TMA371 Partial Differential Equations TM, 2004-12-14. Solutions

1. Let α and β be positive constants. Give the piecewise linear finite element approximation procedure, on the uniform mesh, for the problem

$$-u''(x) = 1, \quad 0 < x < 1; \qquad u(0) = \alpha, \quad u'(1) = \beta.$$

Solution: Multiply the pde by a test function v with $v(0) = 0$, integrate over $x \in (0,1)$ and use partial integration to get

$$\begin{aligned}
& -[u'v]_0^1 + \int_0^1 u'v'\,dx = \int_0^1 v\,dx \qquad \iff \\
(1) \qquad & -u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v'\,dx = \int_0^1 v\,dx \qquad \iff \\
& -\beta v(1) + \int_0^1 u'v'\,dx = \int_0^1 v\,dx.
\end{aligned}$$

The continuous variational formulation is now formulated as follows: Find

$$(VF) \qquad u \in V := \{w : \int_0^1 \Big(w(x)^2 + w'(x)^2\Big)\,dx < \infty, \quad w(0) = \alpha\},$$

such that

$$\int_0^1 u'v'\,dx = \int_0^1 v\,dx + \beta v(1), \quad \forall v \in V^0,$$

where

$$V^0 := \{v : \int_0^1 \Big(v(x)^2 + v'(x)^2\Big)\,dx < \infty, \quad v(0) = 0\}.$$

For the discrete version we let $\mathcal{T}_h$ be a uniform partition: $0 = x_0 < x_1 < \ldots < x_{M+1}$ of $[0,1]$ into the subintervals $I_n = [x_{n-1}, x_n]$, $n = 1, \ldots M+1$. Here, we have M interior nodes: $x_1, \ldots x_M$, two boundary points: $x_0 = 0$ and $x_{M+1} = 1$ and hence $M+1$ intervals.

The finite element method (discrete variational formulation) is now formulated as follows: Find

$$(FEM) \qquad U \in V_h := \{w_h : w_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ w_h(0) = \alpha\},$$

such that

$$(2) \qquad \int_0^1 U'v_h'\,dx = \int_0^1 v_h\,dx + \beta v_h(1), \quad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v_h : v_h \text{ is piecewise linear, continuous on } \mathcal{T}_h,\ v_h(0) = 0\}.$$

Using the basis functions φ_j, $j = 0, \ldots M+1$, where $\varphi_1, \ldots \varphi_M$ are the usual *hat-functions* whereas φ_0 and φ_{M+1} are *semi-hat-functions* viz;

$$(3) \qquad \varphi_j(x) = \begin{cases} 0, & x \notin [x_{j-1}, x_j] \\ \frac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h} & x_j \le x \le x_{j+1} \end{cases}, \qquad j = 1, \ldots M.$$

and

$$\varphi_0(x) = \begin{cases} \frac{x_1 - x}{h} & 0 \le x \le x_1 \\ 0, & x_1 \le x \le 1 \end{cases}, \qquad \varphi_{M+1}(x) = \begin{cases} \frac{x - x_M}{h} & x_M \le x \le x_{M+1} \\ 0, & 0 \le x \le x_M. \end{cases}$$

In this way we may write

$$V_h = \alpha\varphi_0 \oplus [\varphi_1, \dots, \varphi_{M+1}], \quad V_h^0 = [\varphi_1, \dots, \varphi_{M+1}].$$

Thus every $U \in V_h$ can ve written as $U = \alpha\varphi_0 + v_h$ where $v_h \in V_h^0$, i.e.,

$$U = \alpha\varphi_0 + \xi_{1\varphi_1} + \dots \xi_{M+1}\varphi_{M+1} = \alpha\varphi_0 + \sum_{i=1}^{M+1} \xi_i\varphi_i \equiv \alpha\varphi_0 + \tilde{U},$$

where $\tilde{U} \in V_h^0$, and hence the problem (2) can equivalently be formulated as to find $\xi_1, \dots \xi_{M+1}$ such that

$$\int_0^1 \Big(\alpha\varphi_0' + \sum_{i=1}^{M+1} \xi_i\varphi_i'\Big)\varphi_j'\,dx = \int_0^1 \varphi_j\,dx + \beta\varphi_j(1), \quad j = 1, \dots M+1,$$

which can be written as

$$\sum_{i=1}^{M+1} \Big(\int_0^1 \varphi_j'\varphi_i'\,dx\Big)\xi_i = -\int_0^1 \varphi_0'\varphi_j'\,dx + \int_0^1 \varphi_j\,dx + \beta\varphi_j(1), \quad j = 1, \dots M+1,$$

or equivalently $A\xi = b$ where $A = (a_{ij})$ is the tridiagonal matrix with entries

$$a_{ii} = 2, \quad a_{i,i+1} = a_{i+1,i} = -1, \quad i = 1, \dots M, \quad \text{and} \quad a_{M+1,M+1} = 1,$$

i.e.,

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & \dots & 0 & 0 \\ -1 & 2 & -1 & 0 & \dots & 0 & 0 \\ \dots \\ \dots \\ \dots \\ 0 & 0 & \dots & 0 & -1 & 2 & -1 \\ 0 & 0 & \dots & 0 & 0 & -1 & 1 \end{bmatrix},$$

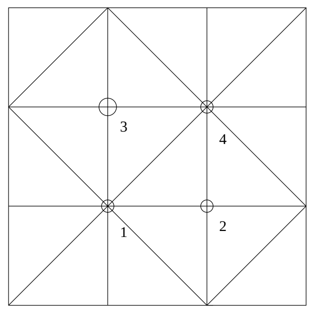
and the unkown ξ and the data b are given by

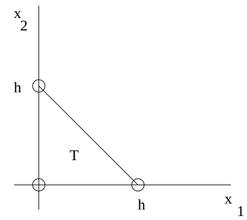
$$\xi = \begin{bmatrix} \xi_1 \\ \xi_2 \\ \cdot \\ \cdot \\ \xi_M \\ \xi_{M+1} \end{bmatrix}, \qquad b = \begin{bmatrix} \int_0^1 \varphi_1\,dx - \alpha\int_0^1 \varphi_0'\varphi_1'\,dx \\ \int_0^1 \varphi_2\,dx \\ \cdot \\ \cdot \\ \int_0^1 \varphi_M\,dx \\ \int_0^1 \varphi_{M+1}\,dx + \beta\varphi_{M+1}(1) \end{bmatrix} = \begin{bmatrix} h + \frac{1}{h}\alpha \\ h \\ \cdot \\ \cdot \\ h \\ \frac{h}{2} + \beta \end{bmatrix}.$$

2. Formulate the $cG(1)$ method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting equation system using the following uniform mesh:





Solution: Let V_h be the usual finite element space cosisting of continuous piecewise linear functions satisfying the boundary condition $v = 0$ on $\partial\Omega$. The $cG(1)$ method is: Find $U \in V_h$ such that

$$(\nabla U, \nabla v) + (U, v) = (f, v) \qquad \forall v \in V_h$$

Making the "Ansatz" $U(x) = \sum_{i=1}^{4} \xi_i \varphi_i(x)$, where φ_i are the standard basis functions, we obtain the system of equations

$$\sum_{i=1}^{4} \xi_i \Big(\int_\Omega \nabla\varphi_i \cdot \nabla\varphi_j \, dx + \int_\Omega \varphi_i \varphi_j \, dx \Big) = \int_\Omega f \varphi_j \, dx, \quad j = 1, \ldots, 4,$$

or, in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_j = (f, \varphi_j)$ is the load vector.

We first compute the mass and stiffness amtrix for the reference triangle T. The local basis functions are

$$\phi_1(x_1, x_2) = 1 - \frac{x_1}{h} - \frac{x_2}{h}, \qquad \nabla\phi_1(x_1, x_2) = -\frac{1}{h}\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$
$$\phi_2(x_1, x_2) = \frac{x_1}{h}, \qquad \nabla\phi_2(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
$$\phi_3(x_1, x_2) = \frac{x_2}{h}, \qquad \nabla\phi_3(x_1, x_2) = \frac{1}{h}\begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12},$$
$$s_{11} = (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2}|T| = 1.$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6}\Big(0 + \frac{1}{4} + \frac{1}{4}\Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoins of the edges. Similarly we can compute the other elements and obtain

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s:

$$M_{11} = M_{44} = 8m_{22} = \frac{8}{12}h^2, \qquad S_{11} = S_{44} = 8s_{22} = 4,$$

$$M_{12} = M_{13} = M_{24} = M_{34} = 2m_{12} = \frac{1}{12}h^2, \quad S_{12} = S_{13} = S_{24} = S_{34} = 2s_{12} = -1,$$

$$M_{14} = 2m_{23} = \frac{1}{12}h^2, \qquad S_{14} = 2s_{23} = 0,$$

$$M_{22} = M_{33} = 4m_{11} = \frac{4}{12}h^2, \qquad S_{22} = S_{33} = 4s_{11} = 4,$$

$$M_{23} = 0, \qquad S_{23} = 0.$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

3. Consider the boundary value problem for the stationary heat flow in $1D$:

$$(BVP) \qquad -(a(x)u'(x))' = f(x), \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

Formulate the corresponding variational formulation (VF), minimization problem (MP) and show that: $\quad (BVP) \Longleftrightarrow (VF) \Longleftrightarrow (MP)$.

Solution: See PDE Lecture Notes, Chapter 8.

4. (a) Prove an a priori and an a posteriori error estimate for the finite element method for the problem

$$-u''(x) + u'(x) = f(x), \quad 0 < x < 1; \qquad u(0) = u(1) = 0.$$

(b) Give an adaptive algorithm based on a posteriori estimates.

Solution: (a) We multiply the differential equation by a test function $v \in H_0^1(I),\ I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I (u'v' + u'v) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{4}$$

A *Finite Element Method* with $cG(1)$ reads as followa: Find $U \in V_h^0$ such that

$$\int_I (U'v' + U'v) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{5}$$

where

$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I (e'v' + e'v) = 0, \quad \forall v \in V_h^0. \tag{6}$$

We note that using $e(0) = e(1) = 0$, we get

$$\int_I e'e = \int_I \frac{1}{2}\frac{d}{dx}\Big(e^2\Big) = \frac{1}{2}(e^2)|_0^1 = 0. \tag{7}$$

Further, usig Poicare inequality we have

$$\|e\|^2 \le \|e'\|^2.$$

A priori error estimate: We use Poicare inequality and (7) to get

$$\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e) = 2\int_I \Big(e'(u-U)' + e'(u-U)\Big) \\
&= 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) + 2\int_I \Big(e'(\pi_h u - U)' + e'(\pi_h u - U)\Big) \\
&= \{v = U - \pi_h u \ \text{ in } (6)\} = 2\int_I \Big(e'(u-\pi_h u)' + e'(u-\pi_h u)\Big) \\
&\le 2\|(u-\pi_h u)'\|\|e'\| + 2\|u-\pi_h u\|\|e'\| \\
&\le 2C_i\{\|hu''\| + \|h^2u''\|\}\|e\|_{H^1},
\end{aligned}$$

this gives that

$$\|e\|_{H^1} \le C_i\{\|hu''\| + \|h^2u''\|\},$$

which is the a priori error estimate.

A posteriori error estimate:

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) \le 2\int_I e'e' = 2\int_I (e'e' + e'e)\\
&= 2\int_I ((u-U)'e' + (u-U)'e) = \{v = e \ \text{ in } (4)\}\\
&= 2\int_I fe - \int_I (U'e' + U'e) = \{v = \pi_h e \ \text{ in } (5)\} \qquad (8)\\
&= \int_I f(e - \pi_h e) - \int_I \Big(U'(e-\pi_h e)' + U'(e - \pi_h e)\Big)\\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e - \pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - U' = f - U'$, (for approximation with picewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$
\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e - \pi_h e)\|\\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}
$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.$$

(b) An adaptive procedure can be formulated in the following steps:

Step I. Start with a given mesh size h and a given error tolerance "TOL". Compute U and $\mathcal{R}(U)$ corresponding to this h

Step II. Compare $C_i\|h\mathcal{R}(U)\|$ with the tolerance "TOL":

IIa). If

$$C_i\|h\mathcal{R}(U)\| < TOL,$$

then accept U as an appropriate cG(1) approximate solution.

IIb). If

$$C_i\|h\mathcal{R}(U)\| \ge TOL,$$

then refine the mish on the subintervals with large $\mathcal{R}(U)$ contributions, thus obain a new mesh and return to Step I.

5. Let Ω be a bounded domain in $\mathbb{R}^d$. Consider the initial-boundary value problem

$$
\begin{cases}
u_t - \Delta u = 0, & \text{in} \quad \Omega \times \mathbb{R}^+,\\
u = 0, & \text{on} \quad \partial\Omega \times \mathbb{R}^+,\\
u(\cdot, 0) = v, & \text{in} \quad \Omega.
\end{cases}
$$

Show the stability estimates

$$(i) \quad ||\nabla u(t)||^2 \le \frac{1}{2t}||v||^2, \quad \text{and} \qquad (ii) \quad \int_0^t s||\Delta u(s)||^2\, ds \le \frac{1}{4}||v||^2.$$

Solution: See PDE Lecture Notes, Chapter 15.

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