TMA682, Extra Excercises in Stability and Error estimates

1. Let $\|\cdot\|$ denote the $L_2(0,1)$-norm. Consider the following heat equation

$$\begin{cases} \dot{u} - u'' = 0, & 0 < x < 1, \quad t > 0, \\ u(0,t) = u_x(1,t) = 0, & \quad t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

   a) Show that the norms: $||u(\cdot,t)||$ and $||u_x(\cdot,t)||$ are non-increasing in time. $||u|| = \Big(\int_0^1 u(x)^2\,dx\Big)^{1/2}$.

   b) Show that $||u_x(\cdot,t)|| \to 0$, as $t \to \infty$.

   c) Give a physical interpretation for a) and b).

2. Consider the problem

$$-\varepsilon u'' + xu' + u = f, \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

   where ε is a positive constant, and $f \in L_2(I)$. Prove that

$$||\varepsilon u''|| \le ||f||.$$

3. Give an a priori error estimate for the following problem:

$$(au_{xx})_{xx} = f, \quad 0 < x < 1, \qquad u(0) = u'(0) = u(1) = u'(1) = 0,$$

   where $a(x) > 0$ on the interval $I = (0,1)$.

4. Prove an a priori error estimate for the finite element method for the problem

$$-u''(x) + u'(x) = f(x), \quad 0 < x < 1, \qquad u(0) = u(1) = 0.$$

5. (a) Prove an *a priori* error estimate for the $cG(1)$ approximation of the boundary value problem

$$-u'' + cu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

where $c \geq 0$ is constant.

(b) For which value of c is the *a priori* error estimate optimal?

6. We modify problem 2 above according to

$$-\varepsilon u'' + c(x)u' + u = f(x) \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

where ε is a positive constant, the function c satisfies $c(x) \geq 0$, $c'(x) \leq 0$, and $f \in L_2(I)$. Prove that there are positive constants C_1, C_2 and C_3 such that

$$\sqrt{\varepsilon}||u'|| \leq C_1||f||, \quad ||cu'|| \leq C_2||f||, \qquad \text{and} \quad \varepsilon||u''|| \leq C_3||f||,$$

where $||\cdot||$ is the $L_2(I)$-norm.

7. Show that for a continuously differentiable function v defined on $(0,1)$ we have that

$$||v||^2 \leq v(0)^2 + v(1)^2 + ||v'||^2.$$

Hint: Use partial integration for $\int_0^{1/2} v(x)^2\,dx$ and $\int_{1/2}^1 v(x)^2\,dx$ and note that $(x - 1/2)$ has the derivative 1.

8. Determine the solution for the wave equation

$$\begin{cases} \ddot{u} - c^2 u'' = f, & x > 0, \qquad\qquad\qquad t > 0, \\ u(x,0) = u_0(x), & u_t(x,0) = v_0(x), \quad x > 0, \\ u_x(1,t) = 0, & \qquad\qquad\qquad\qquad t > 0, \end{cases}$$

in the following cases:

a) $f = 0$.

b) $f = 1, \quad u_0 = 0, \quad v_0 = 0.$