TMA371 Partiella differentialekvationer TM

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Consider the Dirichlet problem
$$-\nabla\cdot(a(x)\nabla u) = f(x), \quad x\in\Omega\subset\mathbb{R}^2, \qquad u=0, \text{ for } x\in\partial\Omega.$$
Assume that c_0 and c_1 are constants such that $c_0 \le a(x) \le c_1$, $\forall x\in\Omega$ and let $U=\sum_{j=1}^N \alpha_j w_j(x)$ be a Galerkin approximation of u in a finite dimensional subspace M of $H_0^1(\Omega)$. Prove the a priori error estimate
$$||u-U||_{H_0^1(\Omega)} \le C \inf_{\chi\in M} ||u-\chi||_{H_0^1(\Omega)}.$$

2. Consider the problem 1 in $1D$, i.e.
$$-(a(x)u')' = f(x),\ 0<x<1, \qquad u(0)=u(1)=0.$$
Prove the following, $cG(1)$, a posteriori error estimate:
$$||u-U|| \le SC_i||h^2R(U)||,$$
where $||\cdot||$ is the usual L_2-norm, $R(U)$ is the residual, S and C_i are stability and interpolation constants, respectively.

3. Consider the boundary value problem
$$\begin{cases} -\Delta u + u = f, & x\in\Omega\subset\mathbb{R}^d,\\ n\cdot\nabla u = g, & \text{on}\quad \Gamma:=\partial\Omega,\end{cases}$$
where n is the outward unit normal to Γ.
(a) Show the following stability estimate: for some constant C,
$$||\nabla u||^2_{L_2(\Omega)} + ||u||^2_{L_2(\Omega)} \le C(||f||^2_{L_2(\Omega)} + ||g||^2_{L_2(\Gamma)}).$$
(b) Formulate a finite element method for the $1D$-case and derive the resulting system of equations for $\Omega=[0,1]$, $f(x)=1$, $g(0)=3$ and $g(1)=0$.

4. Consider the initial-boundary value problem
$$\begin{cases} \dot u - \Delta u = 0, & x\in\Omega, & t>0,\\ u=0, & x\in\partial\Omega, & t>0,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases}$$
Show the stability estimates:
$$||u(t)||^2 + \int_0^t ||\nabla u(s)||^2\,ds \le ||u_0||^2 + C\int_0^t ||f(s)||^2\,ds,$$
$$||\nabla u(t)||^2 + \int_0^t ||\Delta u(s)||^2\,ds \le ||\nabla u_0||^2 + C\int_0^t ||f(s)||^2\,ds.$$

5. Prove an a priori and an a posteriori error estimate for a finite element method for problem
$$-u'' + u' + u = f, \quad \text{in} \quad (0,1), \qquad u(0)=u(1)=0.$$
OBS! Resultaten beräknas komma upp ca 10 maj.

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