TMA372/MAN660 Partiella differentialekvationer TM, IMP, E3, GU

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Consider the boundary value problem for the stationary heat flow in $1D$:

$$(BVP) \qquad \begin{cases} -(a(x)u'(x))' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Formulate the corresponding variational formulation (VF), minimization problem (MP) and show that: $(BVP) \Longleftrightarrow (VF) \Longleftrightarrow (MP)$.

2. Consider the initial value problem: $\dot{u}(t)+au(t) = 0, \quad t > 0, \quad u(0) = 1.$

a) Let $a = 40$, and the time step $k = 0.1$. Draw the graph of $U_n := U(nk)$, $k = 1, 2, \ldots$, approximating u using (i) explicit Euler, (ii) implicit Euler, and (iii) Cranck-Nicholson methods.

b) Consider the case $a = i$, $(i^2 = -1)$, having the complex solution $u(t) = e^{-it}$ with $|u(t)| = 1$ for all t. Show that this property is preserved in Cranck-Nicholson approximation, (i.e. $|U_n| = 1$), but NOT in any of the Euler approximations.

3. Consider the problem

$$-\varepsilon u'' + xu' + u = f \quad \text{in } I = (0,1), \qquad u(0) = u'(1) = 0,$$

where ε is a positive constant, and $f \in L_2(I)$. Prove that

$$||\varepsilon u''|| \le ||f||, \qquad (||\cdot|| \text{ is the } L_2(I) - \text{norm}).$$

4. Let u be the solution of the following Neumann problem:

$$\begin{cases} -\Delta u = f, & \text{in} \Omega \subset \mathbb{R}^d, \\ -\partial_n u = ku, & \text{on} \Gamma := \partial\Omega. \end{cases}$$

where $\partial_n u = n \cdot \nabla u$ with n being the outward unit normal, and $k \ge 0$.

a) Show that $||u||_\Omega \le C_\Omega(||u||_\Gamma + ||\nabla u||_\Omega)$.

b) Use the estimate in a), and show that $||u||_\Gamma \to 0$ as $k \to \infty$.

5. Consider the initial-boundary value problem

$$\begin{cases} \dot{u} - \Delta u = f, & x \in \Omega, & t > 0, \\ u = 0, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Show the stability estimates:

$$||u(t)||^2 + \int_0^t ||\nabla u(s)||^2\, ds \le ||u_0||^2 + C\int_0^t ||f(s)||^2\, ds,$$

$$||\nabla u(t)||^2 + \int_0^t ||\Delta u(s)||^2\, ds \le ||\nabla u_0||^2 + C\int_0^t ||f(s)||^2\, ds.$$

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