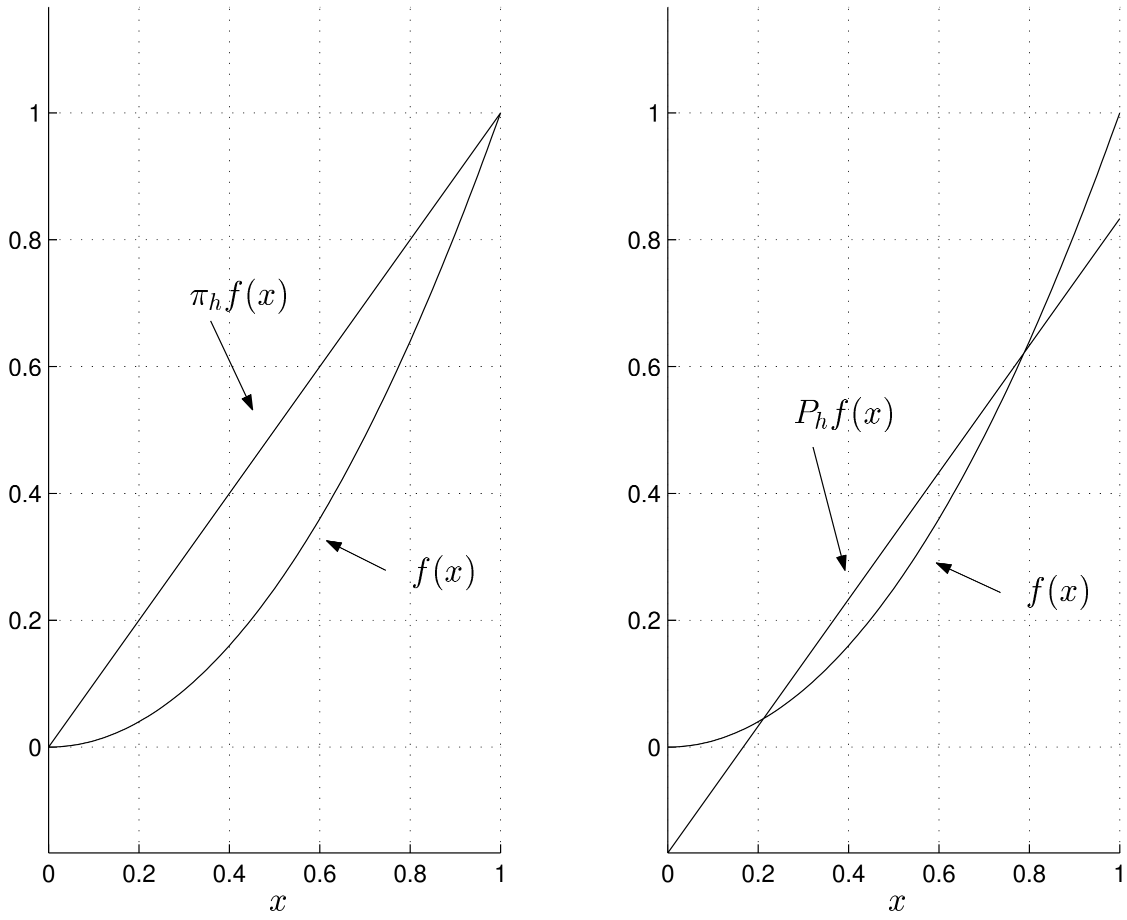

Figure 1: Problem 3(a) (Week 2). Plots of $f(x) = x^2$, $\pi_h f(x)$ and $P_h f(x)$.

(a) Assume $h_j = 1/N$ for all j. Calculate the mass matrix M.
(b) Calculate the mass matrix M in the general case.
Solution: The $(N+1) \times (N+1)$-matrix $M = (m_{ij})_{i,j=0}^N$ with elements

$$m_{ij} = \int_I \varphi_j \, \varphi_i \, dx, \tag{1}$$

where $\{\varphi_i\}_{i=0}^N \subset V_h$ are the nodal basis functions ("hat-functions"), is called the *mass matrix*.
(a) Look at the interval between say x_3 and x_4. On this interval there exist two non-zero basis functions φ_3 and φ_4. For $x \in [x_3, x_4]$ we have the following analytical expressions:

$$\varphi_3(x) = 1 - \frac{x - x_3}{h}, \quad \varphi_4(x) = \frac{x - x_3}{h}.$$

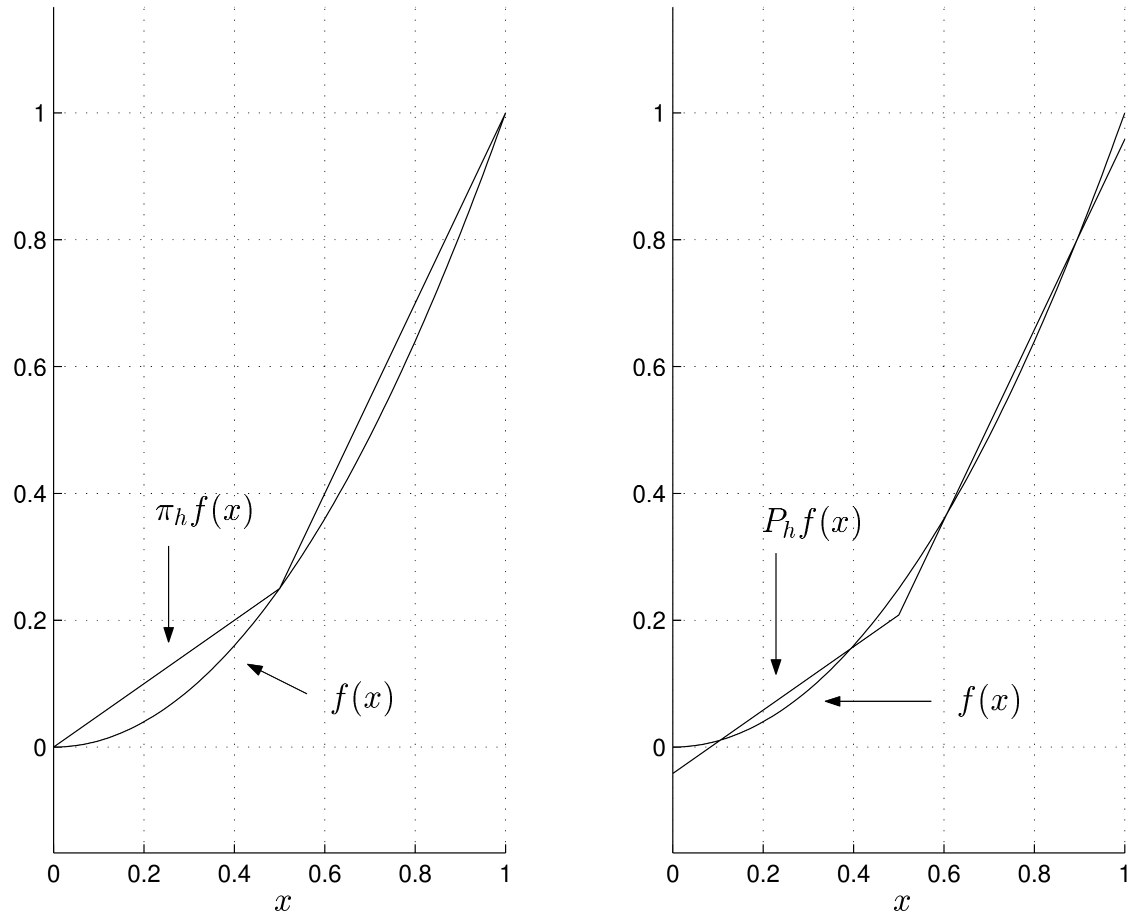

Figure 2: Problem 3(b) (Week 2). Plots of $f(x) = x^2$, $\pi_h f(x)$ and $P_h f(x)$.

This yields for the matrix elements m_{34} and m_{43}:

$$m_{34} = m_{43} = \int_0^1 \varphi_3(x)\,\varphi_4(x)\,dx = \int_{x_3}^{x_4} \varphi_3(x)\,\varphi_4(x)\,dx =$$

$$\int_{x_3}^{x_4} \left(1 - \frac{x - x_3}{h}\right) \cdot \frac{x - x_3}{h}\,dx = \{\text{Make a change of variables: } y = x - x_3\} =$$

$$\int_0^h \left(1 - \frac{y}{h}\right) \cdot \frac{y}{h}\,dy = \frac{h}{6},$$

since the integrand $\varphi_3(x)\,\varphi_4(x)$ is non-zero *only* for $x \in [x_3,\, x_4]$.

The interval $[x_3,\, x_4]$ also contributes to the matrix elements $m_{33} = \int_0^1 \varphi_3(x)\,\varphi_3(x)\,dx$ and $m_{44} = \int_0^1 \varphi_4(x)\,\varphi_4(x)\,dx$:

$$\frac{1}{2} \cdot m_{33} = \{\text{By symmetry }\} = \frac{1}{2} \cdot m_{44} = \int_{x_3}^{x_4} \varphi_4(x)\,\varphi_4(x)\,dx =$$

$$\int_{x_3}^{x_4} \frac{(x-x_3)^2}{h^2}\,dx = \{\text{Make a change of variables: } y = x - x_3\} = \int_0^h \frac{y^2}{h^2}\,dy = \frac{h}{3},$$

i.e., $m_{33} = m_{44} = 2h/3$, where the factor 2 compensates for the fact that φ_3 is non-zero on the interval $[x_2, x_4]$ and φ_4 is non-zero on the interval $[x_3, x_5]$. Thus, m_{33} and m_{44} get only half of their total value from the interval $[x_3, x_4]$.

Due to symmetry we may generalize to $m_{ii} = 2h/3,\ i = 1, \ldots, N-1$, $m_{00} = m_{NN} = h/3$, $m_{i,i+1} = m_{i+1,i} = h/6,\ i = 0, \ldots, N-1$, and $m_{ij} = 0$, otherwise. The exceptions for m_{00} and m_{NN} are due to the fact that the basis functions φ_0 and φ_N are just “half hats”.

We summarize:

$$M = \begin{bmatrix} h/3 & h/6 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ h/6 & 2h/3 & h/6 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & h/6 & 2h/3 & h/6 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & h/6 & 2h/3 & h/6 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & h/6 & 2h/3 & h/6 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & h/6 & h/3 \end{bmatrix}$$

(b) We now look at the case where the interval $I = [0, 1]$ is non-uniformly partitioned. Consider once more the subinterval $[x_3, x_4]$. Simply replacing h by h_4 throughout in the computations in (a) gives $m_{34} = m_{43} = h_4/6$, and that the contribution from this subinterval to m_{33} and m_{44} is $h_4/3$. Adding the contributions from all subintervals now immediately generalizes the mass matrix computed in (a): $M =$

$$\begin{bmatrix} h_1/3 & h_1/6 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\ h_1/6 & (h_1+h_2)/3 & h_2/6 & 0 & \ldots & 0 & 0 & 0 & 0 \\ 0 & h_2/6 & (h_2+h_3)/3 & h_3/6 & \ldots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & h_{N-2}/6 & (h_{N-2}+h_{N-1})/3 & h_{N-1}/6 & 0 \\ 0 & 0 & 0 & 0 & \ldots & 0 & h_{N-1}/6 & (h_{N-1}+h_N)/3 & h_N/6 \\ 0 & 0 & 0 & 0 & \ldots & 0 & 0 & h_N/6 & h_N/3 \end{bmatrix}$$

$\square$

Problem 5. Recall that $(f, g) = \int_I fg\,dx$ and $\|f\|^2_{L^2(I)} = (f, f)$ are the L^2-scalar product and norm, respectively. Let $I = (0, \pi)$, $f = \sin x$, $g = \cos x$ for $x \in I$.

(a) Calculate (f, g).
(b) Calculate $\|f\|_{L^2(I)}$ and $\|g\|_{L^2(I)}$.
Solution:
(a) $(f, g) = \int_0^\pi \sin x \cos x \, dx = \frac{1}{2}[(\sin x)^2]_0^\pi = 0$.
(b) Recall the relations

$$\sin^2 x = \frac{1 - \cos 2x}{2}; \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

Using these, we get:

$$\begin{aligned} ||f||_{L^2(I)} &= \sqrt{\int_0^\pi \sin^2 x \, dx} = \sqrt{\int_0^\pi \frac{1 - \cos 2x}{2} \, dx} = \sqrt{\frac{1}{2}\int_0^\pi dx - \frac{1}{2}\int_0^\pi \cos 2x \, dx} \\ &= \sqrt{\frac{\pi}{2} - \frac{1}{4}[\sin 2x]_0^\pi} = \sqrt{\frac{\pi}{2}}, \end{aligned}$$

and, similarly,

$$||g||_{L^2(I)} = \sqrt{\int_0^\pi \cos^2 x \, dx} = \sqrt{\int_0^\pi \frac{1 + \cos 2x}{2} \, dx} = \sqrt{\frac{1}{2}\int_0^\pi dx + \frac{1}{2}\int_0^\pi \cos 2x \, dx} = \sqrt{\frac{\pi}{2}}.$$

$\square$

Problem 6. Show that $(f - P_h f, v) = 0, \ \ \forall v \in V_h$, if and only if $(f - P_h f, \varphi_i) = 0, \ \ i = 0, \ldots, N$; where $\{\varphi_i\}_{i=0}^N \subset V_h$ is the basis of hat-functions.
Solution:
$\Rightarrow$ Follows immediately since $\varphi_i \in V_h$ for $i = 0, \ldots, N$.
$\Leftarrow$ Assume that $(f - P_h f, \varphi_i) = 0$ for $i = 0, \ldots, N$. Since $v \in V_h$ and $\{\varphi_i\}_{i=0}^N$ is a *basis* for V_h, v can be written as $v = \sum_{i=0}^N \alpha_i \varphi_i$. This gives $(f - P_h f, v) = (f - P_h f, \sum_{i=0}^N \alpha_i \varphi_i) = \sum_{i=0}^N \alpha_i (f - P_h f, \varphi_i) = 0$ which proves the statement. $\square$

Problem 7. Let V be a linear subspace of $\mathbf{R}^n$ with basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ with $m < n$. Let $P\boldsymbol{x} \in V$ be the orthogonal projection of $\boldsymbol{x} \in \mathbf{R}^n$ onto the subspace V. Derive a linear system of equations that determines $P\boldsymbol{x}$. Note that your results are analogous to the L^2-projection when the usual scalar product in $\mathbf{R}^n$ is replaced by the scalar product in $L^2(I)$. Compare this method of computing the projection $P\boldsymbol{x}$ to the method used for computing the projection of a three dimensional vector onto a two dimensional subspace. What happens if the basis $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}$ is orthogonal?
Solution: Let $(\boldsymbol{u}, \boldsymbol{v})$ denote the usual scalar product in $\boldsymbol{R}^n$. Since $P\boldsymbol{x}$ is the orthogonal projection of $\boldsymbol{x} \in \boldsymbol{R}^n$ onto the subspace V of $\boldsymbol{R}^n$, we have

$$(\boldsymbol{x} - P\boldsymbol{x}, \boldsymbol{y}) = 0, \quad \text{for all } \boldsymbol{y} \in V.$$

Since $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is a basis for V we may equivalently write (cf. *Problem 6 (Week 2)*)

$$(\boldsymbol{x} - P\boldsymbol{x}, \boldsymbol{v}_i) = 0, \quad i = 1, ..., m,$$

which leads to

$$(P\boldsymbol{x}, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m.$$

But since $P\boldsymbol{x} \in V$ and $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is a basis for V, $P\boldsymbol{x}$ can be written as a linear combination of elements in the basis, that is, $P\boldsymbol{x} = \sum_{j=1}^{m} \alpha_j \boldsymbol{v}_j$, $\alpha_j \in \boldsymbol{R}$. Inserting this above gives

$$(\sum_{j=1}^{m} \alpha_j \boldsymbol{v}_j, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

or, using the linearity property of the scalar product,

$$\sum_{j=1}^{m} \alpha_j (\boldsymbol{v}_j, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

which is a quadratic linear system of equations $A\alpha = b$, where $a_{ij} = (\boldsymbol{v}_j, \boldsymbol{v}_i)$ and $b_i = (\boldsymbol{x}, \boldsymbol{v}_i)$.

If the basis $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_m\}$ is *orthogonal*, that is, $(\boldsymbol{v}_j, \boldsymbol{v}_i) = 0$ if $i \neq j$, the matrix A becomes *diagonal* and the equations simplify to

$$\alpha_i (\boldsymbol{v}_i, \boldsymbol{v}_i) = (\boldsymbol{x}, \boldsymbol{v}_i), \quad i = 1, ..., m,$$

which immediately gives

$$P\boldsymbol{x} = \sum_{j=1}^{m} \frac{(\boldsymbol{x}, \boldsymbol{v}_j)}{(\boldsymbol{v}_j, \boldsymbol{v}_j)} \boldsymbol{v}_j.$$

In the special case $n = 3$ and $m = 2$, which means computing the projection of a three dimensional vector $\boldsymbol{x}$ onto a two dimensional subspace, i.e., onto a *plane* through the origin, one usually computes $P\boldsymbol{x} = \boldsymbol{x} - \frac{(\boldsymbol{x}, \boldsymbol{n})}{(\boldsymbol{n}, \boldsymbol{n})} \boldsymbol{n}$, where $\boldsymbol{n}$ is a normal to the plane.

To compare the two methods, consider the case $\boldsymbol{n} = \boldsymbol{e}_3$, i.e., the plane $x_3 = 0$. Choosing the standard basis $\boldsymbol{v}_1 = \boldsymbol{e}_1$ and $\boldsymbol{v}_2 = \boldsymbol{e}_2$, we get $P\boldsymbol{x} = \boldsymbol{x} - (\boldsymbol{x}, \boldsymbol{e}_3)\, \boldsymbol{e}_3 = \boldsymbol{x} - x_3 \boldsymbol{e}_3 = x_1 \boldsymbol{e}_1 + x_2 \boldsymbol{e}_2 = (\boldsymbol{x}, \boldsymbol{e}_1)\, \boldsymbol{e}_1 + (\boldsymbol{x}, \boldsymbol{e}_2)\, \boldsymbol{e}_2$.

(Cf. *Applied Mathematics: B&S*, Part II, Section 21.17 *Projection of a point onto a plane.*) $\square$