Mathematic Chalmers & GU

TMA372/MAN660: Partial Differential Equations, 2006–12–18

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Calculators, formula notes and other subject related material are not allowed.

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1. Prove the following interpolation error estimate

$$||f - \pi_1 f||_{L_\infty(a,b)} \leq C_i (b-a)^2 ||f''||_{L_\infty(a,b)}.$$

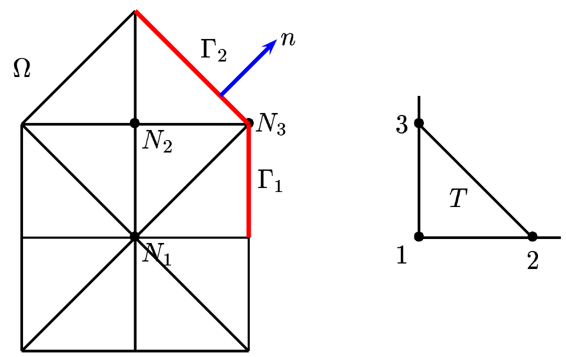
2. Prove an a priori and an a posteriori error estimate for the cG(1) finite element method for

$$-u''(x) + u'(x) = f, \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

3. Formulate the cG(1) piecewise continuous Galerkin method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \quad \nabla u \cdot n = 0, \quad x \in \Gamma_1 \cup \Gamma_2,$$

on the domain Ω, with outward unit normal n at the boundary (see fig.). Write the matrices for the resulting equation system using the following mesh with nodes at N_1, N_2 and N_3.
Hint: You may first compute the matrices for a standard triangle-element T.



4. Consider the initial value problem ($u = u(x,t)$)

$$\dot{u} + Au = f, \quad t > 0; \qquad u(t=0) = u_0.$$

Show that if there is a constant $\alpha > 0$ such that

$$(Av, v) \geq \alpha ||v||^2, \qquad \forall v,$$

then the solution u of the initial value problem satisfies the stability estimate

$$||u(t)||^2 + \alpha \int_0^t ||u(s)||^2\, ds \leq ||u_0||^2 + \frac{1}{\alpha}\int_0^t ||f(s)||^2\, ds.$$

5. Consider the boundary value problem

$$\Delta u = 0, \quad \text{in } \Omega \subset \mathbb{R}^2, \qquad \frac{\partial u}{\partial n} + u = g, \quad \text{on } \Gamma = \partial\Omega, \quad n \text{ is outward unit normal to } \Gamma.$$

a) Show the stability estimate

$$||\nabla u||^2_{L_2(\Omega)} + \frac{1}{2}||u||^2_{L_2(\Gamma)} \leq \frac{1}{2}||g||^2_{L_2(\Gamma)}.$$

b) Discuss, concisely, the conditions for applying the Lax-Milgram theorem to this problem.
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1. See the book or Lecture Notes; Chapter 5.

2. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{1}$$

A *Finite Element Method* with $cG(1)$ reads as follows: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{2}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0. \tag{3}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I e'e = \int_I \frac{1}{2}\frac{d}{dx}\Big(e^2\Big) = \frac{1}{2}(e^2)|_0^1 = 0. \tag{4}$$

Further, using Poincare inequality we have

$$||e||^2 \le ||e'||^2.$$

A priori error estimate: We use Poincare inequality and (4) to get

$$\begin{aligned}
||e||_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) = 2\int_I \Big(e'(u-U)' + e'(u-U)\Big) \\
&= 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) + 2\int_I \Big(e'(\pi_h u - U)' + e'(\pi_h u - U)\Big) \\
&= \{v = U - \pi_h u \ \text{ in } (6)\} = 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) \\
&\le 2||(u-\pi_h u)'||\,||e'|| + 2||u-\pi_h u||\,||e'|| \\
&\le 2C_i\{||hu''|| + ||h^2u''||\}||e||_{H^1},
\end{aligned}$$

this gives that

$$||e||_{H^1} \le C_i\{||hu''|| + ||h^2u''||\},$$

which is the a priori error estimate.

A posteriori error estimate:

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) \\
&= 2\int_I ((u-U)'e' + (u-U)'e) = \{v = e \ \text{ in } (4)\} \\
&= 2\int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \ \text{ in } (5)\} \qquad (5) \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + U'(e - \pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - U' = f - U'$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$
\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\| \\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}
$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

3. Let V be the linear function space defined by

$$V_h := \{v : v \ \text{ is continuous in } \Omega, \ \ v = 0, \ \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get that

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$
\begin{aligned}
-(\Delta u, \nabla v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (n\cdot\nabla u)v\, ds \\
&= (\nabla u, \nabla v) - \int_{\partial\Omega\setminus(\Gamma_1\cup\Gamma_2)} (n\cdot\nabla u)v\, ds - \int_{\Gamma_1\cup\Gamma_2} (n\cdot\nabla u)v\, ds \\
&= (\nabla u, \nabla v), \qquad \forall v \in V.
\end{aligned}
$$

Thus the variational formulation is:

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the usual finite element space consisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{i=1}^3 \xi_i \varphi_i(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^3 \xi_i \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j\, dx + \int_\Omega \varphi_i\varphi_j\, dx\Big) = \int_\Omega f\varphi_j\, dx, \quad j = 1, 2, 3,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness matrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1,x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1,x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1,x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1,x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1,x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = h^2\int_0^1\int_0^{1-x_2}(1-x_1-x_2)^2\,dx_1dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1,\nabla\phi_1) = \int_T |\nabla\phi_1|^2\,dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1,\phi_1) = \int_T \phi_1^2\,dx = \frac{|T|}{3}\sum_{j=1}^{3}\phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0+\frac{1}{4}+\frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$M_{11} = 8m_{22} = \frac{8}{12}h^2, \qquad S_{11} = 8s_{22} = 4,$$
$$M_{12} = 2m_{12} = \frac{1}{12}h^2, \qquad S_{12} = 2s_{12} = -1,$$
$$M_{13} = 2m_{23} = \frac{1}{12}h^2, \qquad S_{13} = 2s_{23} = 0,$$
$$M_{22} = 4m_{11} = \frac{4}{12}h^2, \qquad S_{22} = 4s_{11} = 4,$$
$$M_{23} = 2m_{12} = \frac{1}{12}h^2,, \qquad S_{23} = 2s_{12} = -1,$$
$$M_{33} = 3m_{22} = \frac{3}{12}h^2,, \qquad S_{33} = 3s_{22} = 3/2.$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3/2 \end{bmatrix}.$$

4. Multiply the differential equation by $u(t)$ and integrate over the space domain to get

$$(f,u) = (\dot{u},u) + (Au,u) \geq \frac{1}{2}\frac{d}{dt}\|u\|^2 + \alpha\|u\|^2.$$

Now

$$\Big|\big(\frac{1}{\sqrt{2\varepsilon}}f, \sqrt{2\varepsilon}u\big)\Big| \leq \frac{1}{2}\Big(\frac{1}{2\varepsilon}\|f\|^2 + 2\varepsilon\|u\|^2\Big) = \frac{1}{4\varepsilon}\|f\|^2 + \varepsilon\|u\|^2.$$

With $\varepsilon = \alpha/2$ we get

$$\frac{1}{\alpha}\|f\|^2 + \alpha\|u\|^2 \geq \frac{d}{dt}\|u\|^2 + 2\alpha\|u\|^2.$$

Integrating in time yields

$$\|u(t)\|^2 - \|u_0\|^2 + a\int_0^t \|u(s)\|^2\,ds \leq \frac{1}{\alpha}\int_0^t \|f(s)\|^2\,ds.$$

5. a) Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u \cdot \nabla u = -\int_\Omega (\Delta u) u + \int_{\partial\Omega} \frac{\partial u}{\partial n} u = \int_{\partial\Omega} (g-u)u.$$

In other words

$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Gamma)} = \int_{\partial\Omega} gu \le ||g||^2_{L_2(\Gamma)} ||u||^2_{L_2(\Gamma)} \le \frac{1}{2}||g||^2_{L_2(\Gamma)} + \frac{1}{2}||u||^2_{L_2(\Gamma)},$$

which gives the desired estimate.

To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u,v)$ is a scalar product with the corresponding norm $||v||_a = a(v,v)^{1/2}$. For instance we have that $||v||_a = 0$, only if $v = 0$:

$$0 = ||v||^2_a = a(u,v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2 \ge \alpha \int_\Omega v^2, \qquad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $||g||_{\partial\Omega} < \infty$, then

$$|L(v)| \le ||g||_{\partial\Omega} ||v||_{\partial\Omega} \le ||g||_{\partial\Omega} ||v||_a.$$

We can also apply Riesz theorem in the sense that there existes u such that

$$a(u,v) = L(v), \qquad \forall v,$$

and u is uniquely determined by

$$||u||_a = ||g||_{\partial\Omega}.$$

Moreover since

$$a(u,v) = -\int_\Omega \Delta u v + \int_{\partial\Omega} (\frac{\partial u}{\partial n} + u) v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

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