TMA372/MAN660 Partial Differential Equations TM

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

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1. Let $\pi_1 f$ be the linear interpolant of a twice continuously differentiable function f on the interval $I = (a, b)$. Prove that

$$||f - \pi_1 f||_{L_1(I)} \le (b-a)^2 ||f''||_{L_1(I)}.$$

2. (a) Derive the stiffness matrix and load vector in the global polynomial approximation $U(x) = \sum_{i=0}^{q} \xi_i t^i$ for the following ODE,

$$\dot{u}(t) = \lambda u(t), \quad 0 < t \le 1, \qquad u(0) = u_0.$$

(b) Let $u_0 = 1$ and $\lambda = 2$ and determine the approximate solution $U(t)$, for $q = 1$ and $q = 2$.

3. Consider the Laplace equation with the Dirichlet boundary condition

$$\begin{cases} -\Delta u = f, & \text{in } \Omega = \{(x,y) : 0 < x < 1,\, 0 < y < 1\}, \\ u = 0, & \text{on } \partial\Omega, \end{cases}$$

a) Show that $||D^2 u|| = ||\Delta u||$, where $(D^2 u)^2 = u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2$.

b) Show the same result for the Neumann boundary condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$ (instead of $u = 0$ on $\partial\Omega$).

c) Show in the Dirichlet case (when $u = 0$ on $\partial\Omega$) that $||u|| \le C_\Omega ||\nabla u||$ (Poincare's inequality). What is the numerical value of the constant C_Ω?

4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $||e||_{H^1}^2 = ||e'|| + ||e||$) for a finite element method for problem

$$-u'' + pxu' + (1 + \frac{p}{2})u = f, \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.$$

5. Consider the initial boundary value problem for the heat equation

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^2, & 0 < t \le T, \\ u(x,t) = 0 & x ßn \partial\Omega, & 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Prove the following stability estimates

$$\|u\|^2(T) + 2\int_0^T \|\nabla u\|^2(t)\, dt = \|u_0\|^2, \tag{1}$$

$$\int_0^T t\|\Delta u\|^2(t)\, dt \le \frac{1}{4}\|u_0\|^2, \tag{2}$$

$$\|\nabla u\|(T) \le \frac{1}{\sqrt{2T}}\|u_0\|. \tag{3}$$