Chapter 15. The Poisson Equation

Solve the Poisson equation

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in R^2, with polygonal boundary $\Gamma = \partial\Omega$.

To derive stability estimates we multiply the equation by u and integrate over Ω to obtain

$$-\int_\Omega (\Delta u) u dx = \int_\Omega f \; u dx, \quad x \in \Omega \text{ and } u \in V.$$

Using Green's formula and the boundary condition: $u = 0$ on Γ, we get that

$$\|\nabla u\|^2 = \int_\Omega fu \leq \|f\| \, \|u\|, \tag{1}$$

where $\|\cdot\|$ denotes the usual $L_{2(\Omega)}$-norm.

Poincaré inequality (2D-version):

$$\|u\| \leq C_\Omega \|\nabla u\| \tag{2}$$

Proof. Let φ be a function such that $\Delta\varphi = 1$ in Ω, and $2|\nabla\varphi| \leq C_\Omega$ in Ω, (such a function exists), then again by the use of Green's formula and the boundary condition we get

$$\|u\|^2 = \int_\Omega u^2 \Delta\varphi = -\int_\Omega 2u(\nabla u \cdot \nabla\varphi) \leq C_\Omega \|u\| \, \|\nabla u\|.$$

Thus

$$\|u\| \leq C_\Omega \|\nabla u\|.$$

Now combining with formula (1) we get that the following *weak stability estimate*:

$$\|\nabla u\| \leq C_\Omega \|f\|. \qquad \square \tag{3}$$

Exercise: Derive corresponding estimates for following problem:

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma = \partial\Omega \end{cases}$$

Error estimates for FEM for the Poisson equation:

$$\begin{cases} -\Delta u = f, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma = \partial\Omega \end{cases}$$

where $\Omega \subset R^d, d = 1, 2, 3$, with following variational formulation:

Find $U(x)$ such that $u(x) = 0$ on $\Gamma = \partial\Omega$ and

$$(V): \qquad \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega f v \, dx \quad \text{for all } v \text{ such that } v = 0 \text{ on } \Gamma.$$

FEM: Let $\mathcal{T} = \{K : \cup K = \Omega\}$ be a triangulation of Ω and $\varphi_j, j = 1, 2, \ldots, n$ be the corresponding basis functions, such that $\varphi_j(x)$ is continuous, linear in x on each K and

$$\varphi_j(N_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j \end{cases}$$

where $N_1, N_2, \ldots, N_n$ are the inner nodes in the triangulation.

Now we set the approximate solution:

$$U(x) = U_1\varphi_1(x) + U_2\varphi_2(x) + \ldots + U_n\varphi_n(x),$$

and seek the coefficients $U_i = U(N_i)$, i.e., the nodal values of $U(x)$, at the nodes $N_i, \ 1 \leq i \leq n$, so that

$$\text{(FEM)} \qquad \int_\Omega \nabla U \cdot \nabla \varphi_j \, dx = \int_\Omega f \cdot \varphi_j \, dx, \quad j = 1, 2, \ldots n$$

or equivalently

$$(V_h^0) \qquad \int_\Omega \nabla U \cdot \nabla v \, dx = \int_\Omega f \cdot v \, dx, \quad \forall v \in V_h^0.$$

Recall that

$V_h^0 = \{v(x) : v \text{ is continuous, piecewise linear in } x \text{ (on } \mathcal{T}), \text{ and } v = 0 \text{ on } \Gamma = \partial\Omega\}$.

Note that for $v \in V_h^0$ we have

$$v(x) = v(N_1)\varphi_1(x) + v(N_2)\varphi_2(x) + \ldots + v(N_n)\varphi_n(x).$$

For the error $e = u - U$ we have $\nabla e = \nabla u - \nabla U = \nabla(u - U)$. We observe that subtracting the formula (V_h^0) from the (V); we obtain the *Galerkin Orthogonality:*

$$(4) \qquad \int_\Omega (\nabla u - \nabla U)\nabla v \, dx = \int_\Omega \nabla e \cdot \nabla v \, dx = 0, \qquad \forall v \in V_h^0.$$

On the other hand we may write

$$\|\nabla e\|^2 = \int_\Omega \nabla e \cdot \nabla e \, dx = \int_\Omega \nabla e \cdot \nabla (u - U) \, dx = \int_\Omega \nabla e \cdot \nabla u \, dx - \int_\Omega \nabla e \cdot \nabla U \, dx,$$

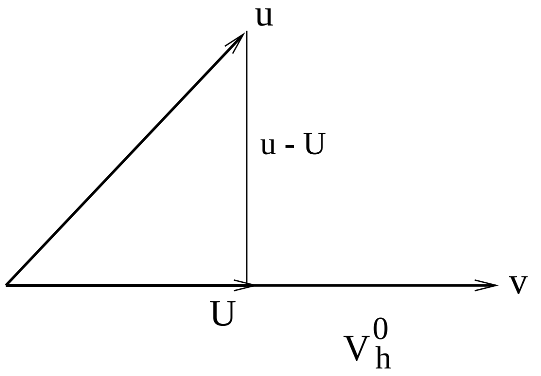
where using the Galerkin orthogonality (4), since $U(x) \in V_h^0$ we have the last integral above: $\int_\Omega \nabla e \cdot \nabla U \, dx = 0$. Thus inserting $\int_\Omega \nabla e \cdot \nabla v \, dx = 0$, $\forall v \in V_h^0$ we have that

$$\|\nabla e\|^2 = \int_\Omega \nabla e \cdot \nabla u dx - \int_\Omega \nabla e \cdot \nabla v dx = \int_\Omega \nabla e \cdot \nabla (u - v) dx \leq \|\nabla e\| \cdot \|\nabla (u - v)\|.$$

Hence

$$\|\nabla (u - U)\| \leq \|\nabla (u - v)\|, \quad \forall v \in V_h^0, \tag{5}$$

that is, U is closer to u than any other v in V_h^0.



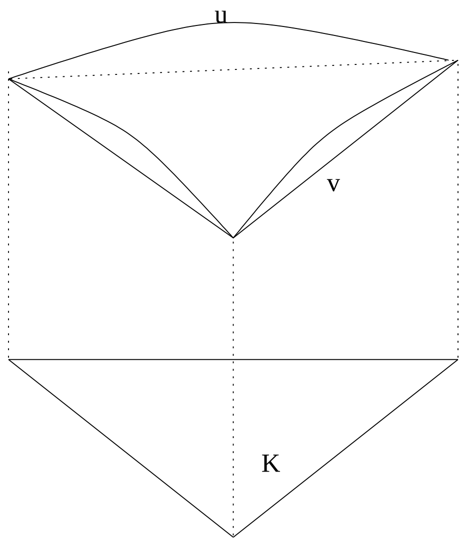
In other words the error $u - U$ is orthogonal to V_h^0.

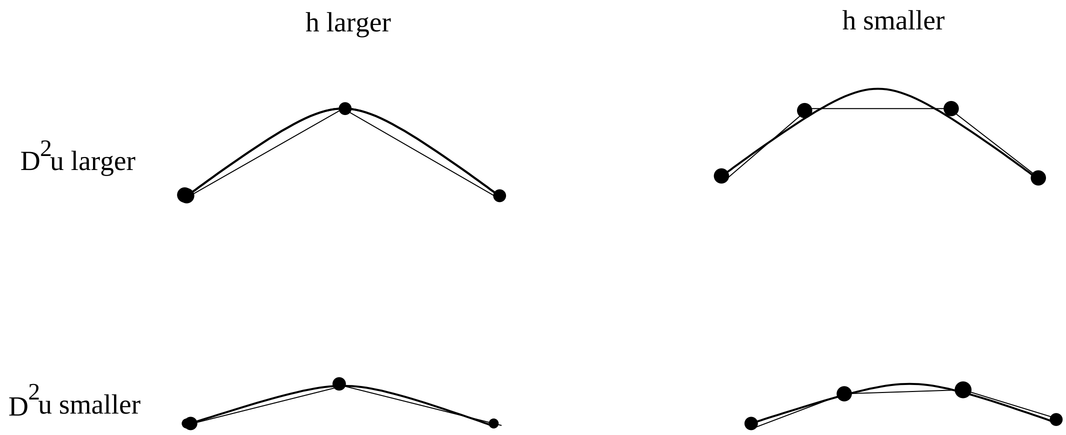
It is possible to show that there is a $v \in V_h^0$ (an interpolant), such that

$$\|\nabla (u - v)\| \leq C \|h \, D^2 u\|, \tag{6}$$

where $h = h(x) = \text{diam}(K)$ for $x \in K$ and C is a constant, independent of h.

This is the case, for example, if v interpolates u at the nodes N_i





Combining (5) and (6) we get

$$\|\nabla e\| = \|\nabla(u - U) \leq C\|h\, D^2 u\|, \tag{7}$$

which is indicating that the error is small if $h(x)$ is sufficiently small depending on D^2u:

<u>Estimate of the error $e = u - U$</u>:

Let φ be the solution of the dual problem

$$\begin{cases} -\Delta\varphi = e, & \text{in } \Omega \\ \varphi = 0, & \text{on } \partial\Omega \end{cases}$$

Then

$$
(8) \quad \begin{aligned}
\|e\|^2 &= \int_\Omega e(-\Delta\varphi)dx = \{\text{Green's formula}\} = \int_\Omega \nabla e\cdot\nabla\varphi\, dx,\\
&= \{\text{Galerkin orthogonality}\} = \int_\Omega \nabla e\cdot\nabla(\varphi - v)\, dx\\
&\le \|\nabla e\|\cdot\|\nabla(\varphi - v)\|, \quad \forall v\in V_h^0.
\end{aligned}
$$

We now choose v such that

$$
(9) \qquad \|\nabla(\varphi - v)\| \le C\|h\cdot D^2\varphi\| \le C\max_\Omega h\cdot\|D^2\varphi\|.
$$

<u>Lemma:</u> *Assume that Ω has no re-intrents. We have for $u\in H^2(\Omega)$; with $u=0$ or $(\frac{\partial u}{\partial n}=0)$ on $\partial\Omega$. that*

$$
\|D^2u\| \le c_\Omega\cdot\|\Delta u\|,
$$

where

$$
D^2u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}.
$$

We postpone the proof of this lemma and first derive the error estimate:

Applying the lemma to φ, we get

$$
(10) \qquad \|D^2\varphi\| \le C_\Omega\cdot\|\Delta\varphi\| = C_\Omega\|e\|
$$

Now (7)-(10) implies that

$$
\begin{aligned}
\|e\|^2 &\le (8) \le \|\nabla e\|\cdot\|\nabla(\varphi - v)\| \le (9) \le \|\nabla e\|\cdot C\max_\Omega h\,\|D^2\varphi\|\\
&\le (10) \le \|\nabla e\|\cdot C\max_\Omega hC_\Omega\|e\| \le (7) \le C^2\,C_\Omega\,\max_\Omega h\|e\|\|h\,D^2u\|.
\end{aligned}
$$

Thus we have obtained the following *a priori error estimate:*

$$
\|e\| = \|u - U\| \le C^2\,C_\Omega\,(\max_\Omega h)\cdot\|h\,D^2u\|,
$$

which using the Lemma, for a uniform (constant h), can be written as an stability estimate viz,

$$
\|u - U\| \le C^2\,C_\Omega^2\,(\max_\Omega h)^2\,\|f\|.
$$

A posteriori error estimate. For simplicity we consider a one dimensional case with $\Omega = (0,1)$ and study the problem:

$$
(11) \qquad \begin{cases} -\varphi''(x) = e(x), & 0 < x < 1, \\ \varphi(0) = \varphi(1) = 0, & e(x) = u(x) - U(x). \end{cases}
$$

Using (11) the $L2$-norm of the error can be written as:

$$
\|e\|^2 = \int_\Omega e \cdot e \, dx = \int_\Omega e(-\varphi'')dx = \int_\Omega e' \cdot \varphi' \, dx.
$$

Thus, using the one-dimensional version of the Galerkin orthogonality: $\int_\Omega e' \cdot v' dx = 0$, and the boundary data: $\varphi(0) = \varphi(1) = 0$ (in a partial integration) we can write

$$
\begin{aligned}
\|e\|^2 &= \int_\Omega e' \cdot \varphi' dx - \int_\Omega e' \cdot v' dx = \int_\Omega e' \cdot (\varphi - v)' dx = \int_\Omega (-e'')(\varphi - v) \, dx = \\
&\le \|h^2 r\| \cdot \|h^{-2}(\varphi - v)\| \le C \cdot \|h^2 r\| \cdot \|\varphi''\| \le C \cdot \|h^2 r\| \cdot \|e\|,
\end{aligned}
$$

where we use the fact that the $-e'' = -u'' + U'' = f + U''$ is the residual r and v is an interpolant of φ. Thus, for this problem, the final *a posteriori error estimate* is:

$$
(12) \qquad \|u - U\| \le C \, \|h^2 r\|.
$$

Observe that for piecewise linear approximations $U'' = 0$ on each element K and hence $r \equiv f$ and our a posteriori error estimate above can be viewed as a stability estimate viz,

$$
\|e\| \le C \, \|h^2 f\|.
$$

Exercise 1: Show that $\|(u - U)'\| \le C\|hr\|$

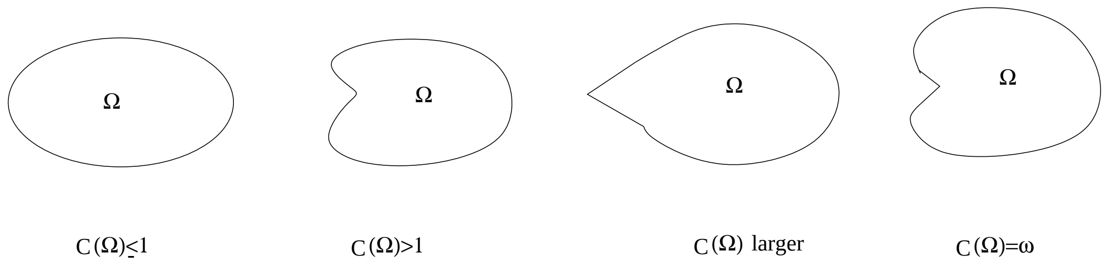
Exercise 2: Verify that for v being the interpolant of φ, we have

$$
\begin{aligned}
\|h^{-2}(\varphi - v)\| &\le C \, \|\varphi''\|, \quad \text{and} \\
\|h^{-1}(\varphi - v)\| &\le C \, \|\varphi'\|
\end{aligned}
$$

Exercise 3: Derive the corresponding estimate to (12) in the 2-dimensional case ($d = 2$).
Note that now is $\nabla e(\varphi - v) \neq 0$ on the enter-element boundaries.

Now we return to the proof of Lemma 1:

First note that for convex Ω, the constant $C_\Omega \le 1$ in lemma 1, otherwise the constant $C_\Omega > 1$ and increases from left to right for the Ω:s below.



Proof. Let Ω be a rectangular domain and set $u = 0$ on $\partial\Omega$. We have then

$$\|\Delta u\|^2 = \int_\Omega (u_{xx} + u_{yy})^2 dxdy = \int_\Omega (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2)\, dxdy.$$

Further applying Green's formula: $\int_\Omega (\Delta u) v\, dx = \int_\Gamma (\nabla u \cdot n) v\, ds - \int_\Omega \nabla u \cdot \nabla v\, dx$ to our rectangular domain Ω we have

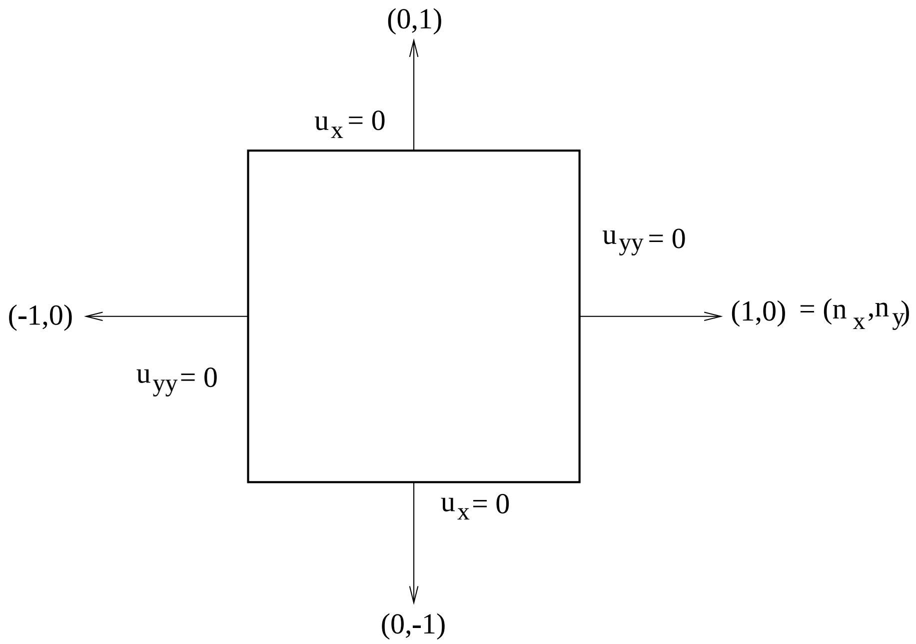
$$(13) \qquad \int_\Omega u_{xx}u_{yy} dxdy = \int_{\partial\Omega} u_x (u_{yy} \cdot n_x) ds - \int_\Omega u_x \underbrace{u_{yyx}}_{=u_{xyy}} dxdy$$

using Green's formula once again (with "$v = u_x$", "$\Delta u = u_{xyy}$") we have

$$\int_\Omega u_x u_{xyy} dxdy = \int_{\partial\Omega} u_x (u_{yx} \cdot n_y) ds - \int_\Omega u_{xy}u_{xy}\, dxdy,$$

which inserting in (13) gives that

$$\int_\Omega u_{xx}u_{yy}\, dxdy = \int_{\partial\Omega} (u_x u_{yy} n_x - u_x u_{yx} n_y) ds + \int_\Omega u_{xy}u_{xy}\, dxdy.$$



Now, as we can see from the figure that $(u_x u_{yy} n_x - u_x u_{yx} n_y) = 0$, on $\partial\Omega$ and hence we have

$$\int_\Omega u_{xx} u_{yy} dxdy = \int_\Omega u_{xy} u_{xy} dxdy = \int_\Omega u_{xy}^2 \, dxdy.$$

Thus, in this case,

$$\|\Delta u\|^2 = \int_\Omega (u_{xx} + u_{yy})^2 dxdy = \int_\Omega (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) dxdy = \|D^2 u\|^2,$$

and the proof is complete. □