TMA372/MAN660 Partial Differential Equations TM

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

1. Formulate the interpolation error estimates in the L_p-norm, $p = 1, 2, \infty$, in an interval (a, b). Prove the L_∞ error estimate for the linear interpolant:

$$||\pi_1 f - f||_{L_\infty(a,b)} \leq C_i(b-a)^2||f''||_{L_\infty(a,b)}$$

2. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method with piecewise linear approximation is applied to the following Poisson's equation with mixed boundary conditions:

$$\begin{cases} -\Delta u = 1, & \text{on} \quad \Omega = (0,1) \times (0,1), \\ \frac{\partial u}{\partial n} = 0, & \text{for} \quad x_1 = 1, \\ u = 0, & \text{for} \quad x \in \partial\Omega \setminus \{x_1 = 1\}, \end{cases}$$

on a triangulation with triangles of side length $1/4$ in the x_1-direction and $1/2$ in the x_2-direction.

3. Give a variational formulation for the boundary value problem (with periodic boundary conditions).

$$\begin{cases} -u'' + \alpha u = f, & 0 < x < 1, \\ u(0) = u(1), & u'(0) = u'(1), \end{cases}$$

where α is a constant and $f \in L_2(0,1)$. Show that, with an appropriate condition on α, the hypothesis in the Lax-Milgram theorem are fullfild.

4. Prove an a priori and an a posteriori error estimate for the finite element method for the problem

$$-u'' + \alpha u = f, \quad \text{in } I = (0,1), \qquad u(0) = u(1) = 0,$$

where $\alpha = \alpha(x)$ is a bounded positive coefficient on $[0, 1]$.

5. Consider the heat equation

$$\begin{cases} \dot{u} - u'' = f(x), & 0 < x < 1, \quad t > 0, \\ u(0,t) = u'(1,t) = 0, & t > 0, \\ u(x,0) = u_0(x), & 0 < x < 1. \end{cases}$$

a) Show, for the homogeneous case: i.e., $f \equiv 0$ (with $\|u\| = (\int_0^1 u^2(x)\,dx)^{1/2}$), the estimates

$$(E1) \quad \frac{d}{dt}\|u\|^2 + 2\|u'\|^2 = 0, \qquad (E2) \quad \|u(.,t)\| \leq e^{-t}\|u_0\|.$$

b) Let $u_s = u_s(x)$ be the solution of the corresponding stationary problem:

$$-u_s'' = f, \quad 0 < x < 1, \qquad u(0) = u'(1) = 0,$$

show that $\quad \|u - u_s\| \to 0, \quad$ as $t \to \infty$.