TMA371 Partial Differential Equations TM, 2003-12-16. Solutions

1. Consider the boundary value problem for the stationary heat flow in $1D$:

$$(BVP) \qquad \begin{cases} -(a(x)u'(x))' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Formulate the corresponding variational formulation (VF), minimization problem (MP) and show that: $\quad (BVP) \Longleftrightarrow (VF) \Longleftrightarrow (MP)$.

Solution: See Lecture Notes Chapter 8.

2. Consider the initial value problem: $\quad \dot{u}(t) + au(t) = 0, \quad t > 0, \quad u(0) = 1.$
a) Let $a = 40$, and the time step $k = 0.1$. Draw the graph of $U_n := U(nk)$, $k = 1, 2, \ldots$, approximating u using (i) explicit Euler, (ii) implicit Euler, and (iii) Cranck-Nicolson methods.
b) Consider the case $a = i$, $(i^2 = -1)$, having the complex solution $u(t) = e^{-it}$ with $|u(t)| = 1$ for all t. Show that this property is preserved in Cranck-Nicolson approximation, (i.e. $|U_n| = 1$), but NOT in any of the Euler approximations.

Solution: a) With $a = 40$ and $k = 0.1$ we get the explicit Euler:

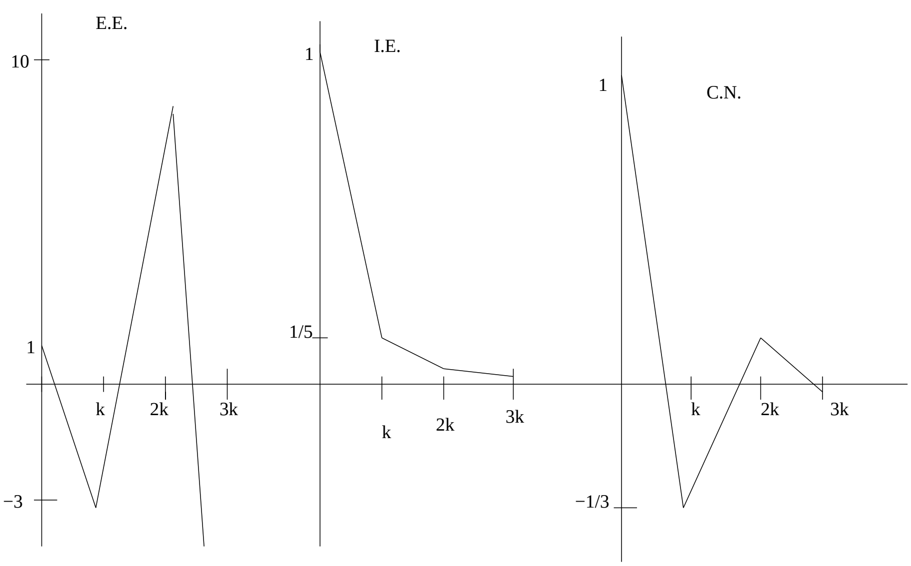
$$\begin{cases} U_n - U_{n-1} + 40 \times (0.1) U_{n-1} = 0, & n = 1.2.3, \ldots \\ U_0 = 1. \end{cases} \Longrightarrow \begin{cases} U_n = -3U_{n-1}, \\ U_0 = 1. \end{cases}$$

Implicit Euler:

$$\begin{cases} U_n = \frac{1}{1+40\times(0.1)} U_{n-1} = \frac{1}{5} U_{n-1}, \\ U_0 = 1. \end{cases}$$

Cranck-Nicolson:

$$\begin{cases} U_n = \frac{1-\frac{1}{2}\times 40\times(0.1)}{1+\frac{1}{2}\times 40\times(0.1)} U_{n-1} = -\frac{1}{3} U_{n-1}, \\ U_0 = 1. \end{cases}$$



b) With $a = i$ we get
Explicit Euler

$$|U_n| = |1 - (0.1) \times i||U_{n-1}| = \sqrt{1 + 0.01}|U_{n-1}| \Longrightarrow |U_n| \geq |U_{n-1}|.$$

Implicit Euler

$$|U_n| = \Big|\frac{1}{1 + (0.1) \times i}\Big||U_{n-1}| = \frac{1}{\sqrt{1 + 0.01}}|U_{n-1}| \leq |U_{n-1}|.$$

Crank-Nicolson

$$|U_n| = \Big|\frac{1 - \frac{1}{2}(0.1) \times i}{1 + \frac{1}{2}(0.1) \times i}\Big||U_{n-1}| = |U_{n-1}|.$$

3. Consider the problem

$$-\varepsilon u'' + xu' + u = f \quad \text{in } I = (0, 1), \qquad u(0) = u'(1) = 0,$$

where ε is a positive constant, and $f \in L_2(I)$. Prove that

$$||\varepsilon u''|| \leq ||f||, \qquad (|| \cdot || \text{ is the } L_2(I) - \text{norm}).$$

Solution: Multiply the equation by $-\varepsilon u''$ and integrate over I to get:

(1) $$||\varepsilon u''||^2_{L_2(I)} - \varepsilon \int_0^1 xu'u''\, dx - \varepsilon \int_0^1 uu''\, dx = \int_0^1 (-\varepsilon u'')f\, dx.$$

But using the boundary condition we have

$$\int_0^1 xu'u''\, dx = [PI] = [xu'^2]_0^1 - \int_0^1 (u' + xu'')u'\, dx = \{u'(1) = 0\}$$
$$= -\int_0^1 u'^2\, dx - \int_0^1 xu'u''\, dx.$$

which implies that

(2) $$\int_0^1 xu'u''\, dx = -\frac{1}{2}\int_0^1 u'^2\, dx.$$

Further

(3) $$\int_0^1 uu''\, dx = [uu']_0^1 - \int_0^1 u'^2\, dx = -\int_0^1 u'^2\, dx.$$

Inserting (2) and (3) in (1) we get

(4) $$\begin{aligned} &||\varepsilon u''||^2_{L_2(I)} + \frac{\varepsilon}{2}\int_0^1 u'^2\, dx + \varepsilon \int_0^1 u'^2\, dx = \int_0^1 (-\varepsilon u'')f\, dx \\ \Longrightarrow\ &||\varepsilon u''||^2_{L_2(I)} \leq \int_0^1 (-\varepsilon u'')f\, dx \leq \{\text{Cauchy-Schwartz}\} \\ &\leq ||\varepsilon u''||_{L_2(I)}||f||_{L_2(I)}. \end{aligned}$$

Thus we have

$$||\varepsilon u''||_{L_2(I)} \leq ||f||_{L_2(I)}.$$

4. Let u be the solution of the following Neumann problem:

$$\begin{cases} -\Delta u = f, & \text{in}\Omega \subset \mathbb{R}^d, \\ -\partial_n u = ku, & \text{on}\Gamma := \partial\Omega. \end{cases}$$

where $\partial_n u = n \cdot \nabla u$ with n being the outward unit normal, and $k \geq 0$.
a) Show that $\quad ||u||_\Omega \leq C_\Omega(||u||_\Gamma + ||\nabla u||_\Omega)$.
b) Use the estimate in a), and show that $||u||_\Gamma \to 0$ as $k \to \infty$.

Solution: a) Assume that φ is a smooth function with $\Delta\varphi = 1$, then using the fact that $\nabla u^2 = 2u\nabla u$ we have

$$\begin{aligned} \|u\|_\Omega^2 &= \int_\Omega u^2 \Delta\varphi = \int_\Gamma u^2 \partial_n \varphi - \int_\Omega \nabla u^2 \cdot \nabla\varphi \\ &\leq C_1\|u\|_\Gamma^2 + C_2\|u\|\|\nabla u\| \leq C_1\|u\|_\Gamma^2 + \frac{1}{2}\|u\|_\Omega^2 + \frac{1}{2}C_2^2\|\nabla u\|_\Omega^2. \end{aligned}$$

Thus

$$\|u\|_\Omega^2 \leq 2C_1\|u\|_\Gamma^2 + C_2^2\|\nabla u\|_\Omega^2 \leq C^2(\|u\|_\Gamma + \|\nabla u\|_\Omega)^2,$$

where

$$C^2 = \max(2C_1, C_2^2), \quad C_1 = \max_\Gamma |\partial_n\varphi|, \quad C_2 = \max(2|\nabla\varphi|).$$

b) Multiply the equation $-\Delta u = f$ by u, integrate over Ω, use partial integration and the boundary condition $-\partial_n u = ku$ to get

$$\begin{aligned} \|\nabla u\|_\Omega^2 + k\|u\|_\Gamma^2 &= \int_\Omega \nabla u \cdot \nabla u + \int_\Gamma u(-\partial_n u) = \int_\Omega u(-\Delta u) = \int_\Omega uf \\ &\leq \|u\|_\Omega\|f\|_\Omega \leq \{\text{use } a)\} \leq C_\Omega(\|u\|_\Gamma + \|\nabla u\|_\Omega)\|f\| \\ &= \|u\|_\Gamma C_\Omega\|f\|_\Omega + \|\nabla u\|_\Omega C_\Omega\|f\|_\Omega \leq \frac{1}{2}|u\|_\Gamma^2 + \frac{1}{2}\|\nabla u\|_\Omega^2 + C_\Omega^2\|f\|_\Omega^2. \end{aligned}$$

Subtracting $\frac{1}{2}|u\|_\Gamma^2 + \frac{1}{2}\|\nabla u\|_\Omega^2$ from both sides gives that

$$(k - \frac{1}{2})|u\|_\Gamma^2 \leq \frac{1}{2}\|\nabla u\|_\Omega^2 + (k - \frac{1}{2})|u\|_\Gamma^2 \leq C_\Omega^2\|f\|_\Omega^2,$$

which gives that $|u\|_\Gamma \to 0$ as $k \to \infty$.

5. Consider the initial-boundary value problem

$$\begin{cases} \dot{u} - \Delta u = f, & x \in \Omega, & t > 0, \\ u = 0, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Show the stability estimates:

$$||u(t)||^2 + \int_0^t ||\nabla u(s)||^2\, ds \leq ||u_0||^2 + C\int_0^t ||f(s)||^2\, ds,$$

$$||\nabla u(t)||^2 + \int_0^t ||\Delta u(s)||^2\, ds \leq ||\nabla u_0||^2 + C\int_0^t ||f(s)||^2\, ds.$$

Solution: Multiplication by u gives

$$(\dot{u}, u) + ||\nabla u||^2 = (f, u) \le ||f||\,||u|| \le C||f||\,||\nabla u|| \le \frac{1}{2}C||f||^2 + \frac{1}{2}||\nabla u||^2.$$

Here $(\dot{u}, u) = \frac{1}{2}\frac{d}{dt}||u||^2$ and hence

$$\frac{d}{dt}||u||^2 + ||\nabla u||^2 \le C||f||^2.$$

Integrating $\int_0^t \cdot\, ds$ gives the first inequality. To get the second one we multiply by $-\Delta u$:

$$(\dot{u}, -\Delta u) + ||\Delta u||^2 \le ||f||\,||\Delta u|| \le \frac{1}{2}||f||^2 + \frac{1}{2}||\Delta u||^2.$$

Here $(\dot{u}, -\Delta u) = \frac{1}{2}\frac{d}{dt}||\nabla u||^2$ and hence

$$\frac{d}{dt}||\nabla u||^2 + ||\Delta u||^2 \le ||f||^2.$$

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