TMA372 Partial Differential Equations TM, 2005-12-13. Solutions

1. Let $\pi_1 f$ be the linear interpolant of a twice continuously differentiable function f on the interval $I=(a,b)$. Prove that

$$||f-\pi_1 f||_{L_1(I)} \le (b-a)^2 ||f''||_{L_1(I)}.$$

Solution: Let $\lambda_0(x)=\frac{\xi_1-x}{\xi_1-x_0}$ and $\lambda_1(x)=\frac{x-\xi_0}{\xi_1-x_0}$ be two linear base functions. Then by the integral form of the Taylor formula we may write

$$\begin{cases} f(\xi_0) &= f(x)+f'(x)(\xi_0-x)+\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy,\\ f(\xi_1) &= f(x)+f'(x)(\xi_1-x)+\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy,\end{cases}$$

Therefore

$$\begin{aligned}\Pi_1 f(x) &= f(\xi_0)\lambda_0(x)+f(\xi_1)\lambda_1(x)\\ &= f(x)+\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\end{aligned}$$

and by the triangle inequality we get

$$\begin{aligned}|f(x)-\Pi_1 f(x)| &= \Big|\lambda_0(x)\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy+\lambda_1(x)\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le |\lambda_0(x)|\Big|\int_x^{\xi_0}(\xi_0-y)f''(y)\,dy\Big|+|\lambda_1(x)|\Big|\int_x^{\xi_1}(\xi_1-y)f''(y)\,dy\Big|\\ &\le |\lambda_0(x)|\int_x^{\xi_0}|\xi_0-y||f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}|\xi_1-y||f''(y)|\,dy\\ &\le |\lambda_0(x)|\int_x^{\xi_0}(b-a)|f''(y)|\,dy+|\lambda_1(x)|\int_x^{\xi_1}(b-a)|f''(y)|\,dy\\ &\le (b-a)\Big(|\lambda_0(x)|+|\lambda_1(x)|\Big)\int_a^b|f''(y)|\,dy\\ &= (b-a)\Big(\lambda_0(x)+\lambda_1(x)\Big)\int_a^b|f''(y)|\,dy=(b-a)\int_a^b|f''(y)|\,dy.\end{aligned}$$

Consequently

$$\int_a^b|f(x)-\Pi_1 f(x)|dx \le \int_a^b(b-a)\Big(\int_a^b|f''(y)|\,dy\Big)\,dx,=(b-a)^2||f''||_{L_1(I)}.$$

2. (a) Derive the stiffness matrix and load vector in the global polynomial approximation $U(x)=\sum_{i=0}^q \xi_i t^i$ for the following ODE,

$$\dot u(t)=\lambda u(t),\quad 0<t\le 1,\qquad u(0)=u_0.$$

(b) Let $u_0=1$ and $\lambda=2$ and determine the approximate solution $U(t)$, for $q=1$ and $q=2$.

Solution: (a) We insert U in the equation (note $\dot U(t)=\sum_{j=1}^q j\xi_j t^{j-1}$), multiplying the reult by $t^i,\ i=1,\dots,q$ and integrating over $(0,1)$ we get

$$\sum_{j=1}^q \xi_j\int_0^1\Big(jt^{i+j-1}-\lambda t^{i+j}\Big)\,dt=\lambda\xi_0\int_0^1 t^i\,dt, \tag{1}$$

where the contribution for $j=0$ which corresponds to $\xi_0 = u(0)$ is on the right hand side. After integration we get

$$(2) \quad \sum_{j=1}^{q}\Big(\frac{j}{j+i} - \frac{\lambda}{j+i+1}\Big)\xi_j = \frac{\lambda}{i+1}\xi_0, \quad i,j = 1,\dots,q. \iff A\xi = b, \quad \text{where}$$

$$\begin{cases} a_{ij} = \frac{j}{j+i} - \frac{\lambda}{j+i+1}, & i,j = 1,\dots,q. \\ b_i = \frac{\lambda}{i+1}\xi_0, & i = 1,\dots,q. \end{cases}$$

(b) Thus with $u_0 = 1$ and $\lambda = 2$ we get

$$A\xi = b, \quad \text{where} \begin{cases} a_{ij} = \frac{j}{j+i} - \frac{2}{j+i+1}, & i,j = 1,\dots,q. \\ b_i = \frac{2}{i+1}, & i = 1,\dots,q. \end{cases}$$

$$q=1: \implies -\frac{1}{6}\xi_1 = 1 \implies \xi_1 = -6. \implies U(t) = 1 - 6t.$$

$$q=2: \implies \begin{bmatrix} -\frac{1}{6} & \frac{1}{6} \\ -\frac{1}{6} & \frac{1}{10} \end{bmatrix}\begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{2}{3} \end{bmatrix} \iff \begin{bmatrix} \xi_1 \\ \xi_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \end{bmatrix}.$$

Hence

$$U(t) = 1 + t + 5t^2.$$

3. Consider the Laplace equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x,y) : 0 < x < 1,\ 0 < y < 1\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

a) Show that $||D^2u|| = ||\Delta u||$, where $(D^2u)^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$.
b) Show the same result for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ (instead of $u = 0$ on $\partial\Omega$).
c) Show in the Dirichlet case (when $u = 0$ on $\partial\Omega$) that $||u|| \le C_\Omega ||\nabla u||$ (Poincare's inequality). What is the numerical value of the constant C_Ω?

Solution: (a) *A general approach* Let $\Gamma = \partial\Omega$ be the boundary of Ω. We have that

$$||\Delta u||^2 = \int_\Omega u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}.$$

Now an application of the Green's formula (partial integration first in y and then in x) gives

$$\int_\Omega u_{xx}u_{yy} = \int_\Gamma u_{xx}u_y n_y - \int_\Omega u_{xxy}u_y = \int_\Gamma u_{xx}u_y n_y - \int_\Gamma u_{xy}u_y n_x + \int_\Omega u_{xy}u_{xy},$$

where $n = (n_x, n_y)$ is the outward unit normal at the boundary. Now, on the part of the boundary Γ, where $n_y \neq 0$, we have $u_{xx} = 0$, since $u = 0$. Likewise, $u_y = 0$ on the part of the boundary, where $n_x \neq 0$. Thus $\int_\Omega u_{xx}u_{yy} = \int_\Omega u_{xy}u_{xy}$, which gives the desired identity.

Alternatively, let $\Omega = [0,1] \times [0,1]$. Then

$$||\Delta u|| = \int_\Omega (\Delta u)^2 = \int_\Omega (u_{xx} + u_{yy})^2 = \int_\Omega \Big(u_{xx}^2 + u_{yy}^2 + 2u_{xx}u_{yy}\Big).$$

$$\int_\Omega u_{xx}u_{yy} = \int_\Gamma u_x u_{yy} n_1 - \int_\Omega u_x u_{yyx} = \int_\Gamma u_x u_{yy} n_1 - \int_\Gamma u_x u_{xy} n_2 + \int_\Omega u_{xy}u_{xy},$$

where we have used partial integration and the identities $u_{yyx} = u_{yxy} = u_{xyy}$. Further we have that

$$-\int_\Gamma u_x u_{xy} n_2 = -\int_\Gamma u_x (u_y n_2)_x = -\int_\Gamma u_x (-u|n_2|)_x = \int_\Gamma u_x^2 |n_2| \geq 0,$$

where we have used the boundary condition $n \cdot \nabla u = -u$. Note that $n_2 = 0$ on the vertical boundary lines. Similarly we get

$$\int_\Gamma u_x u_{xy} n_1 = \int_\Gamma u_{yy}(-u)|n_1|,$$

which after integration by parts over the vertical boundary lines (where $n_1 \neq 0$) gives that

$$\int_\Gamma u_y u_y |n_1| - u_y(1,1)u(1,1) + u_y(1,0)u(1,0) - u_y(0,1)u(0,1) + u_y(0,0)u(0,0)$$
$$= \int_\Gamma u_y u_y |n_1| + u(1,1)u(1,1) + u(1,0)u(1,0) + (0,1)u(0,1) + u(0,0)u(0,0) \geq 0,$$

where we have agauin used the boundary condition $n \cdot \nabla u = -u$. In other words we have proved that

$$||\Delta u|| \geq ||D^2 u||.$$

The reversed inequality is trivial.
(b) In the case of Neumann boundary condition: $\frac{\partial u}{\partial n} = 0$ on the boundary, we have that $u_y = 0$ on the part of Γ where $n_y \neq 0$, similarly $u_{xy} = 0$ on the part of Γ where $n_x \neq 0$ (because, then $u_x = 0$ in y-direction). Thus we obtain the same identity as in (a).
(c) See proof of Poincare inequality in Chapter 15.

4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $\|e\|_{H^1}^2 = \|e'\| + \|e\|$) for a finite element method for problem

$$-u'' + pxu' + (1 + \frac{p}{2})u = f, \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.$$

Solution: We multiply the differential equation by a test function $v \in H_0^1(I),\ I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\text{(3)} \qquad \int_I \Big(u'v' + pxu'v + (1 + \frac{p}{2})uv\Big) = \int_I fv, \quad \forall v \in H_0^1(I).$$

A *Finite Element Method* with $cG(1)$ reads as followa: Find $U \in V_h^0$ such that

$$\text{(4)} \qquad \int_I \Big(U'v' + pxU'v + (1 + \frac{p}{2})Uv\Big) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I),$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$

Now let $e = u - U$, then (1)-(2) gives that

$$\text{(5)} \qquad \int_I \Big(e'v' + pxe'v + (1 + \frac{p}{2})ev\Big) = 0, \quad \forall v \in V_h^0.$$

A posteriori error estimate: We note that using $e(0)=e(1)=0$, we get

$$\int_I pxe'e = \frac{p}{2}\int_I x\frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2}\int_I e^2 = -\frac{p}{2}\int_I e^2, \tag{6}$$

so that

(7)

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \Big(e'e' + pxe'e + (1+\frac{p}{2})ee\Big)\\
&= \int_I \Big((u-U)'e' + px(u-U)'e + (1+\frac{p}{2})(u-U)e\Big) = \{v = e \text{ in}(1)\}\\
&= \int_I fe - \int_I \Big(U'e' + pxU'e + (1+\frac{p}{2})Ue\Big) = \{v = \pi_h e \text{ in}(2)\}\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + pxU'(e-\pi_h e) + (1+\frac{p}{2})U(e-\pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned}$$

where $\mathcal{R}(U) := f + U'' - pxU' - (1+\frac{p}{2})U = f - pxU' - (1+\frac{p}{2})U$, (for approximation with picewise linears, $U\equiv 0$, on each subinterval). Thus (5) implies that

$$\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\|\\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

A priori error estimate: We use (4) and write

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + pxe'e + (1+\frac{p}{2})ee)\\
&= \int_I \Big(e'(u-U)' + pxe'(u-U) + (1+\frac{p}{2})e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\}\\
&= \int_I \Big(e'(u-\pi_h u)' + pxe'(u-\pi_h u) + (1+\frac{p}{2})e(u-\pi_h u)\Big)\\
&\le \|(u-\pi_h u)'\|\,\|e'\| + p\|u-\pi_h u\|\,\|e'\| + (1+\frac{p}{2})\|u-\pi_h u\|\,\|e\|\\
&\le \{\|(u-\pi_h u)'\| + (1+p)\|u-\pi_h u\|\}\|e\|_{H^1}\\
&\le C_i\{\|hu''\| + (1+p)\|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + (1+p)\|h^2u''\|\},$$

which is the a priori error estimate.

5. Consider the initial boundary value problem for the heat equation

$$\begin{cases}
\dot u - \Delta u = 0, & x\in\Omega\subset\mathbb{R}^2, \quad 0<t\le T,\\
u(x,t) = 0 & x in \partial\Omega, \quad 0<t\le T,\\
u(x,0) = u_0(x), & x\in\Omega.
\end{cases}$$

Prove the following stability estimates

$$\|u\|^2(T) + 2\int_0^T \|\nabla u\|^2(t)\,dt = \|u_0\|^2, \tag{8}$$

$$\int_0^T t\|\Delta u\|^2(t)\,dt \leq \frac{1}{4}\|u_0\|^2, \tag{9}$$

$$\|\nabla u\|(T) \leq \frac{1}{\sqrt{2T}}\|u_0\|. \tag{10}$$

Solution See Lecture Notes or text book Chapter 16.

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