TMA371 Partial Differential Equations TM, 2004-04-13. Solutions

1. Formulate the interpolation error estimates in the L_p-norm, $p = 1, 2, \infty$, in an interval (a, b). Prove the L_∞ error estimate for the linear interpolant:

$$||\pi_1 f - f||_{L_\infty(a,b)} \le C_i(b-a)^2||f''||_{L_\infty(a,b)}.$$

Solution: See proof of theorem 5.3 in CDE or lecture note chapter 5.

2. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method with piecewise linear approximation is applied to the following Poisson's equation with mixed boundary conditions:

$$\begin{cases} -\Delta u = 1, & \text{on} \quad \Omega = (0,1) \times (0,1), \\ \frac{\partial u}{\partial n} = 0, & \text{for} \quad x_1 = 1, \\ u = 0, & \text{for} \quad x \in \partial\Omega \setminus \{x_1 = 1\}, \end{cases}$$

on a triangulation with triangles of side length $1/4$ in the x_1-direction and $1/2$ in the x_2-direction.

Solution: Let $\Gamma_1 := \partial\Omega \setminus \Gamma_2$ where $\Gamma_2 := \{(1, x_2) : 0 \le x_2 \le 1\}$. Define

$$V = \{v : v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma_1\}.$$

Multiply the equation by $v \in V$ and integrate over Ω; using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n} v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v,$$

where we have used $\Gamma = \Gamma_1 \cup \Gamma_2$ and the fact that $v = 0$ on Γ_1 and $\frac{\partial u}{\partial n} = 0$ on Γ_2.

Variational formulation:

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V.$$

FEM: cG(1):

Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V_h \subset V, \tag{1}$$

where

$$V_h = \{v : v \text{ is piecewise linear and continuous in } \Omega,\ v = 0 \text{ on } \Gamma_1, \text{ on above mesh } \}.$$

A set of bases functions for the finite dimensional space V_h can be written as $\{\varphi_i\}_{i=1}^4$, where

$$\begin{cases} \varphi_i \in V_h, & i = 1, 2, 3, 4 \\ \varphi_i(N_j) = \delta_{ij}, & i, j = 1, 2, 3, 4. \end{cases}$$

Then the equation (2) is equivalent to: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla \varphi_i = \int_\Omega \varphi_i, \qquad i = 1, 2, 3, 4. \tag{2}$$

Set $U = \sum_{j=1}^4 \xi_j \varphi_j$. Invoking in the relation (3) above we get

$$\sum_{j=1}^4 \xi_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \int_\Omega \varphi_i, \qquad i = 1, 2, 3, 4.$$

Now let $a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$ and $b_i = \int_\Omega \varphi_i$, then we have that

$$A\xi = b, \quad A \text{ is the stiffness matrix } \; b \text{ is the load vector.}$$

Below we compute a_{ij} and b_i

$$b_i = \int_\Omega \varphi_i = \begin{cases} 6 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/8, & i = 1, 2, 3 \\ 3 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/16, & i = 4 \end{cases}$$

and

$$a_{ii} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_i = \begin{cases} 2 \cdot (\frac{5}{4} + 1 + \frac{1}{4}) = 5, & i = 1, 2, 3 \\ \frac{5}{4} + 1 + \frac{1}{4} = 5/2, & i = 4 \end{cases}$$

Further

$$a_{i,i+1} = \int_\Omega \nabla\varphi_{i+1} \cdot \nabla\varphi_i = 2 \cdot (-1) = -2 = a_{i+1,i}, \quad i = 1, 2, 3,$$

and

$$a_{ij} = 0, \quad |i - j| > 1.$$

Thus we have

$$A = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix} \qquad b = \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

3. Give a variational formulation for the boundary value problem (with periodic boundary conditions).

$$\begin{cases} -u'' + \alpha u = f, & 0 < x < 1, \\ u(0) = u(1), & u'(0) = u'(1), \end{cases}$$

where α is a constant and $f \in L_2(0,1)$. Show that, with an appropriate condition on α, the hypothesis in the Lax-Milgram theorem are fullfild.

Solution: Let $V = \{v \in H^1(0,1) : v(0) = v(1) = 0\}$ with

$$||v||_V = ||v||_{H^1} = \sqrt{||v||^2 + ||v'||^2}.$$

Multiplication of the differential equation by $v \in V$, integration by parts taking the boundary conditions into account, leeads to the variational formulation

$$u \in V, \qquad a(u, v) = L(v) \quad \forall v \in V,$$

with $\quad a(u, v) = (u', v') + \alpha(u, v) \quad$, and $\quad L(v) = (f, v)$.
If $\alpha > 0$ then we have

$$a(u, u) = ||u'||^2 + \alpha||u||^2 \geq \min(1, \alpha)||u||^2_{H^1},$$
$$a(u, v) \leq ||u'||||v'|| + \alpha||u||||v|| \leq (1 + \alpha)||u||_{H^1}||v||_{H^1},$$
$$|L(v)| \leq ||f||||v|| \leq ||f||||v||_{H^1}.$$

4. Prove an a priori and an a posteriori error estimate for the finite element method for the problem

$$-u'' + \alpha u = f, \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

where $\alpha = \alpha(x)$ is a bounded positive coefficient on $[0,1]$.

Solution: We consider

$$-u'' + \alpha u = f, \quad \text{in} \quad I = (0,1), \qquad u(0) = u(1) = 0,$$

with $0 \le \alpha(x) \le M,\ x \in I$. Multiply the equation by $v \in H_0^1(I)$ and integrate over I, using partial integration we get

$$\int_I (u'v' + \alpha uv) = \int_I fv.$$

Variational formulation: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + \alpha uv) = \int_I fv, \qquad \forall v \in H_0^1(I). \tag{3}$$

FEM; cG(1): Find $U \in V_h^0$ such that

$$\int_I (U'v' + \alpha Uv) = \int_I fv, \qquad \forall v \in V_h^0. \tag{4}$$

Let $e = u - U$ be the error then $(8) - (9)$ implies that

$$\int_I (e'v' + \alpha ev) = 0, \qquad \forall v \in V_h^0 \subset H_0^1(I). \tag{5}$$

Defining the energy scalar product and the energy norm as

$$(v,w)_E = \int_I (v'w' + \alpha vw), \quad \|v\|_E^2 = (v,v)_E = \int_I ((v')^2 + \alpha v^2)$$

we note that

$$(e,v)_E = 0, \qquad \forall v \in V_h^0. \tag{6}$$

A posteriori error estimates:

$$\begin{aligned}
\|e\|_E^2 &= \int_I (e'e' + \alpha ee) = \int_I (u-U)'e' + \int_I \alpha(u-U)e \\
&= \{v = e \text{ in } (3)\} = \int_I fe - \int_I (U'e' + \alpha Ue) = \{v = \pi_h e \text{ in } (4)\} \\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + \alpha U(e - \pi_h e)\Big) = [PI] \\
&= \int_I R(U)(e - \pi_h e),
\end{aligned}$$

where $R(U) := f + U'' - \alpha U = f - \alpha U$. Thus we may estimate the energy norm of e as follows:

$$\begin{aligned}
\|e\|_E^2 &\le \|hR(U)\|_{L_2(I)} \|h^{-1}(e - \pi_h e)\|_{L_2(I)} \\
&\le C_i \|hR(U)\|_{L_2(I)} \|e'\|_{L_2(I)} \le C_i \|hR(U)\|_{L_2(I)} \|e\|_E,
\end{aligned}$$

and hence

$$\|e\|_E \le C_i \|hR(U)\|_{L_2(I)}.$$

A priori error estimate:

$$\|e\|_E^2 = (e,e)_E = (e, u-U)_E = \{v = U - \pi_h u \text{ in } (6)\}$$
$$= (e, u - \pi_h u)_E \le \{C - S\} \le \|e\|_E \|u - \pi_h u\|_E,$$

so that

$$\|e\|_E \le \|u - \pi_h u\|_E.$$

Recall that $\alpha(x) \le M$ and the definition of the energy norm implies that

$$\|u - \pi_h u\|_E^2 = \|(u-\pi_h u)'\|_{L_2(I)}^2 + \|\sqrt{\alpha}(u - \pi_h u)\|_{L_2(I)}^2$$
$$\le C_i^2 \|hu''\|_{L_2(I)}^2 + C_i^2 M \|h^2 u''\|_{L_2(I)}^2,$$

which gives

$$\|e\|_E \le C_i\Big(\|hu''\|_{L_2(I)} + \sqrt{M}\|h^2 u''\|_{L_2(I)}\Big).$$

5. Consider the heat equation

$$\begin{cases} \dot u - u'' = f(x), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u'(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

a) Show, for the homogeneous case: i.e., $f \equiv 0$ (with $\|u\| = (\int_0^1 u^2(x)\,dx)^{1/2}$), the estimates

$$(E1) \quad \frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0, \qquad (E2) \quad \|u(.,t)\| \le e^{-t}\|u_0\|.$$

b) Let $u_s = u_s(x)$ be the solution of the corresponding stationary problem:

$$-u_s'' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

show that $\quad \|u - u_s\| \to 0, \quad$ as $t \to \infty$.

Solution: a) Multiply the equation for u by u and integrate with respect to x. Using partial integration and boundary conditions we get

$$0 = \int_0^1 fu = \int_0^1 (\dot u - u'')u = \int_0^1 \dot u u + \int_0^1 u'u' = \frac{1}{2}\frac{d}{dt}\int_0^1 u^2 + \int_0^1 (u')^2,$$

which is the desired identity (E1).

Now (E1) together with Poincare inequality $\|u\| \le \|u'\|$ gives that

$$\frac{d}{dt}\|u\|^2 + 2\|u\|^2 \le 0, \Longleftrightarrow \frac{d}{dt}(\|u\|^2 e^{2t}) = (\frac{d}{dt}\|u\|^2 + 2\|u\|^2)e^{2t} \le 0.$$

Integrating with respect to time variable from 0 to t leads to

$$\|u\|^2 e^{2t} - \|u_0\|^2 \le 0, \text{ i.e., } \|u\|^2 \le e^{-2t}\|u_0\|^2,$$

which, taking the square root, gives the estimate (E2).

b) Let $w = u - u_s$, then w satisfies the differential equation

$$\dot w - w'' = \dot u - u'' + u_s'' = f - f = 0,$$

so that we can apply (E2) to w to get

$$\|u - u_s\| \le e^{-t}\|u_0 - u_s\| \to 0, \quad \text{as } t \to \infty.$$

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