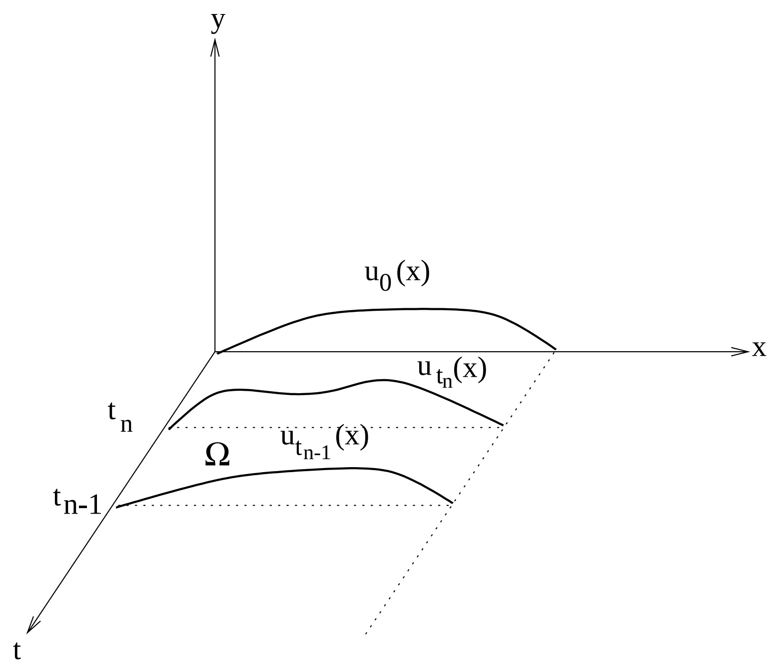
Chapter 16. The heat equation

Consider the initial boundary value problem:

$$
(1) \qquad \begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \subset R^2 \text{ (or } R^d, d = 1, 2, 3) \quad (DE) \\ u = 0, & \text{on } \Gamma := \partial\Omega, \quad (BC) \\ u(0, x) = u_0, & \text{for } x \in \Omega, \quad (IC) \end{cases}
$$

where $\dot{u} = \frac{\partial u}{\partial t}$. Here is an illustration in $1 - D$ case:



Energy estimates:

To derive stability and energy estimates we multiply (1) by u and integrate over Ω viz

$$
(2) \qquad \int_\Omega \dot{u} u \, dx - \int_\Omega (\Delta u) u \, dx = 0.
$$

Note that $\dot{u}u = \frac{1}{2}\frac{d}{dt}u^2$ and using Green's formula:

$$
-\int_\Omega (\Delta u) u \, dx = -\int_\Gamma (\nabla u \cdot n) \, u \, ds + \int_\Omega \nabla u \cdot \nabla u dx \ \text{ (and with } u = 0 \text{ on } \Gamma),
$$

$$
-\int_\Omega (\Delta u) u \, dx = \int_\Omega \nabla u \cdot \nabla u \, dx = \int_\Omega |\nabla u|^2 \, dx.
$$

Thus equation (2) can be written in the following, equivalent, form:

(3) $$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = 0 \iff \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = 0,$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$ norm.

Integrate equation (3) over $s \in (0,t)$ we get

$$\frac{1}{2}\int_0^t \frac{d}{ds}\|u\|^2(s)ds + \int_0^t \|\nabla u\|^2(s)ds = \frac{1}{2}\|u\|^2(t) - \frac{1}{2}\|u\|^2(0) + \int_0^t \|\nabla u\|^2 ds = 0,$$

thus with $u(0) = u_0$ we have

$$\|u\|^2(t) + 2\int_0^t \|\nabla u\|^2(s)\, ds = \|u_0\|^2.$$

In particular, we have the stability estimates

(3a) $$\|u\|(t) \le \|u_0\|,$$

and

(3b) $$\int_0^t \|\nabla u\|^2(s)\, ds \le \frac{1}{2}\|u_0\|.$$

Exercise 1: Show that $\|\nabla u(t)\| \le \|\nabla u_0\|$ (the stability estimate for the gradient).
(Hint: Multiply (1) by $-\Delta u$ and integrate over Ω).
Is this inequality valid for $u_0 =$ constant?

Exercise 2: Derive the corresponding estimate with (BC): $\frac{\partial u}{\partial n} = 0$.

Now we multiply (1): $\dot u - \Delta u = 0$, by $-t \cdot \Delta u$ and integrate over Ω to obtain

(4) $$-t\int_\Omega \dot u \cdot \Delta u\, dx + t\int_\Omega (\Delta u)^2\, dx = 0.$$

Using Green's formula ($u = 0$ on Γ) we have

$$\int_\Omega \dot u \Delta u\, dx = -\int_\Omega \nabla \dot u \cdot \nabla u\, dx = -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2,$$

so that (4) can be written as

$$t\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + t\|\Delta u\|^2 = 0,$$

and by using the obvious relation $t\frac{d}{dt}\|\nabla u\|^2 = \frac{d}{dt}(t\|\nabla u\|^2) - \|\nabla u\|^2$ we get

$$\frac{d}{dt}(t\|\nabla u\|^2) + 2t\|\Delta u\|^2 = \|\nabla u\|^2,$$

Integration in t gives:

$$\int_0^t \frac{d}{ds}(s\|\nabla u\|^2(s))\,ds + 2\int_0^t s\|\Delta u\|^2(s)ds = \int_0^t \|\nabla u\|^2(s)ds \leq \frac{1}{2}\|u_0\|^2,$$

where in the last inequality we use (3b), consequently

$$t\|\nabla u\|^2(t) + 2\int_0^t s\|\Delta u\|^2(s)\,ds \leq \frac{1}{2}\|u_0\|^2. \tag{5}$$

In particular, we have:

$$\|\nabla u\|(t) \leq \frac{1}{\sqrt{2t}}\|u_0\| \tag{5a}$$

$$\Big(\int_0^t s\|\Delta u\|^2(s)\,ds\Big)^{1/2} \leq \frac{1}{2}\|u_0\| \tag{5b}$$

Analogously we can show that

$$\|\Delta u\|(t) \leq \frac{1}{\sqrt{2}\;t}\|u_0\| \tag{6}$$

<u>Exercise 3</u>: Prove (6).

Hint: Multiply (1) by $t^2\,(\Delta^2 u)$ and note that $\Delta u = \dot{u} = 0$ on Γ, or alternatively: differentiate $\dot{u} - \Delta u = 0$ with respect to t and multiply the resulting equation by $t^2\,\dot{u}$.

Now using (1): ($\dot{u} = \Delta u$) and (6) we obtain

$$\int_\varepsilon^t \|\dot{u}\|(s)ds \leq \frac{1}{\sqrt{2}}\|u_0\|\int_\varepsilon^t \frac{1}{s}\,ds = \frac{1}{\sqrt{2}}\ln\frac{t}{\varepsilon}\|u_0\|$$

or more carefully

(7)

$$\begin{aligned}\int_\varepsilon^t \|\dot{u}\|(s)ds &= \int_\varepsilon^t \|\Delta u\|(s)ds = \int_\varepsilon^t 1\cdot\|\Delta u\|(s)ds = \int_\varepsilon^t \frac{1}{\sqrt{s}}\cdot\sqrt{s}\|\Delta u\|(s)ds\\ &\leq \{\text{Cauchy Schwartz}\} \leq \Big(\int_\varepsilon^t s^{-1}\,ds\Big)^{1/2}\cdot\Big(\int_\varepsilon^t s\|\Delta u\|^2(s)\,ds\Big)^{1/2}\\ &\leq \{(5b)\} \leq \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\|u_0\|.\end{aligned}$$

Summing up: For the initial boundary value problem

$$\begin{cases} \dot{u} - \Delta u = 0, & \text{in } \Omega \subset R^2 \text{ (or } R^d, d = 1, 2, 3) \\ u = 0, & \text{on } \Gamma := \partial\Omega \\ u(0, x) = u_0, & \text{for } x \in \Omega \end{cases}$$

we have the stability estimates:

$$\begin{array}{ll} \text{(3a)} & \|u\|(t) \leq \|u_0\| \\ \text{(3b)} & \int_0^t \|\nabla u\|^2(s)\, ds \leq \frac{1}{2}\|u_0\|^2 \\ \text{(5a)} & \|\nabla u\|(t) \leq \frac{1}{\sqrt{2t}}\|u_0\| \\ \text{(5b)} & (\int_0^t s\|\Delta u\|^2(s)\, ds)^{1/2} \leq \frac{1}{2}\|u_0\| \\ \text{(6)} & \|\Delta u\|(t) \leq \frac{1}{\sqrt{2}\cdot t}\|u_0\| \\ \text{(7)} & \int_\varepsilon^t \|\dot{u}\|(s)\, ds \leq \frac{1}{2}\sqrt{\ln \frac{t}{\varepsilon}}\|u_0\|. \end{array}$$

Error analysis

Consider the one-dimensional heat equation with Dirichlet boundary condition:

$$\begin{cases} \dot{u} - u'' = f, & \text{in } \Omega = (0, 1) \quad t > 0 \\ u = 0, & \text{on } \partial\Omega \quad t > 0, \quad \text{i.e.} \quad u(0, t) = u(1, t) = 0 \\ u = u_0, & \text{in } \Omega \quad t = 0, \quad \text{i.e.} \quad u(x, 0) = u_0(x). \end{cases}$$

For an illustration se Fig. on page 1.

Variational formulation: For every time interval I_n find $u(x, t),\ t \in I_n$, such that

$$\text{(VF)} \qquad \int_{I_n}\int_0^1 (\dot{u}v + u'v')dxdt = \int_{I_n}\int_0^1 fvdxdt, \quad \forall v: \ v(0, t) = v(1, t) = 0.$$

A piecewise linear Galerkin approximation: For each time interval $I_n = (t_{n-1}, t_n]$, with $t_n - t_{n-1} = k$, let

$$U(x,t) = U_{n-1}(x)\Psi_{n-1}(t) + U_n(x)\Psi_n(t),$$

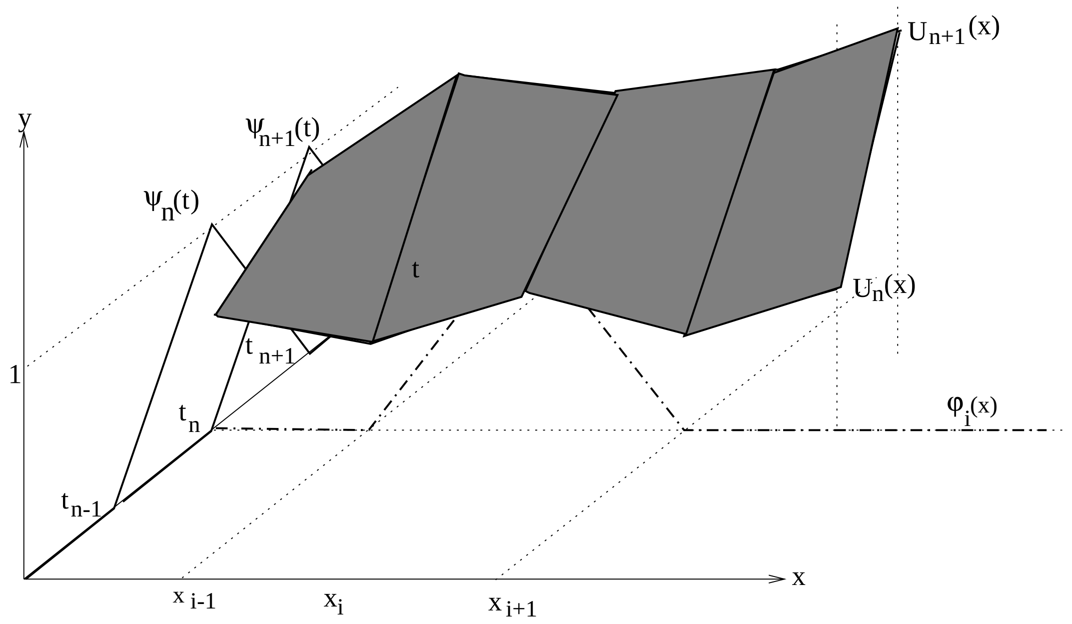
where

$$\Psi_n(t) = \frac{t - t_{n-1}}{k}, \qquad \Psi_{n-1}(t) = \frac{t_n - t}{k},$$

and

$$U_n(x) = U_{n,1}\varphi_1(x) + U_{n,2}\varphi_2(x) + \ldots + U_{n,m}\varphi_m(x),$$

with $\varphi(x_j) = \delta_{ij}$ being the usual finite element basis corresponding to a partition of $\Omega = (0,1)$, with $0 = x_1 < \cdots < x_k < x_{k+1} < \cdots < x_m = 1$.



i.e., U is piecewise linear in both space and time variables. Now the unknowns are the coefficients $U_{n,k}$ satisfying the discrete variational formulation:

$$\text{(8)} \qquad \int_{I_n}\int_0^1 (\dot{U}\varphi_j + U'\varphi_j')\,dxdt = \int_{I_n}\int_0^1 f\varphi_j\,dxdt, \qquad j = 1, 2, \ldots, m$$

Note $I_n = (t_{n-1}, t_n]$ and on I_n we have

$$\dot{U}(x,t) = U_{n-1}(x)\dot{\Psi}_{n-1}(t) + U_n(x)\dot{\Psi}_n(t) = U_{n-1}(x)\Big(-\frac{1}{k}\Big) + U_n(x)\Big(\frac{1}{k}\Big) = \frac{U_n - U_{n-1}}{k}.$$

and

$$U'(x,t) = U'_{n-1}(x)\Psi_{n-1}(t) + U'_n(x)\Psi_n(t).$$

Inserting in (8) we get using $\int_{I_n} dt = k$ and $\int_{I_n} \Psi_n dt = \int_{I_n} \Psi_{n-1} dt = \frac{k}{2}$ that

$$\underbrace{\int_0^1 U_n\varphi_j dx}_{M\cdot U_n} - \underbrace{\int_0^1 U_{n-1}\varphi_j\, dx}_{M\cdot U_{n-1}} + \underbrace{\int_{I_n} \Psi_{n-1}\, dt}_{\frac{k}{2}} \underbrace{\int_0^1 U'_{n-1}\varphi'_j\, dx}_{S\cdot U_{n-1}}$$

$$+ \underbrace{\int_{I_n} \Psi_n\, dt}_{\frac{k}{2}} \underbrace{\int_0^1 U'_n\varphi'_j\, dx}_{S\cdot U_n} = \underbrace{\int_{I_n}\int_0^1 f\varphi_j\, dxdt}_{F}$$

which can be written in a compact form as the Crank- Nicolson system (CNS)

$$\text{(CNS)} \qquad \Big(M + \frac{k}{2}S\Big)U_n = \Big(M - \frac{k}{2}S\Big)U_{n-1} + F,$$

with the solution U_n given by

$$U_n = \underbrace{\Big(M + \frac{k}{2}S\Big)^{-1}}_{B^{-1}} \underbrace{\Big(M - \frac{k}{2}S\Big)}_{A} U_{n-1} + \underbrace{\Big(M + \frac{k}{2}S\Big)^{-1}}_{B^{-1}} F,$$

where

$$U_n = \begin{bmatrix} U_{n,1} \\ U_{n,2} \\ \dots \\ U_{n,m} \end{bmatrix}$$

Thus with a given source term f we can determine the source vector F and then, for each $n = 1, 2, \dots N$, given the vector U_{n-1} we seek the vector U_n (nodal values of n at time level t_n), using the CNS above.

<u>Exercise 4:</u> Derive a corresponding equation system, as above, for the dG(0).

For general space domain Ω (8) can be written as

$$\text{(9)} \qquad \int_{I_n}\int_\Omega (\dot{U}v + U'v')dxdt = \int_{I_n}\int_\Omega fvdxdt \quad \text{for all } v \in V_h,$$

where $V_h = \{v(x) : v \text{ is continuous, piecewise linear, and } v(0) = v(1) = 0\}$.

Note that this variational formulation is valid for the exact solution u and for all $v(x,t)$ such that $v(0,t) = v(1,t) = 0$:

$$(10) \qquad \int_{I_n}\int_\Omega (\dot{u}v + u'v')dxdt = \int_{I_n}\int_\Omega f v\, dxdt, \quad \forall v \in V_h,$$

This implies that for the error $e = u - U$, we have subtracting (9) from (10), the following Galerkin orthogonality relation:

$$(11) \qquad \int_{I_n}\int_\Omega (\dot{e}v + e'v')\, dxdt = 0, \quad \text{for all } v \in V_h.$$

To derive error estimates we let $\varphi(x,t)$ be the solution of the following dual problem:

$$\begin{cases} -\dot{\varphi} - \varphi'' = 0, & \text{in } \Omega \quad t < T \\ \varphi = 0, & \text{on } \partial\Omega \quad t < T \\ \varphi = e, & \text{in } \Omega \quad \text{for } t = T \end{cases},$$

where $e = e(t) = e(\cdot, T) = u(\cdot, T) - U(\cdot, T),\ T = t_N$.

Note that for $w(x,t) = \varphi(x, T-t)$, $(t > 0)$, we can write the backward dual problem as a forward problem:

$$\begin{cases} \dot{w} - w'' = 0, & \text{in } \Omega \quad t > 0 \\ w = 0, & \text{on } \partial\Omega \quad t > 0 \\ w = e, & \text{in } \Omega \quad \text{for } t = 0. \end{cases}$$

For this problem we have shown that, see (7)

$$(12) \qquad \int_\varepsilon^T \|\dot{w}\| \le \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\ \|e\|,$$

and consequently (let $s = T - t$ then $\varepsilon \xrightarrow{t} T \Leftrightarrow T - \varepsilon \xrightarrow{s} 0$, and $ds = -dt$) we have for φ:

$$(13) \qquad \int_0^{T-\varepsilon} \|\dot{\varphi}\| \le \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\ \|e\|.$$

Now since $-\varphi'' = \dot{\varphi}$ we get also

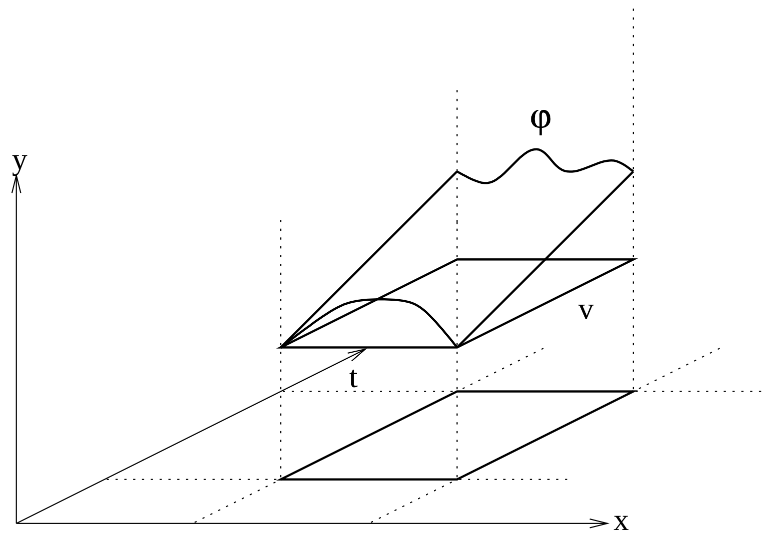
$$(14) \qquad \int_0^{T-\varepsilon} \|\varphi''\| \le \frac{1}{2}\sqrt{\ln\frac{T}{\varepsilon}}\|e\|$$

To continue we assume that $u_0 \in V_h$ then, since $(-\dot\varphi - \varphi'') = 0$, we can write

$$
\begin{aligned}
\|e(T)\|^2 &= \int_\Omega e(T)\cdot e(T)\,dx + \int_0^T \int_\Omega e(-\dot\varphi - \varphi'')\ dxdt = [\text{PI in } t] \\
&= \int_\Omega e(T)\cdot e(T)\,dx - \int_\Omega e(T)\cdot e(T)\,dx + \int_\Omega \underbrace{e(0)\cdot e(0)}_{=0}\,dx \\
&+ \int_0^T \int_\Omega (\dot e\varphi + e'\varphi')\,dxdt = \{\text{Galerkin Orthogonality (11)}\} \\
&= \int_0^T \int_\Omega \dot e(\varphi - v) + e'(\varphi - v)'\ dxdt = \{\text{PI in } x\text{, in 2ed term}\} \\
&= \int_0^T \int_\Omega (\dot e - e'')(\varphi - v)\ dxdt + \int_0^T e'\underbrace{(\varphi - v)\Big|_{\partial\Omega}}_{=0}\,dt \\
&= \int_0^T \int_\Omega (f - \dot U + U'')(\varphi - v)\ dxdt = \int_0^T \int_\Omega r(U)(\varphi - v)\ dxdt,
\end{aligned} \tag{15}
$$

where we use $\dot e = \dot u - \dot U$ and $e'' = u'' - U''$ to write $\dot e - e'' = \dot u - u'' - \dot U - U'' = f - \dot U - U' := r(U)$ which is the residual. Now with mesh variables $h = h(x,t)$ and $k = k(t)$ in x and t, respectively we can derive an interpolation estimate of the form:

$$(\varphi - v) \le k\dot\varphi + h^2\varphi'' \le (k + h^2)\dot\varphi + (k + h^2)\varphi'',$$



Summing up we have basically:

$$
\begin{aligned}
\|e(T)\|^2 &\leq \int_0^T \|(k+h^2)r(U)\|(\|\dot{\varphi}\| + \|\varphi''\|) \\
&\leq \max_{[0,T]} \|(k+h^2)r(U)\| \Big[\int_0^{T-\varepsilon} (\|\dot{\varphi}\| + \|\varphi''\|) + 2 \max_{[T-\varepsilon,T]} \|\varphi\| \Big] \\
&\leq \{\text{maximum principle }, \ (13)-(14)\} \\
&\leq \max_{[0,T]} \|(k+h^2)r(U)\| \Big(\sqrt{\ln \frac{T}{\varepsilon}} \ \|e\| + 2\|e\| \Big).
\end{aligned} \tag{16}
$$

This gives our final estimate:

$$
\|e(t)\| \leq \Big(2 + \sqrt{\ln \frac{T}{\varepsilon}}\Big) \max_{[0,T]} \|(k+h^2)r(U)\|. \tag{17}
$$

Adaptivity:

Starting from the a posteriori estimate of the error $e = u - U$ for example for

$$
\begin{cases}
-\Delta u = f, & \text{in } \Omega \\
u = 0, & \text{on } \partial\Omega
\end{cases}
$$

i.e.

$$
\|\nabla e\| \leq c\|hr(U)\|,
$$

where $r(U) = |f| + \max_{\partial K} |[\nabla u]|$, and [ ] denotes the jump, we have the following Algorithm:

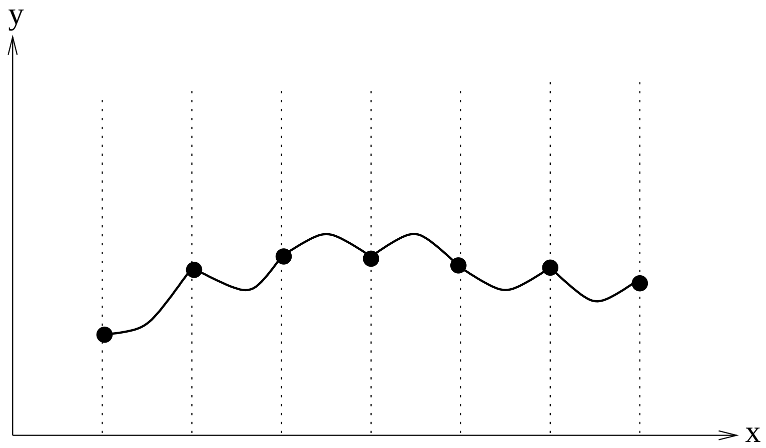
(1) Choose an arbitrary $h = h(x)$ and a tolerance Tol > 0.

(2) Given h, compute the corresponding U.

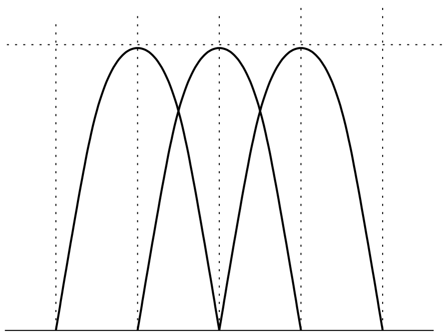
(3) If $C\|hr(U)\| \leq$ Tol, accept U. Otherwise choose a new (refined) $h = h(x)$ and return to step (2) above. $\square$

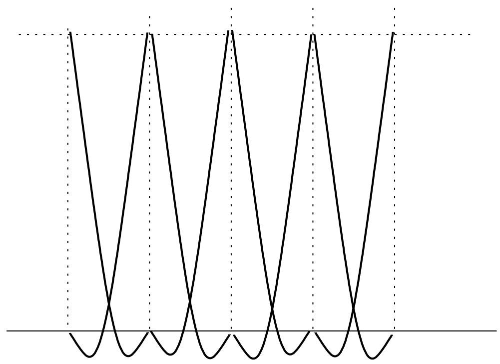
Higher order elements:

Ex. $cG(2)$: Piecewise polynomials of degree 2



is determined by the values of the approximate solution at the end-points of the subintervals. The constructing is through the bases functions of the form:





Error estimates. (A simple case): For $-u'' = f, \;\; 0 < x < 1$ associated with Dirichlet (or Neumann) boundary condition we have

(1) $\|(u-U)'\| \le C\|h^2 D^3 u\|$

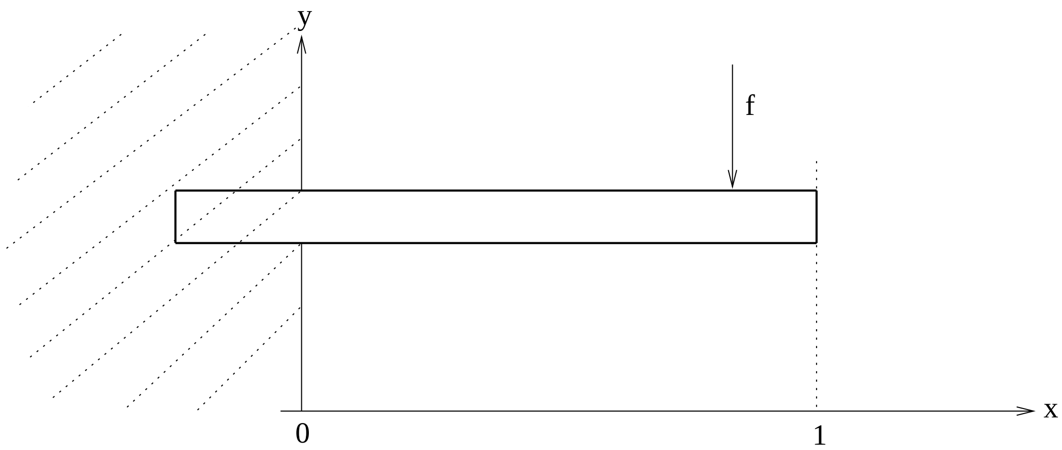
(2) $\|u-U\| \le C \max h \|h^2 D^3 u\|$

(3) $\|u-U\| \le C\|h^2 r(U)\|, \quad \text{where} \quad |r(U)| \le Ch.$

These estimates can be extended to, for example, the space-time discretization of the heat equation.

The equation of an elastic beam

$$\begin{cases} (au'')'' = f, & \Omega = (0,1) \\ u(0) = 0, & u'(0) = 0 \qquad \text{(Dirichlet)} \\ u''(1) = 0, & (au'')'(1) = 0, \quad \text{(Neumann)} \end{cases}$$



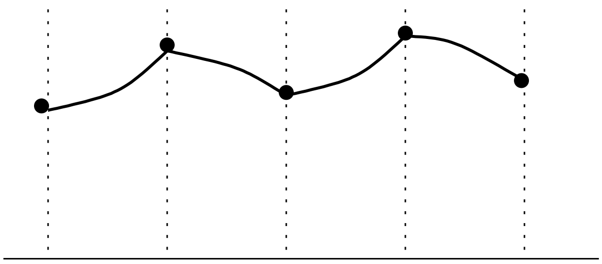
where	a	is the bending stiffness
	au''	is the moment
	f	is the function load
	$u = u(x)$	is the vertical deflection

Variational form:

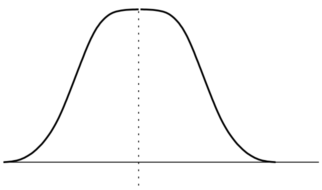
$$\int_0^1 au''v''dx = \int_0^1 fvdx, \text{ for all } v(x) \text{ such that } v(0) = v'(0) = 0.$$

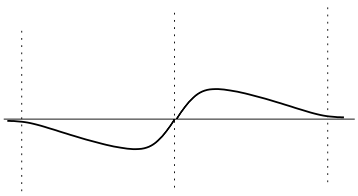
FEM: Piecewise linear functions won't work (inadequate).

Exercise 5: Work out the details with piecewise cubic polynomials having continuous first derivatives: i.e., two degrees of freedom on each node.



A cubic polynomial in (a,b) is uniquely determined by $\varphi(a), \varphi'(a), \varphi(b)$ and $\varphi'(b)$, where the basic functions would have the following form:



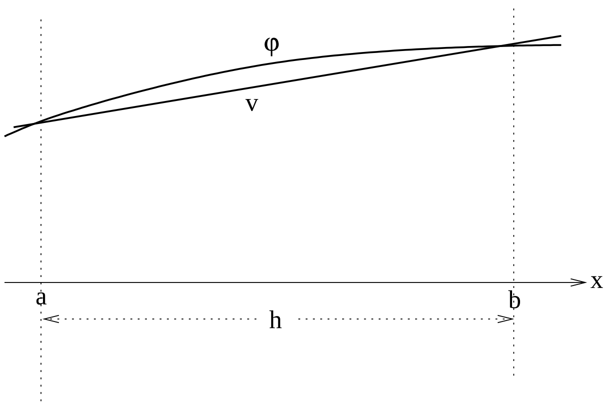


<u>Some basic estimates:</u>

<u>Exercise 6:</u> Let $x, \bar{x} \in I = [a,b]$ and $w(\bar{x}) = 0$. Show that

(E1) $\quad |w(x)| \leq \int_I |w'| dx.$

<u>Exercise 7:</u> Assume that v interpolates φ, at $a,\ b$.



Show, using (E1) that

$$\text{(i)} \qquad |(\varphi - v)(x)| \le \int_I |(\varphi - v)'|\, dx,$$

$$\text{(ii)} \qquad |(\varphi - v)'(x)| \le \int_I |(\varphi - v)''|\, dx = \int_I |\varphi''|\, dx,$$

$$\text{(iii)} \qquad \text{(E2)} \quad \max_I |\varphi - v| \le \max_I |h^2 \varphi''|,$$

$$\text{(iv)} \qquad \int_I |\varphi - v|\, dx \le \int_I |h^2 \varphi''|\, dx,$$

$$\text{(v)} \qquad \|\varphi - v\|_I \le \|h^2 \varphi''\|_I \quad \text{and} \quad \|h^{-2}(\varphi - v)\|_I \le \|\varphi''\|_I,$$

where $\|w\|_I = \left(\int_I w^2\, dx \right)^{1/2}$ is the $L_2(I)$-norm.

Use

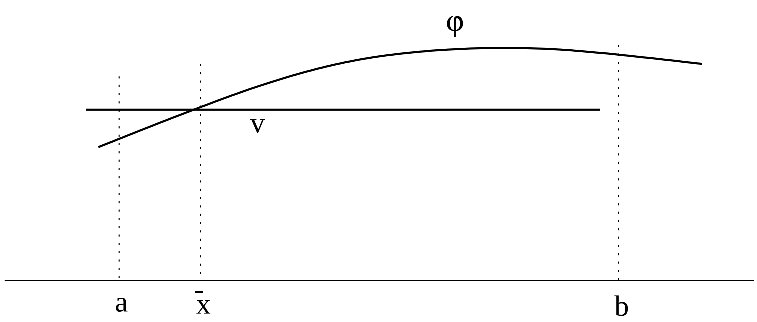
$$v' = \frac{\varphi(b) - \varphi(a)}{h} = \frac{1}{h} \int_a^b \varphi' dx \quad (\varphi' \text{ is constant on } I),$$

and show that

$$\text{(vi)} \qquad |(\varphi - v)(x)| \le 2 \int_I |\varphi'|\, dx,$$

$$\text{(vii)} \qquad \int_I h^{-1} |\varphi - v|\, dx \le 2 \int_I |\varphi'|\, dx \quad \text{and} \quad \|h^{-1}(\varphi - v)\| \le 2 \|\varphi'\|_I.$$

<u>Exercise 8:</u> Let now $v(t)$ be the constant interpolant of φ on I.



Show that

$$\text{(E3)} \qquad \int_I h^{-1} |\varphi - v|\, dx \le \int_I |\varphi'|\, dx.$$

Lemma 1. Let U be the cG(1) approximation of u satisfying

$$\dot{u} + u = f, \quad t > 0, \quad u(0) = u_0.$$

Then we have that

$$|(u - U)(T)| \leq \max_{[0,T]} |k(f - \dot{U} - U)|,$$

where k is the time step.

Proof. The error $e = u - U$ satisfies Galerkin orthogonality:

$$\int_0^T (\dot{e} + e)v dt = 0, \qquad \text{for all piecewise constants } v(t).$$

Let φ satisfy the dual equation

$$-\dot{\varphi} + \varphi = 0, \quad t < T, \quad \varphi(T) = e(T).$$

Then $\underline{\varphi(t) = e(T) \cdot e^{t-T}}$. We show this in the following lines:

Note that integrating $-\dot{\varphi} + \varphi = 0$ gives

$$\int \frac{\dot{\varphi}}{\varphi} dt = \int 1 \cdot dt.$$

Thus $\ln \varphi = t + C$. Let now $C = \ln C_1$, then $\ln \varphi - \ln C_1 = \ln \frac{\varphi}{C_1} = t$, and hence $\varphi(t) = C_1 \cdot e^t$. Since $\varphi(T) = e(T)$ we have then $\varphi(T) = C_1 \cdot e^T = e(T)$, i.e. $C_1 = e(T) \cdot e^{-T}$, and therefore

$$\underline{\varphi(t) = e(T) \cdot e^{t-T}}.$$

To continue we have

$$|e(T)|^2 = e(T) \cdot e(T) + \int_0^T \underbrace{e(-\dot{\varphi} + \varphi) dt}_{=0} = e(T) \cdot e(T) - \int_0^T e\dot{\varphi}\, dt + \int_0^T e\varphi dt.$$

Note that

$$\int_0^T e\dot{\varphi} dt = [PI] = [e \cdot \varphi]_{t=0}^T - \int_0^T \dot{e}\varphi dt = e(T)\varphi(T) - e(0)\varphi(0) - \int_0^T \dot{e}\varphi dt.$$

Using $\varphi(T) = e(T)$, and $e(0) = 0$, we thus have

$$|e(T)|^2 = e(T)\cdot e(T) - e(T)\cdot e(T) + \int_0^T \dot{e}\varphi\, dt + \int_0^T e\varphi\, dt = \int_0^T (\dot{e}+e)\varphi\, dt$$
$$= \int_0^T (\dot{e}+e)(\varphi - v)dt = \int_0^T \Big(\underbrace{\dot{u}+u}_{=f} - \dot{U} - U\Big)(\varphi - v)dt.$$

We have that $\dot{U}+U-f := r(U)$, is the residual and

$$|e(T)|^2 = -\int_0^T r(U)\cdot(\varphi - v)dt \le \max_{[0,T]} |k\cdot r(U)| \int_0^T \frac{1}{k}|\varphi - v|dt.$$

Recall that

$$\text{(E3)} \qquad \int_I h^{-1}|\varphi - v|dx \le \int_I |\varphi'|dx.$$

Further $-\dot{\varphi}+\varphi = 0$ implies $\dot{\varphi} = \varphi$, and $\varphi(t) = e(T)\cdot e^{t-T}$. Thus we can write

$$|e(T)|^2 \le \max_{[0,T]} |k\cdot r(U)| \int_0^T |\dot{\varphi}|dt = \max_{[0,T]} |k\cdot r(U)| \int_0^T |\varphi(t)|\, dt$$
$$\le \max_{[0,T]} |kr(U)||e(T)| \int_0^T e^{t-T}dt,$$

and since $\int_0^T e^{t-T}dt = [e^{t-T}]_0^T = e^0 - e^{-T} = 1 - e^{-T} \le 1, \quad T>0$, we finally get

$$\underline{|e(T)| \le \max_{[0,T]} |k\cdot r(U)|} \qquad \square$$

<u>Exercise 9:</u> Generalize the Lemma to the problem $\dot{u}+au = f$, with $a =$ positive constant.

Is the statement of Lemma 1 valid for $\dot{u} - u = f$?

<u>Exercise 10:</u> Study the dG(0)-case for $\dot{u}+au = f, a>0$

Lemma 2: Let $\dot{u}+u=f, t>0$. Show for the cG(1)-approximation $U(t)$ that

$$|(u-U)(T)| \leq \max_{[0,T]} |k^2\ddot{u}|T.$$

Proof. "Sketchy", via dual equation $\dot{\varphi}+\varphi=0,\ t<T,\ \ \varphi(T)=e(T)$

$$\begin{aligned}|e(T)|^2 = |\Theta(T)|^2 &= \Theta(T)\varphi(T) + \underbrace{\int_0^T \bar{\Theta}(-\dot{\Phi}+\Phi)\,dt}_{=0} = \int_0^T (\dot{\Theta}+\Theta)\bar{\Phi}dt \\ &= -\int_0^T (\dot{\rho}+\rho)\bar{\Phi}\,dt = -\int_0^T \rho\cdot\bar{\Phi}\,dt \leq \max_{[0,T]}|k^2\ddot{u}|\int_0^T |\bar{\Phi}|\,dt \\ &\leq \max_{[0,T]}|k^2\ddot{u}|\cdot T\cdot|e(T)|. \qquad \square\end{aligned}$$

Here $\rho=u-\hat{u}, \Theta=\hat{u}-U$ and Φ is cG(1)-approximation of ϕ such that

$\int_0^T v(-\dot{\Phi}+\Phi)\,dt=0$ for all piecewise constant $v(t)$.

$\hat{u}$ is the piecewise linear interpolant of u,

$\bar{w}=$ piecewise constant mean value.