TMA371 Partiella differentialekvationer TM

OBS! Skriv namn och personnummer på samtliga inlämnade papper.

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1. Determine the stiffness matrix and load vector if the $cG(1)$ finite element method with piecewise linear approximation is applied to Poisson's equation
$$\begin{cases} -\Delta u = 1, & \text{on} \quad \Omega = (0,1) \times (0,1), \\ \frac{\partial u}{\partial n} = 0, & \text{for} \quad x_1 = 1, \\ u = 0, & \text{for} \quad x \in \partial\Omega \setminus \{x_1 = 1\}, \end{cases}$$
on a triangulation with triangles of side length $1/4$ in the x_1-direction and $1/2$ in the x_2-direction.

2. Formulate and prove the Lax-Milgram theorem in the case of a symmetric bilinear form. Determine if the assumptions of the Lax-Milgram theorem are satisfied in the following case: $I = (0,1)$, $f \in L_2(I)$, $V = H^1(I)$ and
$$a(v,w) = \int_I v'w'\,dx + v(0)w(0), \quad L(v) = \int_I fv\,dx.$$

3. Consider the initial-boundary value problem
$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega, & t > 0, \\ u = 0, & x \in \partial\Omega, & t > 0, \\ u(x,0) = u_0(x), & \dot{u}(x,0) = u_1(x), & x \in \Omega. \end{cases}$$
a) Define a suitable energy for this problem and prove that the energy is conserved.

b) Reformulate the problem as a system of 2 equations with time derivatives of at most order one. Why such a reformulation is necessary?

4. Consider the boundary value problem
$$\begin{cases} -\varepsilon u'' + \alpha(x)u' + u = f(x), & 0 < x < 1, \\ u(0) = 0, & u'(1) = 0, \end{cases}$$
where ε is a positive constant and α is a function satisfying $\alpha(x) \geq 0$, $\alpha'(x) \leq 0$. Show, using the L_2-norm $||v|| = \left(\int_0^1 v^2\,dx\right)^{1/2}$, that
$$\sqrt{\varepsilon}||u'|| \leq C_1||f||, \quad ||\alpha u'|| \leq C_2||f||, \quad \varepsilon||u''|| \leq C_3||f||.$$

5. Show that for a continuously differentiable function v defined on $(0,1)$ we have that
$$||v||^2 \leq v(0)^2 + v(1)^2 + ||v'||^2.$$
Hint: Use partial integration for $\int_0^{1/2} v(x)^2\,dx$ and $\int_{1/2}^1 v(x)^2\,dx$ and note that $(x - 1/2)$ has the derivative 1.