TMA371 Partial Differential Equations TM, 1999-12-21. Solutions

1. Consider

$$
\left\{
\begin{array}{lll}
-\Delta u = 1, & \text{on} & \Omega = (0,1)\times(0,1), \\
\frac{\partial u}{\partial n} = 0, & \text{for} & x \in \Gamma_2 : \; x_1 = 1, \\
u = 0, & \text{for} & x \in \Gamma_1 := \partial\Omega \setminus \{x_1 = 1\},
\end{array}
\right. \tag{1}
$$

Define

$$V = \{v : v \in H^1(\Omega),\; v = 0 \;\text{ on }\; \Gamma_1\}.$$

Multiply the equation by $v \in V$ and integrate over Ω; using Green's formula

$$\int_\Omega \nabla u \cdot \nabla v - \int_\Gamma \frac{\partial u}{\partial n} v = \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v,$$

where we have used $\Gamma = \Gamma_1 \cup \Gamma_2$ and the fact that $v = 0$ on Γ_1 and $\frac{\partial u}{\partial n} = 0$ on Γ_2.

Variational formulation:

Find $u \in V$ such that

$$\int_\Omega \nabla u \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V.$$

FEM: cG(1):

Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla v = \int_\Omega v, \qquad \forall v \in V_h \subset V, \tag{2}$$

where

$V_h = \{v : v$ is piecewise linear and continuous in $\Omega,\; v = 0$ on $\Gamma_1,$ on above mesh $\}$.

A set of bases functions for the finite dimensional space V_h can be written as $\{\varphi_i\}_{i=1}^4$, where

$$
\left\{
\begin{array}{ll}
\varphi_i \in V_h, & i = 1,2,3,4 \\
\varphi_i(N_j) = \delta_{ij}, & i,j = 1,2,3,4.
\end{array}
\right.
$$

Then the equation (2) is equivalent to: Find $U \in V_h$ such that

$$\int_\Omega \nabla U \cdot \nabla \varphi_i = \int_\Omega \varphi_i, \qquad i = 1,2,3,4. \tag{3}$$

Set $U = \sum_{j=1}^4 \xi_j \varphi_j$. Invoking in the relation (3) above we get

$$\sum_{j=1}^4 \xi_j \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i = \int_\Omega \varphi_i, \qquad i = 1,2,3,4.$$

Now let $a_{ij} = \int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i$ and $b_i = \int_\Omega \varphi_i$, then we have that

$$A\xi = b, \quad A \text{ is the stiffness matrix } \ b \text{ is the load vector.}$$

Below we compute a_{ij} and b_i

$$b_i = \int_\Omega \varphi_i = \begin{cases} 6 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/8, & i = 1,2,3 \\ 3 \cdot \frac{1}{3} \cdot \frac{1/4 \cdot 1/2}{2} \cdot 1 = 1/16, & i = 4 \end{cases}$$

and

$$a_{ii} = \int_\Omega \nabla\varphi_i \cdot \nabla\varphi_i = \begin{cases} 2 \cdot (\frac{5}{4} + 1 + \frac{1}{4}) = 5, & i = 1,2,3 \\ \frac{5}{4} + 1 + \frac{1}{4} = 5/2, & i = 4 \end{cases}$$

Further

$$a_{i,i+1} = \int_\Omega \nabla\varphi_{i+1} \cdot \nabla\varphi_i = 2 \cdot (-1) = -2 = a_{i+1,i}, \quad i = 1,2,3,$$

and

$$a_{ij} = 0, \quad |i - j| > 1.$$

Thus we have

$$A = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix} \qquad b = \frac{1}{16} \begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

2. For the proof of the Lax-Milgram theorem see the book, Chapter 21.
As for the given case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and

$$a(v,w) = \int_I v'w'\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx,$$

it is trivial to show that $a(\cdot,\cdot)$ is bilinear and $b(\cdot)$ is linaer. We have that

$$a(v,v) = \int_I (v')^2\,dx + v(0)^2 \geq \frac{1}{2}\int_I (v')^2\,dx + \frac{1}{2}v(0)^2 + \frac{1}{2}\int_I (v')^2\,dx. \tag{4}$$

Furthet

$$v(x) = v(0) + \int_0^x v'(y)\,dy, \quad \forall x \in I$$

implies

$$\begin{aligned} v^2(x) &\leq 2\Big(v(0)^2 + (\int_0^x v'(y)\,dy)^2\Big) \\ &\leq \{C - S\} \leq 2v(0)^2 + 2\int_0^1 v'(y)^2\,dy, \end{aligned}$$

so that

$$\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}v^2(x), \quad \forall x \in I.$$

Integrating over x we get

$$\frac{1}{2}v(0)^2 + \frac{1}{2}\int_0^1 v'(y)^2\,dy \geq \frac{1}{4}\int_I v^2(x)\,dx. \tag{5}$$

Now combining (4) and (5) we get

$$\begin{aligned} a(v,v) &\geq \frac{1}{4}\int_I v^2(x)\,dx + \frac{1}{2}\int_I (v')^2(x)\,dx \\ &\geq \frac{1}{4}\Big(\int_I v^2(x)\,dx + \int_I (v')^2(x)\,dx\Big) = \frac{1}{4}||v||_V^2, \end{aligned}$$

so that we can take $\kappa_1 = 1/4$. Further

$$\begin{aligned} |a(v,w)| &\leq \Big|\int_I v'w'\,dx\Big| + |v(0)w(0)| \\ &\leq \{C-S\} \leq ||v'||_{L_2(I)}||w'||_{L_2(I)} + |v(0)||w(0)| \\ &\leq ||v||_V||w||_V + |v(0)||w(0)|. \end{aligned}$$

Now we have that

$$v(0) = -\int_0^x v'(y)\,dy + v(x), \quad \forall x \in I, \tag{6}$$

and by the Mean-value theorem for the integrals: $\exists \xi \in I$ so that $v(\xi) = \int_0^1 v(y)\,dy$. Choose $x = \xi$ in (6) then

$$\begin{aligned} |v(0)| &= \Big| -\int_0^\xi v'(y)\,dy + \int_0^1 v(y)\,dy\Big| \\ &\leq \int_0^1 |v'|\,dy + \int_0^1 |v|\,dy \leq \{C-S\} \\ &\leq ||v'||_{L_2(I)} + ||v||_{L_2(I)} \leq 2||v||_V, \end{aligned}$$

implies that

$$|v(0)||w(0)| \leq 4||v||_V||w||_V,$$

and consequently

$$|a(u,w)| \leq ||v||_V||w||_V + 4||v||_V||w||_V = 5||v||_V||w||_V,$$

so that we can take $\kappa_2 = 5$. Finally

$$|L(v)| = \Big|\int_I fv\,dx\Big| \leq ||f||_{L_2(I)}||v||_{L_2(I)} \leq ||f||_{L_2(I)}||v||_V,$$

taking $\kappa_3 = ||f||_{L_2(I)}$ all the conditions in the Lax-Milgram theorem are fulfilled.

3. a) Multiply the equation by $\dot{u}$ and integrate to obtain

$$\begin{aligned} &(\ddot{u},\dot{u}) - (\Delta u,\dot{u}) + (u,\dot{u}) = 0, \\ &(\ddot{u},\dot{u}) + (\nabla u,\nabla\dot{u}) + (u,\dot{u}) = 0, \\ &\frac{1}{2}\frac{d}{dt}(||\dot{u}||^2 + ||\nabla u||^2 + ||u||^2) = 0, \\ &\frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2) = \frac{1}{2}(||u_1||^2 + ||\nabla u_0||^2 + ||u_0||^2). \end{aligned}$$

This means that the energy $E = \frac{1}{2}(||\dot{u}(t)||^2 + ||\nabla u(t)||^2 + ||u(t)||^2)$ is conserved.

b) Set $v_1 = \dot{u}$, $v_2 = u$. Then

$$\begin{aligned} &\dot{v}_1 - \Delta v_2 + v_2 = 0, \\ &\dot{v}_2 - v_1 = 0. \end{aligned}$$

Now we have asystem $\dot{v}+Av=0$ of first order in t and we can use various techniques developed for such systems, for example, we can apply standard time-discretization methods such as $dG(0)$ or $cG(1)$.

4. Multiplication by u gives

$$\varepsilon||u'||^2+\int_0^1 \alpha u' u\,dx+||u||^2=(f,u)\le ||f||||u||\le \frac{1}{2}||f||^2+\frac{1}{2}||u||^2.$$

Here

$$\begin{aligned}\int_0^1 \alpha u' u\,dx &= \frac{1}{2}\int_0^1 \alpha \frac{d}{dx}u^2\,dx \\ &= \frac{1}{2}\alpha(1)u(1)^2-\frac{1}{2}\int_0^1 \alpha' u^2\,dx \ge 0,\end{aligned} \tag{7}$$

and hence

$$\varepsilon||u'||^2+\frac{1}{2}||u||^2\le \frac{1}{2}||f||^2.$$

This proves

$$\sqrt{\varepsilon}||u'||\le ||f||, \qquad ||u||\le ||f||. \tag{8}$$

Multiply the equation by $\alpha u'$ and integrate over x to obtain

$$-\varepsilon\int_0^1 u''\alpha u'\,dx+||\alpha u'||^2+\int_0^1 \alpha u' u\,dx\le \frac{1}{2}||f||^2+\frac{1}{2}||\alpha u'||^2.$$

Hence by (7)

$$\begin{aligned}||\alpha u'||^2 &\le ||f||^2+\varepsilon\int_0^1 \alpha\frac{d}{dx}(u')^2\,dx \\ &= ||f||^2-\varepsilon\alpha(0)u'(0)^2-\varepsilon\int_0^1 \alpha'(u')^2\,dx \\ &\le ||f||^2+||\alpha'||\varepsilon||u'||^2\le ||f||^2+C\varepsilon||u'||^2.\end{aligned}$$

Using also (8) we conclude

$$||\alpha u'||\le C||f||. \tag{9}$$

Finally, by the differential equation and (8) and (9) we get

$$\varepsilon||u''||=||f-\alpha u'-u||\le ||f||+||\alpha u'||+||u||\le C||f||.$$

5. We have, using the hint, that

$$\begin{aligned}||v||^2=\int_0^1 v^2\,dx &= \int_0^{1/2} v^2\,dx+\int_{1/2}^1 v^2\,dx \\ &=[(x-1/2)v(x)^2]_0^{1/2}+[(x-1/2)v(x)^2]_{1/2}^1-\int_0^1 (x-1/2)2v(x)v'(x)\,dx \\ &\le \frac{1}{2}v(0)^2+\frac{1}{2}v(1)^2+||v||||v'||\le \frac{1}{2}\Big(v(0)^2+v(1)^2+||v'||^2\Big)+\frac{1}{2}||v||^2,\end{aligned}$$

and the proof is complete.

MA