Chapter 17. The wave equation

Consider the wave equation:

$$\begin{cases} \ddot{u} - \Delta u = 0, & \text{in } \Omega \qquad (DE) \\ u = 0, & \text{on } \partial\Omega = \Gamma \qquad (BC) \\ (u = u_0) \wedge (\dot{u} = v_0) & \text{in } \Omega, \text{ for } t = 0, \quad (IC) \end{cases} \qquad \left(\ddot{u} = \frac{\partial^2 u}{\partial t^2}\right)$$

<u>Conservation of energy:</u>

We multiply the equation by $\dot{u}$ and integrate over Ω,

$$\int_\Omega \ddot{u} \cdot \dot{u} dx - \int_\Omega \Delta u \cdot \dot{u} dx = 0.$$

Using Green's formula:

$$-\int_\Omega (\Delta u)\dot{u}\, dx = -\int_\Gamma (\nabla u \cdot n)\dot{u}\, ds + \int_\Omega \nabla u \cdot \nabla \dot{u}\, dx,$$

and the boundary condition $u = 0$ on Γ, (which implies $\dot{u} = 0$ on Γ), we get

$$\int_\Omega \ddot{u} \cdot \dot{u}\, dx + \int_\Omega \nabla u \cdot \nabla \dot{u}\, dx = 0.$$

Consequently we have that

$$\int_\Omega \frac{1}{2}\frac{d}{dt}(\dot{u}^2)\, dx + \int_\Omega \frac{1}{2}\frac{d}{dt}(|\nabla u|^2)\, dx = 0 \Longleftrightarrow \frac{1}{2}\frac{d}{dt}(\|\dot{u}\|^2 + \|\nabla u\|^2) = 0,$$

and hence $\dfrac{1}{2}\|\dot{u}\|^2 + \dfrac{1}{2}\|\nabla u\|^2 =$ constant, independent of t.

Therefore the total energy is conserved. $\square$

Here $\frac{1}{2}\|\dot{u}\|^2$ is the kinetic energy, and $\frac{1}{2}\|\nabla u\|^2$ is the potential (elastic) energy.

<u>Exercise 1:</u> Show that $\|\nabla \dot{u}\|^2 + \|\nabla u\|^2 =$ constant, independent of t.
Hint: Multiply (DE): $\ddot{u} - \Delta u = 0$ by $-\Delta \dot{u}$ and integrate over Ω or
Alternatively: differentiate the equation with respect to x and multiply by $\dot{u}, \ldots$.

<u>Exercise 2:</u> Derive a total conservation of energy relation using the Robin type boundary condition: $\dfrac{\partial u}{\partial n} + u = 0.$

FEM for the wave equation.

We seek the solution $u(x,t)$ for the problem:

$$\begin{cases} \ddot{u} - u'' = 0, & 0 < x < 1 \qquad t > 0 \qquad (DE) \\ u(0,t) = 0, & u'(1,t) = g(t,) \quad t > 0 \qquad (BC) \\ u(x,0) = u_0(x), & \dot{u}(x,0) = v_0(x), \quad 0 < x < 1 \quad (IC). \end{cases}$$

We let $\dot{u} = v$, and reformulate the problem as a system of PDEs:

$$\begin{cases} \dot{u} - v = 0 & \text{(Convection)} \\ \dot{v} - u'' = 0 & \text{(Diffusion)} \end{cases}$$

Remark. We rewrite the above system as $\dot{w} + Aw = 0$ with $w = \begin{pmatrix} u \\ v \end{pmatrix}$, then

$$\dot{w} + aw = \begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} + \underbrace{\begin{pmatrix} a & b \\ c & d \end{pmatrix}}_{A} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ thus } \begin{cases} au + bv = -\dot{u} \\ cu + dv = -\dot{v} \end{cases} \text{ but}$$

$\dot{u} = v$ and $\dot{v} = u''$ gives that $\begin{cases} au + bv = -v \\ cu + dv = -u'' \end{cases}$.

Consequently we have $a = 0, b = -1$ and $c = -\frac{\partial^2}{\partial x^2},\ d = 0$, i.e.

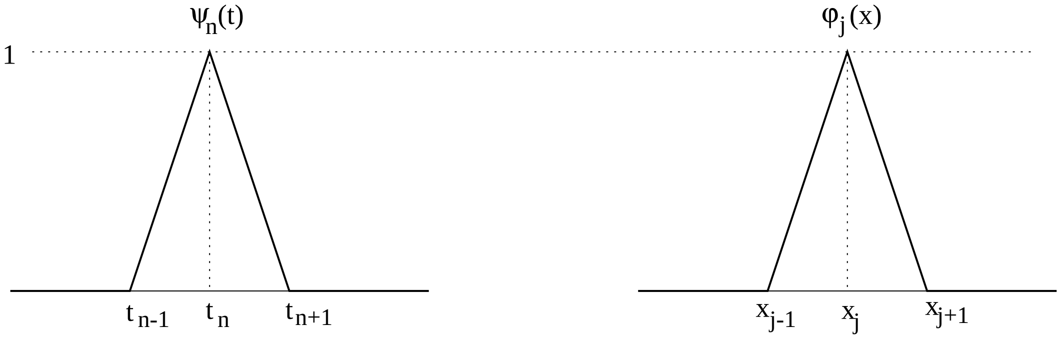
$$\underbrace{\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix}}_{\dot{w}} + \underbrace{\begin{pmatrix} 0 & -1 \\ -\frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} u \\ v \end{pmatrix}}_{w} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \square$$

Let now for each n, the piecewise linear approximations to be defined as

$$\begin{cases} U(x,t) = U_{n-1}(x)\Psi_{n-1}(t) + U_n(x)\Psi_n(t), \\ V(x,t) = V_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t), \end{cases} \qquad 0 < x < 1, \quad t \in I_n,$$

where

$$\begin{cases} U_n(x) = U_{n,1}(x)\varphi_1(x) + \ldots + U_{n,m}(x)\varphi_m(x), \\ V_{n-1}(x) = V_{n-1,1}(x)\varphi_1(x) + \ldots + V_{n-1,m}(x)\varphi_m(x), \end{cases}$$



Note that since $\dot{u} - v = 0,\ t \in I_n$ we have

(1) $$\int_{I_n}\int_0^1 \dot{u}\varphi\, dxdt - \int_{I_n}\int_0^1 v\varphi\, dxdt = 0, \qquad \text{for all } \ \varphi(x,t).$$

Similarly, integrating $\dot{v} - u'' = 0$, we have

(2) $$\int_{I_n}\int_0^1 \dot{v}\varphi\, dxdt - \int_{I_n}\int_0^1 u''\varphi\, dxdt = 0,$$

where, in the second term we use partial integration in x and the boundary condition $u'(1,t) = g(t)$ to obtain

$$\int_0^1 u''\varphi dx = [u'\varphi]_0^1 - \int_0^1 u'\varphi'\, dx = g(t)\varphi(1,t) - u'(0,t)\varphi(0,t) - \int_0^1 u'\varphi'\, dx.$$

Inserting in (2) we get

(3) $$\int_{I_n}\int_0^1 \dot{v}\varphi\, dxdt + \int_{I_n}\int_0^1 u'\varphi'\, dxdt = \int_{I_n} g(t)\varphi(1,t)\, dt, \qquad \text{for all } \varphi \text{ such that } \varphi(0,t) = 0.$$

We therefore seek $U(x,t)$ and $V(x,t)$ as above such that

(4) $$\begin{aligned} &\int_{I_n}\int_0^1 \underbrace{\frac{U_n(x) - U_{n-1}(x)}{k}}_{\dot{U}}\varphi_j(x)\, dxdt - \\ &- \int_{I_n}\int_0^1 \Big(V_{n-1}(x)\Psi_{n-1}(t) + V_n(x)\Psi_n(t)\Big)\varphi_j(x)\, dxdt = 0, \end{aligned}$$

for $j = 1, 2, \ldots, m$,

and

$$\int_{I_n}\int_0^1 \underbrace{\frac{V_n(x)-V_{n-1}(x)}{k}}_{\dot{V}} \varphi_j(x)\,dxdt$$

(5)
$$+\int_{I_n}\int_0^1 \underbrace{\Big(U'_{n-1}(x)\Psi_{n-1}(t)+U'_n(x)\Psi_n(t)\Big)}_{U'} \varphi'_j(x)\,dxdt$$

$$=\int_{I_n} g(t)\varphi_j(1)\,dt, \qquad \text{for } j=1,2,\ldots,m.$$

This is reduced to the *iterative* forms:

(6)
$$\underbrace{\int_0^1 U_n(x)\varphi_j(x)dx}_{MU_n} - \frac{k}{2}\underbrace{\int_0^1 V_n(x)\varphi_j(x)dx}_{MV_n}$$
$$= \underbrace{\int_0^1 U_{n-1}(x)\varphi_j(x)dx}_{MU_{n-1}} + \frac{k}{2}\underbrace{\int_0^1 V_{n-1}(x)\varphi_j(x)\,dx}_{MV_{n-1}}, \text{ for } j=1,2,\ldots,m,$$

and

(7)
$$\underbrace{\int_0^1 V_n(x)\varphi_j(x)dx}_{MV_n} + \frac{k}{2}\underbrace{\int_0^1 U'_n(x)\varphi'_j(x)\,dx}_{SU_n}$$
$$= \underbrace{\int_0^1 V_{n-1}(x)\varphi_j(x)\,dx}_{MV_{n-1}} - \frac{k}{2}\underbrace{\int_0^1 U'_{n-1}(x)\varphi'_j(x)\,dx}_{SU_{n-1}} + g_n, \text{ for } j=1,2,\ldots,m,$$

respectively, where

$$S=\frac{1}{h}\begin{bmatrix} 2 & -1 & \ldots & 0 \\ -1 & 2 & -1 & \ldots \\ 0 & -1 & 2 & \ldots \\ 0 & 0 & -1 & \ldots \end{bmatrix}, \quad M=h\begin{bmatrix} \frac{2}{3} & \frac{1}{6} & \ldots & 0 \\ \frac{1}{6} & \frac{2}{3} & \frac{1}{6} & \ldots \\ \ldots & \frac{1}{6} & \ldots & \ldots \\ 0 & \ldots & \frac{1}{6} & \frac{2}{3} \end{bmatrix},$$

and we use the vector functions:

$$U_n = \begin{pmatrix} U_{n,1} \\ U_{n,2} \\ \dots \\ U_{n,m} \end{pmatrix}, \quad \text{and} \quad g_n = \begin{pmatrix} 0 \\ \dots \\ 0 \\ g_{n,m} \end{pmatrix} \quad \text{where} \quad g_{n,m} = \int_{I_n} g(t)\,dt.$$

<u>Exercise 3:</u> Verify the entries of the matrices S and M.

In the compact form, the vectors U_n and V_n are determined through solving the linear system of equations:

$$\begin{cases} MU_n - \frac{k}{2}MV_n = MU_{n-1} + \frac{k}{2}MV_{n-1} \\ \frac{k}{2}SU_n + MV_n = -\frac{k}{2}SU_{n-1} + MV_{n-1} + g_n. \end{cases}$$

This is a system of $2m$ equations with $2m$ unknowns:

$$\underbrace{\begin{bmatrix} M & -\frac{k}{2}S \\ \frac{k}{2}S & M \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} U_n \\ V_n \end{bmatrix}}_{W} = \underbrace{\begin{bmatrix} M & \frac{k}{2}M \\ -\frac{k}{2}S & M \end{bmatrix} \begin{bmatrix} U_{n-1} \\ V_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ g_n \end{bmatrix}}_{b},$$

with $W = A \setminus b$, $U_n = W(1:m)$, $\quad V_n = W(m+1:2m)$.

<u>Exercise 4:</u> Derive the corresponding linear system of equations in the case of time discretization with dG(0).

<u>Exercise 5:</u> (Conservation of energy)
Show that cG(1)-cG(1) for the wave equation in system form with $g(t) = 0$, conserves energy: i.e.

$$\|U_n'\|^2 + \|V_n\|^2 = \|U_{n-1}'\|^2 + \|V_{n-1}\|^2.$$

Hint: Multiply the first equation by $(U_{n-1} + U_n)^t SM^{-1}$ and the second equation by $(V_{n-1}+V_n)^t$ and add up. Use then, e.g., the fact that $U_n^t S U_n = \|U_n'\|^2$, where

$$U_n = \begin{pmatrix} U_{n,1} \\ U_{n,2} \\ \dots \\ U_{n,m} \end{pmatrix}, \text{ and } U_n = U_n(x) = U_{n,1}(x)\varphi_1(x) + \ldots + U_{n,m}(x)\varphi_m(x).$$

<u>Exercise 6:</u> Apply cG(1) time discretization directly to the wave equation by letting

$$U(x,t) = U_{n-1}\Psi_{n-1}(t) + U_n(x)\Psi_n(t), \qquad t \in I_n.$$

Note that $\dot{U}$ is piecewise constant in time and comment on:

$$\underbrace{\int_{I_n}\int_0^1 \ddot{U}\varphi_j\,dxdt}_{?} + \underbrace{\int_{I_n}\int_0^1 u'\varphi_j'\,dxdt}_{\frac{k}{2}S(U_{n-1}+U_n)} = \underbrace{\int_{I_n} g(t)\varphi_j(1)dt}_{g_n}, \quad j = 1,2,\ldots,m.$$

<u>Exercise 7:</u> Show that the FEM with the mesh size h for the problem:

$$\begin{cases} -u'' = 1 & 0 < x < 1 \\ u(0) = 1 & u'(1) = 0, \end{cases}$$

with

$$U(x) = 7\varphi_0(x) + U_1\varphi_1(x) + \ldots + U_m\varphi_m(x).$$

leads to the linear system of equations: $\tilde{A}\cdot\tilde{U} = \tilde{b}$, where

$$\underset{m\times(m+1)}{\tilde{A} = \frac{1}{h}\begin{bmatrix} -1 & 2 & -1 & & 0 \\ 0 & -1 & 2 & -1 & \dots \\ \dots & \dots & \dots & & \dots \\ 0 & \dots & 0 & & \dots \end{bmatrix}}, \qquad \underset{(m+1)\times 1}{\tilde{U} = \begin{bmatrix} 7 \\ U_1 \\ \dots \\ U_m \end{bmatrix}}, \qquad \underset{m\times 1}{\tilde{b} = \begin{bmatrix} h \\ \dots \\ h \\ \frac{h}{2} \end{bmatrix}}$$

which is reduced to $AU = b$, with

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & \dots & 0 \\ -1 & 2 & -1 & \dots \\ 0 & -1 & 2 & \dots \\ 0 & 0 & -1 & \dots \end{bmatrix}, \quad U = \begin{bmatrix} U_1 \\ U_2 \\ \dots \\ U_m \end{bmatrix}, \quad b = \begin{bmatrix} h + \frac{7}{h} \\ h \\ \dots \\ h \\ \frac{h}{2} \end{bmatrix}$$

<u>Exercise 8:</u> Construct a FEM for the problem

$$\begin{cases} \ddot{u} + \dot{u} - u'' = f, & 0 < x < 1 \quad t > 0, \\ u(0,t) = 0, & u'(1,t) = 0, \quad t > 0, \\ u(x,0) = 0, & \dot{u}(x,0) = 0, \quad 0 < x < 1. \end{cases}$$

<u>Exercise 9:</u> Assume that $u = u(x)$ satisfies

$$\int_0^1 u'v'dx = \int_0^1 fv\,dx, \quad \text{for all } v(x) \text{ such that } v(0) = 0.$$

Show that $-u'' = f$ for $0 < x < 1$ and $u'(1) = 0$.

Hint: See Lecture notes, previous chapters.

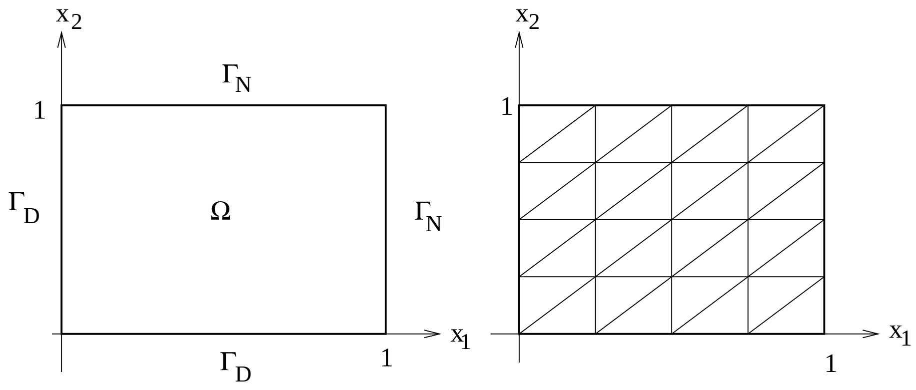
<u>Exercise 10:</u> Consider the following two dimensional problem:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_N \end{cases}$$

See figure below

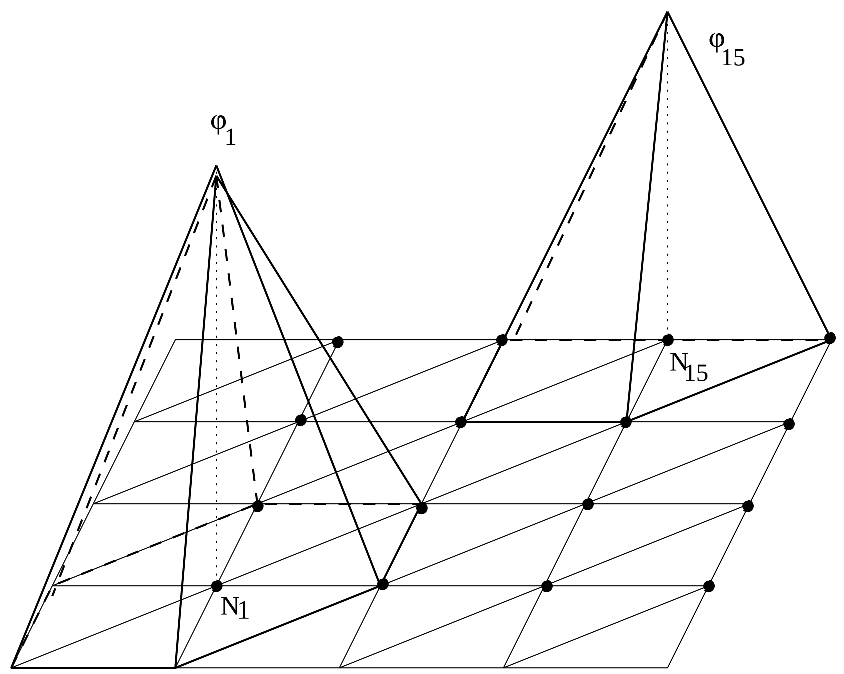
Triangulate Ω as in the figure and let

$$U(x) = U_1\varphi_1(x) + \ldots + U_{16}\varphi_{16}(x),$$



where $x = (x_1, x_2)$ and $\varphi_j,\ j = 1, \ldots 16$ are the basis functions, see Fig. below, and determine $U_1, \ldots U_{16}$ so that

$$\int_\Omega \nabla U \cdot \nabla \varphi_j dx = \int_\Omega \varphi_j dx, j = 1, 2, \ldots, 16$$



Exercise 12: Generalize the whole procedure above to

$$\begin{cases} -\nabla(a\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ a\frac{\partial u}{\partial n} = 7, & \text{on } \Gamma_N \end{cases}, \text{ where } \begin{cases} a = 1 & \text{for } x_1 < \frac{1}{2} \\ a = 2 & \text{for } x_1 > \frac{1}{2} \\ f = x_2 \end{cases}, \quad \text{mesh-size}=h.$$