# Chapter 14. Piecewise polynomials in several dimensions

## Variational formulation in $\mathbb{R}^2$

All the previous studies (1 - dimensional) can be extended to $\mathbb{R}^n$, then the "mathematics of computation" becomes much more cumbersome. On the other hand, two or three dimensional cases are of both physical relevance and practical interest.

A typical problem to study is e.g.

$$\begin{cases} -\Delta u + au = f & \mathbf{x} := (x, y) \in \Omega \subset R^2 \\ u(x, y) = 0 & (x, y) \in \partial\Omega \end{cases}$$

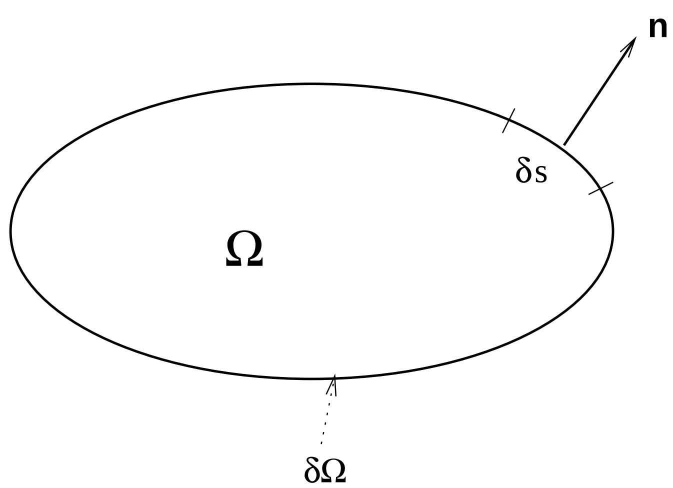
The only difference with the 1-dimensional case is in the performance of the partial integrations.

<u>Green's formula:</u> Let $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$, then

$$\iint_\Omega \Big(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\Big) v\,dxdy = \int_{\partial\Omega} \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big) \cdot \mathbf{n}(x, y) v\,ds - \iint_\Omega \Big(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\Big) \cdot \Big(\frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\Big) dxdy,$$

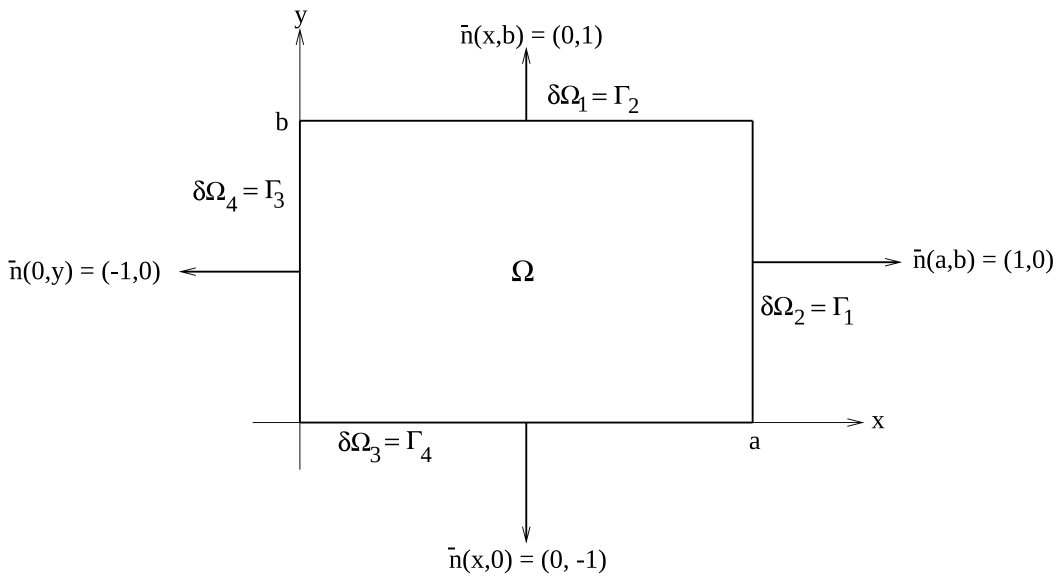
where $\mathbf{n}(x, y)$ is the outward unit normal at the boundary point $\mathbf{x} = (x, y) \in \partial\Omega$ and $ds$ is a curve element on the boundary $\partial\Omega$.

In the figures below $\delta\Omega \equiv \partial\Omega$.




In concise form $\quad \int_\Omega (\Delta u) v dx = \int_\Omega (\nabla u \cdot \mathbf{n}) v ds - \int_\Omega \nabla u \cdot \nabla v dx$

**Proof:** (In the case that $\Omega$ is a rectangular domain)




then we have

$$\iint_\Omega \frac{\partial^2 u}{\partial x^2} v\,dxdy = \int_0^b \int_0^a \frac{\partial^2 u}{\partial x^2}(x,y)\cdot v(x,y)\,dxdy = [P.I.] =$$
$$= \int_0^b \left( \left[ \frac{\partial u}{\partial x}(x,y)\cdot v(x,y) \right]_{x=0}^{a} - \int_0^a \frac{\partial u}{\partial x}(x,y)\cdot \frac{\partial v}{\partial x}(x,y)\,dx \right) dy =$$
$$= \int_0^b \left( \frac{\partial u}{\partial x}(a,y)\cdot v(a,y) - \frac{\partial u}{\partial x}(0,y)\cdot v(0,y) \right) dy -$$
$$- \iint_\Omega \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}(x,y)\,dxdy.$$

Now we have on $\Gamma_1 : \mathbf{n}(a,y) = (1,0)$

on $\Gamma_2 : \mathbf{n}(x,b) = (0,1)$

on $\Gamma_3 : \mathbf{n}(0,y) = (-1,0)$

on $\Gamma_4 : \mathbf{n}(x,0) = (0,-1)$

This implies that the first integral on the right hand side can be written as

$$\int_{\partial\Omega} \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)\cdot \mathbf{n}(x,y) v\,ds = \left( \int_{\Gamma_1} + \int_{\Gamma_2} \right) \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)\cdot \mathbf{n}(x,y) v(x,y)\,ds$$

i.e.

$$(1) \qquad \iint_\Omega \frac{\partial^2 u}{\partial x^2} dxdy = \int_{\Gamma_1 \cup \Gamma_3} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \mathbf{n}(x,y) v(x,y) ds - \iint_\Omega \frac{\partial u}{\partial x} \cdot \frac{\partial v}{\partial x} dxdy$$

Similarly in $y$-direction we get

$$(2) \qquad \iint_\Omega \frac{\partial^2 u}{\partial y^2} v dxdy = \int_{\Gamma_2 \cup \Gamma_4} \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) \cdot \mathbf{n}(x,y) v(x,y) ds - \iint_\Omega \frac{\partial u}{\partial y} \cdot \frac{\partial v}{\partial y} dxdy$$

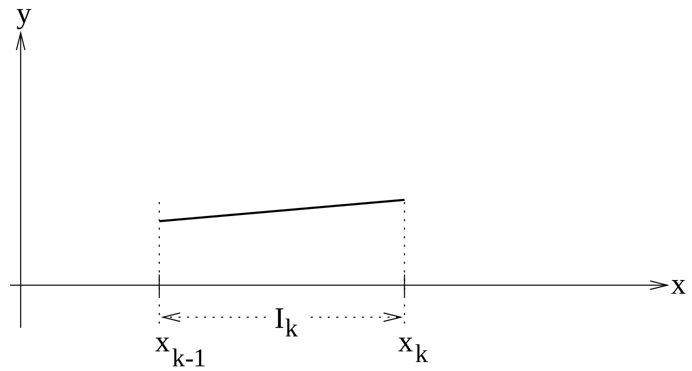
Now (1) + (2) gives the desired result.

For the general domain $\Omega$, see (CDE).

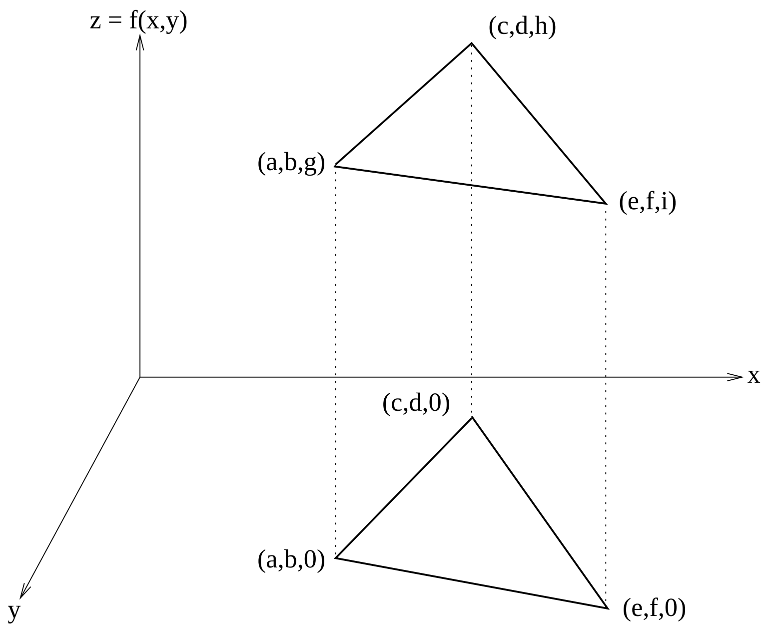
**Basis functions for the piecewise linear case**

We just consider an example:

In the 1-dimensional case a function which is linear on a subinterval is uniquely determined by its values at the endpoints. (There is only one straight line connecting two points)

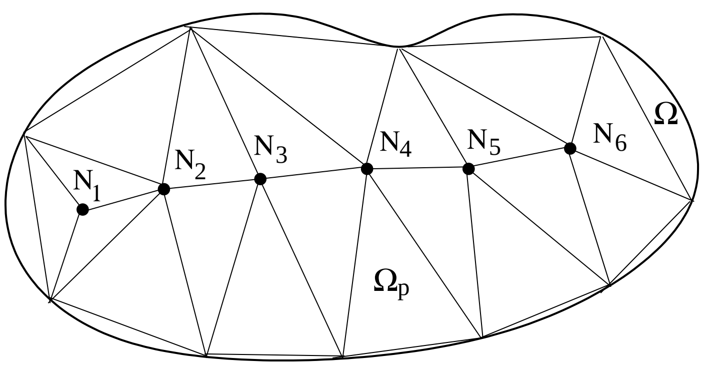



Similarly a plane in $\mathbb{R}^3$ is uniquely determined by three points. Therefore it is natural to make partitions of 2-dimensional domains using triangular elements and letting the sides of the triangles to correspond to the endpoints of the intervals in the 1-dimensional case.




This figure illustrates how piecewise linear function on a triangle is determined by its values at the vertices of the triangle.

Obseve that this "partitioning": *triangulation* works only for the domains with polygonal boundary.




Here we have 6 internal nodes $N_i,\ 1 \leq i \leq 6$. $\Omega_p$ is the *polygonal* domain inside $\Omega$, which is triangulated.

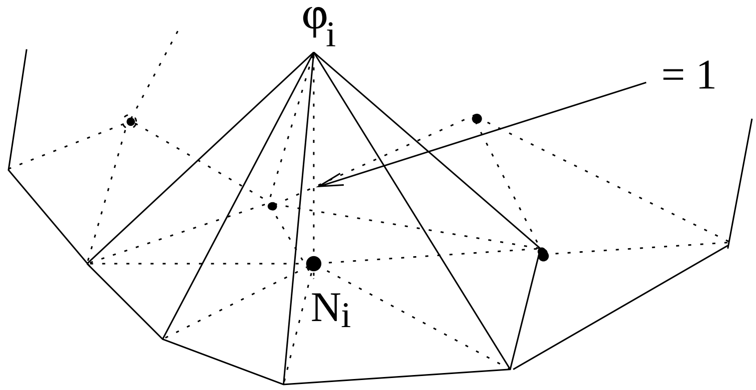
Now for every linear function $U$ on $\Omega_p$ we have

$$U(\mathbf{x}) = U_1\varphi_1(\mathbf{x}) + U_1\varphi_2(\mathbf{x}) + \ldots + U_6\varphi_6(\mathbf{x}),$$

where $U_i = U(N_i) \quad i = 1, 2, \ldots, 6$ are numbers and $\varphi_i(N_i) = 1$, while $\varphi_i(N_j) = 0$ for $j \neq i$. Further $\varphi_i(\mathbf{x})$ is linear in $\mathbf{x}$ in every triangle/element. In other words

$$\varphi_i(N_j) = \left\{ \begin{array}{ll} 1 & j = i \\ & \\ 0 & j \neq i \end{array} \right\} = \delta_{ij} \quad \text{(affin )}$$

and $\varphi_i(\mathbf{x}) = 0$ on $\partial\Omega_p$.




Therefore, e.g., given a differential equation to determine the approximate solution $U$ is now reduced to find the values (numbers) $U_1, U_2, \ldots, U_6$, satisfying the corresponding variational formulation. For instance if we chosse $\mathbf{x} = N_5$, then $U(N_5) = U_1\varphi_1(N_5) + U_2\varphi_2(N_5) + \ldots + U_5\varphi_5(N_5) + U_6\varphi_6(N_5)$, where $\varphi_1(N_5) = \varpi_1(N_5) = \varphi_2(N_5) = \varphi_3(N_5) = \varphi_4(N_5) = \varphi_6(N_5) = 0$ and $\varphi_5(N_5) = 1$, and hence

$$U(N_5) = U_5\varphi_5(N_5) = U_5$$

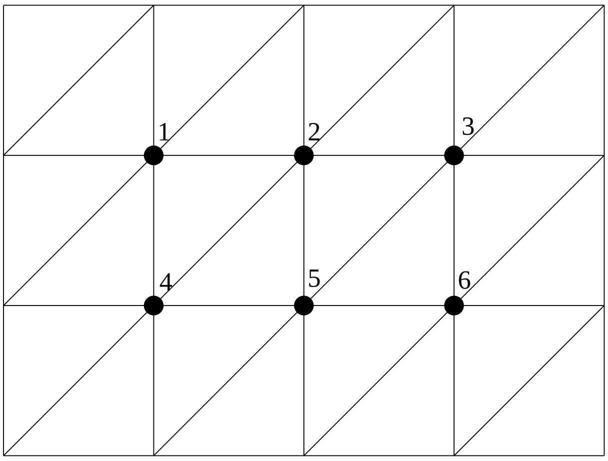
Example, let $\Omega = \{(x, y) : 0 < x < 4, 0 < y < 3\}$ and make a FEM discretization of the following boundary value problem:

$$\left\{ \begin{array}{ll} -\Delta u = f & \text{in } \Omega \\ & \\ u = 0 & \text{on } \partial\Omega \end{array} \right.$$

The variational formulation will be: Find a function $u$ vanishing at the boundary $\Gamma = \partial\Omega$ of $\Omega$ (i.e. $u = 0$, on $\Gamma$), such that

$$\iint_\Omega (\nabla u \cdot \nabla v) dxdy = \iint_\Omega f v dxdy \text{ for all test functions } v \text{ with } v = 0 \text{ on } \partial\Omega.$$

We shall make a test function space of piecewise linear functions.




We triangulate $\Omega$ as in the figure above and let

$$V_h^0 = \text{“Space of all continuous functions, which are linear on each sub-triangle and are 0 on the boundary”}.$$

Since such a function is uniquely determined by its values at the vertices of the triangles and 0 on the boundary, so indeed in our example we have only 6 inner vertices of interest. Now precisely as in the “$1 - D$” case we construct basis functions. (6 of them in this particular case), with values 1 at one of the nodes and zero at the others. Then we get the two-dimensional *telt functions* as above.