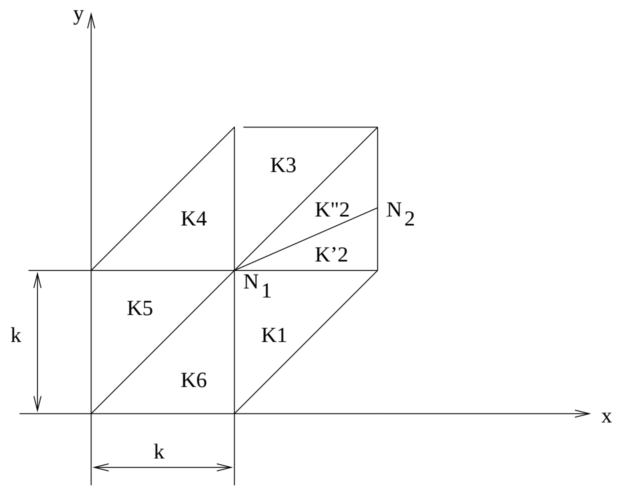
TMA371/MAN660 Partial Differential Equations TM, IMP, E3, GU
2002-12-17. Solutions

1.

Let Ω be the triangulated domain below. Compute the cG(1) solution of

$$\begin{cases} -\Delta u = 1, & \text{on } \Omega \\ u_x(1,y) = 0, \;\; 1/2 \le y \le 1, & u(x,y) = 0, \quad \text{on the rest of boundary.} \end{cases}$$



Solution. We split the boundary $\Gamma := \partial\Omega$ of the domain as $\Gamma = \Gamma_N + \Gamma_D$ with $\Gamma_N = \{(1,y) : 1/2 \le y \le 1\}$ and $\Gamma_D = \Gamma \setminus \Gamma_N$.

Variational Formulation: Using Green's formula we have that

$$\begin{aligned} \int_\Omega 1 \cdot v = \int_\Omega -\Delta u v &= \int_\Omega \nabla u \cdot \nabla v - \int_\Gamma (\partial_n u) v \\ &= \{v = 0 \text{ on } \Gamma_D, \text{ and } \partial_n u = u_x = 0 \text{ on } \Gamma_N\} \\ &= \int_\Omega \nabla u \cdot \nabla v. \end{aligned} \tag{1}$$

Thus we have the finite element formulation: Find piecewise linear function $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega 1 \cdot v, \quad \forall v \in V_h. \tag{2}$$

Remark. It is natural to think $P_1 = (1/2, 1/2)$, $P_2 = (1, 1/2)$ and $P_3 = (1,1)$ as nodes. Then there are two possibilities:

(I) Normally we let the basis functions ψ_2, and ψ_3 (corresponding to the nodes P_2 and P_3, respectively) to have $K_1 \cap K_2$, and $K_2 \cap K_3$ as their supports, respectively. This however extends the Neumann boundary codition from Γ_N to $\Gamma_N \cap \{(x,y) : y = x - 1/2, \; 1/2 \le x \le 1\} \cup \{(x,y) : y = 1, \; 1/2 \le x \le 1\}$, thus removing the Dirichlet condition from the line-segments $\{(x,y) : y = x - 1/2, \; 1/2 \le x \le 1\}$ and $\{(x,y) : y = 1, \; 1/2 \le x \le 1\}$.

(II) To circumvent this difficulty, we restrict the support of both basis functions $\psi_i,\ i = 2, 3$ to ONLY! K_2. And this choice introduce discontinuities and cannot be considered in cG(1).

Hence summing up, because of cG(1) approximation in this problem we cannot choose the boundary points $(1, 1/2)$ and $(1, 1)$ as nodes (this will either destroy the continuity or replace the Dirichlet condition on $\{(x, y) : 1/2 \le x \le 1\}$, with $0 \le y \le 1/2$ and $y = 1$ by the Neumann condition. So the only adequarte way is the *refinement* of K_2 into two triangles K_2' and K_2'', by letting, e.g. $N_2 = (1, 3/4)$, as in the figure. We define the test functions $\varphi_i(Nj) = \delta_{ij},\ i, j = 1, 2$. In order to have zero boundary condition on $\Gamma_D,\ \varphi_2$, has $K_2 = K_2' \cap K_2''$ as its support. Let now

$$U(x, y) = U_1\varphi_1(x, y) + U_2\varphi_2(x, y).$$

where $U_i = U(N_i),\ i = 1, 2$. Observe that φ_is are the bases functions for V_h and thus the equation (2) is equivalent to the following system:

$$\int_\Omega \nabla\varphi_1 \cdot \nabla\varphi_i U_1 + \int_\Omega \nabla\varphi_2 \cdot \nabla\varphi_i U_2 + = \int_\Omega \varphi_i, \quad i = 1, 2. \tag{3}$$

Let now $\varphi_2' = \varphi|_{K_2'}$ and $\varphi_2'' = \varphi|_{K_2''}$, then using a standard triangle K_2 with vertices $(0, 0),\ (k, 0)$, and (k, k), for $\varphi_2'(x, y) = ax + by + c$ we have that

$$\begin{aligned}
&\varphi_2'(0, 0) = 0 \Rightarrow c = 0,\\
&\varphi_2'(k, 0) = 0 \Rightarrow ak = 0 \Leftrightarrow a = 0,\\
&\varphi_2'(k, k/2) = 1 \Rightarrow bk/2 = 1 \Leftrightarrow b = 2/k.
\end{aligned}$$

Thus $\varphi_2'(x, y) = \frac{2}{k}y$. Similarly for $\varphi_2''(x, y) = ax + by + c$ we have that

$$\begin{aligned}
&\varphi_2''(0, 0) = 0 \Rightarrow c = 0,\\
&\varphi_2''(k, k) = 0 \Rightarrow ak + bk = 0 \Leftrightarrow b = -a,\\
&\varphi_2''(k, k/2) = 1 \Rightarrow ak + bk/2 = 1 \Leftrightarrow a = 2/k,\ b = -2/k.
\end{aligned}$$

Thus $\varphi_2''(x, y) = \frac{2}{k}x - \frac{2}{k}y$.
Considering in standard triangles we can easily see from the figure that:

$$\begin{aligned}
&\nabla\varphi_1\Big|_{K_1} = \tfrac{1}{k}(1, 1), && \nabla\varphi_2\Big|_{K_1} = (0, 0), &&\\
&\nabla\varphi_1\Big|_{K_2} = \tfrac{1}{k}(-1, 0), && \nabla\varphi_2\Big|_{K_2'} = \tfrac{1}{k}(0, 2), && \nabla\varphi_2\Big|_{K_2''} = \tfrac{1}{k}(2, -2),\\
&\nabla\varphi_1\Big|_{K_3} = \tfrac{1}{k}(0, -1), && \nabla\varphi_2\Big|_{K_3} = (0, 0), &&\\
&\nabla\varphi_1\Big|_{K_4} = \tfrac{1}{k}(1, -1), && \nabla\varphi_2\Big|_{K_4} = (0, 0), &&\\
&\nabla\varphi_1\Big|_{K_5} = \tfrac{1}{k}(1, 0), && \nabla\varphi_2\Big|_{K_5} = (0, 0), &&\\
&\nabla\varphi_1\Big|_{K_6} = \tfrac{1}{k}(0, 1), && \nabla\varphi_2\Big|_{K_6} = (0, 0). &&
\end{aligned}$$

Now considering the intersections of the supports of the φ functions we have that:

$$\begin{aligned}\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_1 &= \sum_{j=1}^{6}\int_{K_j}\nabla\varphi_1\cdot\nabla\varphi_1\\ &= \frac{k^2}{2}\frac{1}{k^2}\Big\{(-1,1)\cdot(-1,1)+(-1,0)\cdot(-1,0)+(0,-1)\cdot(0,-1)\\ &\quad +(1,-1)\cdot(1,-1)+(1,0)\cdot(1,0)+(1,0)\cdot(1,0)\Big\}\\ &= \frac{1}{2}\{2+1+1+2+1+1\}=4.\end{aligned}$$

Note that $|K'|=|K''|=\frac{1}{2}k\cdot\frac{k}{2}=\frac{k^2}{4}$, thus

$$\begin{aligned}\int_\Omega \nabla\varphi_2\cdot\nabla\varphi_2 &= \int_{K_2}\nabla\varphi_2\cdot\nabla\varphi_2=\int_{K_2'}\nabla\varphi_2'\cdot\nabla\varphi_2'+\int_{K_2''}\nabla\varphi_2''\cdot\nabla\varphi_2''\\ &= \frac{k^2}{4}\frac{1}{k^2}\Big\{(0,2)\cdot(0,2)+(2,-2)\cdot(2,-2)\Big\}=\frac{1}{4}\{4+8\}=3,\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \nabla\varphi_1\cdot\nabla\varphi_2 &= \int_{K_2}\nabla\varphi_1\cdot\nabla\varphi_2=\int_{K_2'}\nabla\varphi_1\cdot\nabla\varphi_2'+\int_{K_2''}\nabla\varphi_1\cdot\nabla\varphi_2''\\ &= \frac{k^2}{4}\frac{1}{k^2}\Big\{(-1,0)\cdot(0,2)+(-1,0)\cdot(2,-2)\Big\}=\frac{1}{4}\{-2\}=-1/2.\end{aligned}$$

As for the right hand side in (3) we have

$$\int_\Omega\varphi_1=6\cdot\frac{1}{3}\frac{k^2}{2}=k^2,\qquad \int_\Omega\varphi_2=\int_{K_2}\varphi_2=\cdots=1\cdot\frac{1}{3}\frac{k^2}{2}=k^2/6.$$

Thus recalling that $k=1/2$, we have the folowing system of equations:

$$\begin{pmatrix}4 & -1/2\\ -1/2 & 3\end{pmatrix}\begin{pmatrix}U_1\\ U_2\end{pmatrix}\begin{pmatrix}1/4\\ 1/24\end{pmatrix}\Rightarrow U_1=6U_2-1/12,\ \text{with}\quad U_2=\frac{7}{6(47)}.$$

2. See stability lemma in chapter 9 of the lecture notes.

3. Prove that if $u=0$ on the boundary of the unit square Ω, then

$$\Big(\int_\Omega |u|^2\,dx\Big)^{1/2}\le\Big(\int_\Omega|\nabla u|^2\,dx\Big)^{1/2}.$$

Solution. We have that

$$\begin{aligned}|u(x)|=|u(x_1,x_2)-u(0,x_2)| &= \Big|\int_0^{x_1}\frac{\partial}{\partial x_1}u(\bar{x_1},x_2)\,d\bar{x_1}\Big|\\ &= \Big|\int_0^{x_1}1\cdot\frac{\partial}{\partial x_1}u(\bar{x_1},x_2)\,d\bar{x_1}\Big|\le\{\text{Cauchy's inequality}\}\\ &\le\Big(\int_0^{x_1}1^2\,d\bar{x_1}\Big)^{1/2}\cdot\Big(\int_0^{x_1}(\frac{\partial}{\partial x_1}u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2}\\ &\le\Big(\int_0^{1}(\frac{\partial}{\partial x_1}u(\bar{x_1},x_2))^2\,d\bar{x_1}\Big)^{1/2}.\end{aligned}$$

This implies that

$$\begin{aligned}
\int_\Omega |u|^2\,dx &\le \int_\Omega \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx \\
&= \int_0^1 \int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx_1\, dx_2 \\
&= \int_0^1 \Big(\int_0^1 (\frac{\partial}{\partial x_1} u(\bar{x_1}, x_2))^2\, d\bar{x_1}\Big)\, dx_2 = \int_0^1 \int_0^1 (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\, dx_1\, dx_2 \\
&= \int_\Omega (\frac{\partial}{\partial x_1} u(x_1, x_2))^2\, dx \le \int_\Omega |\nabla u|^2\, dx,
\end{aligned}$$

which gives the desired result:

$$\Big(\int_\Omega |u|^2\,dx\Big)^{1/2} \le \Big(\int_\Omega |\nabla u|^2\,dx\Big)^{1/2}.$$

4. Prove an a priori and an a posteriori error estimate (in the energy norm: $\|u\|_E^2 := \|u'\|^2 + \|u\|^2$) for the cG(1) finite element method for the problem

$$\begin{cases} -u'' + \alpha u' + u = f, & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

where $\alpha \ge 0$. For which value of α is the a priori error estimate optimal?

Solution. <u>The Variational formulation:</u>
Multiply the equation by $v \in V$, integrate by parts over $(0,1)$ and use the boundary conditions to obtain

$$\text{(4)}\qquad \text{Find } u \in V: \quad \int_0^1 u'v'\,dx + \int_0^1 \alpha u'v\,dx + \int_0^1 uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V.$$

<u>cG(1):</u>

$$\text{(5)}\quad \text{Find } U \in V_h: \quad \int_0^1 U'v'\,dx + \int_0^1 \alpha U'v\,dx + \int_0^1 Uv\,dx = \int_0^1 fv\,dx, \quad \forall v \in V_h.$$

From (1)-(2), we find <u>The Galerkin orthogonality:</u>

$$\text{(6)}\qquad \int_0^1 \Big((u-U)'v' + \alpha(u-U)'v + (u-U)v\Big)\,dx = 0, \quad \forall v \in V_h.$$

We define the inner product $(\cdot,\cdot)_E$ associated to the energy norm to be

$$(v,w)_E = \int_0^1 (v'w' + vw)\,dx, \qquad \forall v, w \in V.$$

Note that

$$\text{(7)}\qquad \int_0^1 e'e\,dx = \frac{1}{2}\int_0^1 \frac{d}{dx}\Big(e^2\Big)\,dx = \frac{1}{2}[e^2]_0^1 = 0.$$

Thus using (7) we have

$$\text{(8)}\qquad ||e||_E^2 = \int_0^1 (e'e' + ee)\,dx = \int_0^1 (e'e' + \alpha e'e + ee)\,dx.$$

We split the second factor e as $e = u - U = u - v + v - U$, with $v \in V_h$ and write

$$(9)\qquad \begin{aligned} ||e||_E^2 &= \int_0^1 \Big(e'(u-U)' + \alpha e'(u-U) + e(u-U)\Big)\,dx = \Big\{v \in V_h\Big\} \\ &= \int_0^1 \Big(e'(u-v)' + \alpha e'(u-v) + e(u-v)\Big)\,dx \\ &+ \int_0^1 \Big(e'(v-U)' + \alpha e'(v-U) + e(v-U)\Big)\,dx \\ &= \int_0^1 \Big(e'(u-v)' + \alpha e'(u-v) + e(u-v)\Big)\,dx, \end{aligned}$$

where, in the last step, we have used the Galerkin orthogonality to eliminate terms involving U. Now we can write

$$(10)\qquad \begin{aligned} ||e||_E^2 &= \int_0^1 \Big(e'(u-v)' + e(u-v) + \alpha e'(u-v)\Big)\,dx \\ &\leq ||e||_E \cdot ||u-v||_E + \alpha ||e'||_{L_2} ||u-v||_{L_2} \\ &\leq ||e||_E \Big(||u-v||_E + \alpha ||u-v||_{L_2}\Big) \leq ||e||_E ||u-v||_E (1+\alpha), \end{aligned}$$

and derive the <u>a priori error estimate:</u>

$$||e||_E \leq ||u-v||_E (1+\alpha), \quad \forall v \in V_h.$$

To obtain <u>a posteriori error estimates</u> the idea is to eliminate u-terms, by using the differential equation, and replacing their contributions by the data f. Then this f combined with the remaining U-terms would yield to the residual error:
<u>A posteriori error estimate:</u>

$$(11)\qquad \begin{aligned} ||e||_E^2 &= \int_0^1 (e'e' + ee)\,dx = \int_0^1 (e'e' + \alpha e'e + ee)\,dx \\ &= \int_0^1 (u'e' + \alpha u'e + ue)\,dx - \int_0^1 (U'e' + \alpha U'e + Ue)\,dx. \end{aligned}$$

Now using the variational formulation (4) we have that

$$\int_0^1 (u'e' + \alpha u'e + ue)\,dx = \int_0^1 fe\,dx.$$

Inserting in (11) and using (5) with $v = \Pi_k e$ we get

$$(12)\qquad \begin{aligned} ||e||_E^2 &= \int_0^1 fe\,dx - \int_0^1 (U'e' + \alpha U'e + Ue)\,dx \\ &+ \int_0^1 (U'\Pi_h e' + \alpha U'\Pi_h e + U\Pi_h e)\,dx - \int_0^1 f\Pi_h e\,dx. \end{aligned}$$

Thus

$$
\begin{aligned}
||e||_E^2 &= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 \Big(U'(e-\Pi_h e)' + \alpha U'(e-\Pi_h e) + U(e-\Pi_h e)\Big)\,dx \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\,dx - \sum_{j=1}^{M+1}\int_{I_j} U'(e-\Pi_h e)'\,dx \\
&= \{\text{partial integration}\} \\
&= \int_0^1 f(e-\Pi_h e)\,dx - \int_0^1 (\alpha U' + U)(e-\Pi_h e)\,dx + \sum_{j=1}^{M+1}\int_{I_j} U''(e-\Pi_h e)\,dx \\
&= \int_0^1 (f + U'' - \alpha U' - U)(e-\Pi_h e)\,dx = \int_0^1 R(U)(e-\Pi_h e)\,dx \\
&= \int_0^1 hR(U)h^{-1}(e-\Pi_h e)\,dx \le ||hR(U)||_{L_2}||h^{-1}(e-\Pi_h e)||_{L_2} \\
&\le C_i||hR(U)||_{L_2}\cdot||e'||_{L_2} \le ||hR(U)||_{L_2}\cdot||e||_E.
\end{aligned}
$$

This gives the <u>a posteriori error estimate:</u>

$$||e||_E \le C_i||hR(U)||_{L_2},$$

with $R(U) = f + U'' - \alpha U' - U = f - \alpha U' - U$ on (x_{i-1}, x_i), $i = 1, \dots, M+1$.

The a priori error estimate is optimal for $\alpha = 0$.

5. Consider the boundary value problem

$$
\begin{cases} -\Delta u = 0, & \text{in a bounded domain } \Omega \subset \mathbb{R}^d,\ d = 2, 3. \\ \frac{\partial u}{\partial n} + u = g, & \text{on } \Gamma = \partial\Omega. \end{cases}
$$

a) Prove the L_2 stability estimate

$$||\nabla u||_{L_2(\Omega)}^2 + \frac{1}{2}||u||_{L_2(\Gamma)}^2 \le \frac{1}{2}||g||_{L_2(\Gamma)}^2.$$

b) Verify the conditions on Riesz/Lax-Milgram theorems for this problem.

Solution: a) Using Greens formula we have that

$$\int_\Omega |\nabla u|^2 = \int_\Omega \nabla u\cdot\nabla u = -\int_\Omega (\Delta u)u + \int_{\partial\Omega}\frac{\partial u}{\partial n}u = \int_{\partial\Omega}(g-u)u.$$

In other words

$$||\nabla u||_{L_2(\Omega)}^2 + ||u||_{L_2(\Gamma)}^2 = \int_{\partial\Omega} gu \le ||g||_{L_2(\Gamma)}^2||u||_{L_2(\Gamma)}^2 \le \frac{1}{2}||g||_{L_2(\Gamma)}^2 + \frac{1}{2}||u||_{L_2(\Gamma)}^2,$$

which gives the desired estimate.

To show the Riesz/Lax-Milgram conditions we introduce the notation

$$a(u,v) = \int_\Omega \nabla u\cdot\nabla v + \int_{\partial\Omega} uv, \quad \text{and} \quad L(v) = \int_{\partial\Omega} gv.$$

Then $a(u,v)$ is a scalar product with the corresponding norm $||v||_a = a(v,v)^{1/2}$. For instance we have that $||v||_a = 0$, only if $v = 0$:

$$0 = ||v||_a^2 = a(u,v) = \int_\Omega |\nabla v|^2 + \int_{\partial\Omega} v^2 \ge \alpha\int_\Omega v^2, \qquad \text{for some } \alpha > 0 \Rightarrow v = 0.$$

Further $L(v)$ is bounded in this norm, e.g. if $||g||_{\partial\Omega} < \infty$, then

$$|L(v)| \leq ||g||_{\partial\Omega}||v||_{\partial\Omega} \leq ||g||_{\partial\Omega}||v||_a.$$

We can also apply Riesz theorem in the sense that there existes u such that

$$a(u,v) = L(v), \qquad \forall v,$$

and u is uniquely determined by

$$||u||_a = ||g||_{\partial\Omega}.$$

Moreover since

$$a(u,v) = -\int_\Omega \Delta uv + \int_{\partial\Omega} (\frac{\partial u}{\partial n} + u)v,$$

we have that

$$\Delta u = 0, \quad \text{in } \Omega \quad \frac{\partial u}{\partial n} + u = g \quad \text{on } \Gamma.$$

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