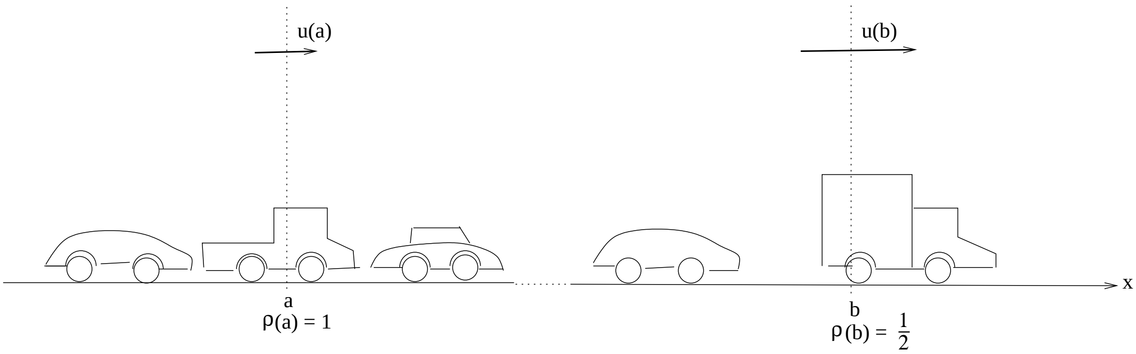
Chapter 18. Stationary convection - diffusion problems

The convection problem:

Example: Consider the traffic flow in a highway.



Let $\rho = \rho(x,t)$ be the density of cars ($0 \le \rho \le 1$) and $u = u(x,t)$ the velocity (speed vector) of the cars.

For a highway path (a,b) the difference between the traffic inflow $u(a)\rho(a)$ at the point $x = a$ and outflow $u(b)\rho(b)$ at $x = b$ gives the density variation on the interval (a,b):

$$\frac{d}{dt}\int_a^b \rho(x,t)dx = \int_a^b \dot{\rho}(x,t)dx = \rho(a)u(a) - \rho(b)u(b) = -\int_a^b (u\rho)'dx$$

or equivalently

$$\int_a^b \Big(\dot{\rho} + (u\rho)'\Big)\, dx = 0.$$

Now since a and b can be chosen arbitrary, thus we have

$$(1) \qquad \underline{\dot{\rho} + (u\rho)' = 0}$$

Let now $u = 1 - \rho$, (motivate this choice), then $\dot{\rho} + ((1-\rho)\rho)' = \dot{\rho} + (\rho - \rho^2)' = 0$, i.e.,

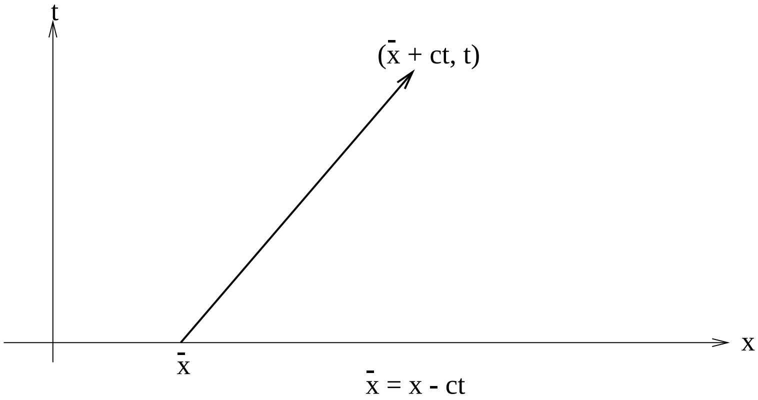
$$\underline{\dot{\rho} + (1-2\rho)\rho' = 0} \qquad \text{(A non-linear convection equation).}$$

Alternatively, we may assume that $u = c - \varepsilon \cdot (\rho'/\rho)$, $c > 0$, $\varepsilon > 0$, (motivate). Then we get from (1) that

$$\dot{\rho} + \Big((c - \varepsilon\frac{\rho'}{\rho})\rho\Big)' = 0, \iff \dot{\rho} + c\rho' - \varepsilon\rho'' = 0. \text{ (A convection - diffusion equation).}$$

Which is convection dominated if $c > \varepsilon$.

For $\varepsilon = 0$ the solution is given by the exact transport $\rho(x,t) = \rho_0(x-ct)$, because then $\rho =$ constant on the $(c,1)$-direction.



Note that differentiating $\rho(x,t) = \rho(\bar{x}+ct,t)$ with respect to t we get

$$\frac{\partial \rho}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial \rho}{\partial t} = 0, \quad \Longleftrightarrow \quad c\rho' + \dot{\rho} = 0.$$

We rewrite our last convection-diffusion equation for ρ, by changing the notation from ρ to u, and replacing c by β and get

$$\underline{\dot{u} + \beta \cdot u' - \varepsilon \cdot u'' = 0}.$$

We compare this equation with the Navier-Stokes equation for incompressible flow:

$$\dot{u} + (\beta \cdot \nabla)u - \varepsilon \Delta u + \nabla P = 0, \quad \wedge \quad \operatorname{div} u = 0,$$

where $\beta = u$, $u = (u_1, u_2, u_3)$ is the velocity vector, with u_1 representing the mass, u_2 momentum, and $u_3 =$ energy. Further P is the pressure and $\varepsilon = \dfrac{1}{Re}$ with Re denoting the Reynold's number.

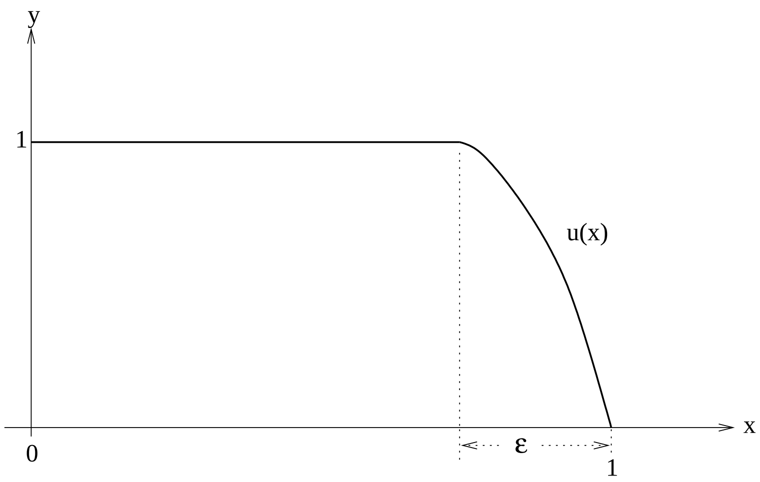
These equations are not easily solvable, for $\varepsilon > 0$ and *small*, because of difficulties related to boundary layer and turbulence. A typical range for the Reynold's number Re is between 10^5 and 10^7.

Example: Consider the

$$\text{(BVP)} \quad \begin{cases} u' - \varepsilon u'' = 0, & 0 < x < 1 \\ u(0) = 1 & u(1) = 0. \end{cases}$$

The exact solution is given by

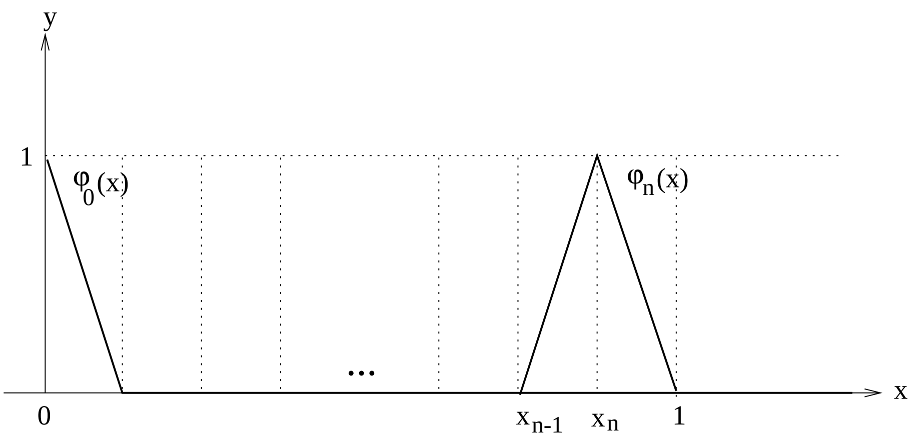
$$u(x) = C\Big(e^{1/\varepsilon} - e^{x/\varepsilon}\Big), \qquad \text{with} \quad C = \frac{1}{e^{1/\varepsilon} - 1}.$$



Note an outflow boundary layer of width $\sim \varepsilon$.

<u>FEM.</u> The finite element discretization would be then

$$U(x) = \varphi_0(x) + U_1\varphi_1(x) + \ldots + U_n\varphi_n(x).$$



The variational formulation:

$$\int_0^1 \Big(U'\varphi_j dx + \varepsilon U'\varphi_j'\Big)\, dx = 0, \quad j = 1, 2, \ldots n$$

gives the equations

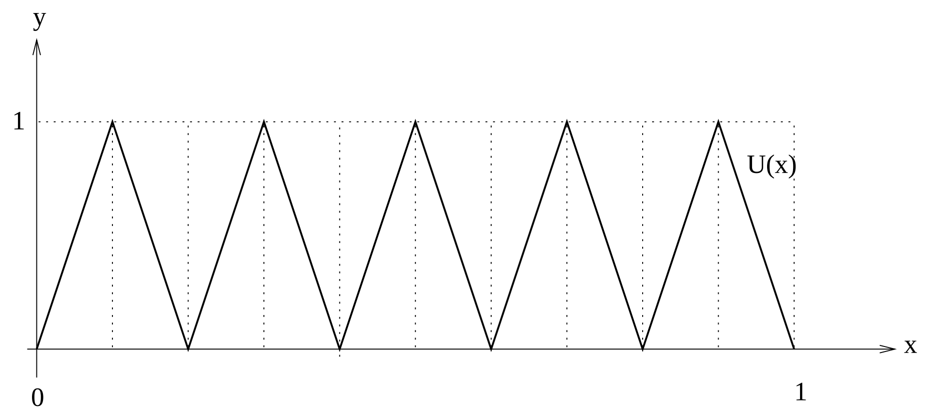
$$(2) \qquad \frac{1}{2}\Big(U_{j+1} - U_{j-1}\Big) + \frac{\varepsilon}{h}\Big(2U_j - U_{j-1} - U_{j+1}\Big) = 0, \quad j = 1, 2, \dots, n,$$

where $U_0 = 1$ and $U_{n+1} = 0$.

Note that we can also write using "Central -differencing"

$$\underbrace{\frac{U_{j+1} - U_{j-1}}{2h}}_{\text{corresp. to } u'(x_j)} - \varepsilon \underbrace{\frac{U_{j+1} - 2U_j + U_{j-1}}{h^2}}_{\text{corresp. to } u''(x_j)} = 0 \quad \Big(\iff \frac{1}{h} \times \text{the equation}(2)\Big).$$

For ε very small this gives that $U_{j+1} \approx U_{j-1}$, giving, for even n values, alternating 0 and 1 as the solution values at the nodes:



i.e., oscillations in U are transported "upstreams" making U a "globally bad approximation" of u.

A better method is to approximate $u'(x_j)$ by an "upwind" derivative:

$$u'(x_j) \approx \frac{U_j - U_{j-1}}{h},$$

which, formally, gives a better stability, however, with low accuracy.

The example itself demonstrates that a high accuracy without stability is indeed useless.

A more systematic method of making FE - solution of the fluid problems stable is through using the:

Streamline - diffusion method (SDM):

The idea is to choose the test functions of the form $(v + \frac{1}{2}\beta h v')$, instead of just v. Then, e.g., for our model problem we obtain the equation ($\beta \equiv 1$)

$$(3) \qquad \int_0^1 \Big[u'(v + \frac{1}{2}hv') - \varepsilon \cdot u''\Big(v + \frac{1}{2}hv'\Big)\Big]dx = \int_0^1 f\Big(v + \frac{1}{2}hv'\Big)\, dx.$$

Here, in the discrete version of the variational formulation, we should interpret the term $\int_0^1 U''v'dx$ as

$$\sum_j \int_{I_j} U''v'dx = 0 \text{ (in the case of approximation with piecewise linears).}$$

Note that $v = \varphi_j$ implies:

$$\int_0^1 U'\frac{1}{2}h\varphi_j' dx = U_j - \frac{1}{2}U_{j+1} - \frac{1}{2}U_{j-1},$$

which combined with (adding the two)

$$\int_0^1 U'\varphi_j dx = \frac{U_{j+1} - U_{j-1}}{2},$$

gives $(U_j - U_{j-1})$, which is approximating the first integral in (3) corresponding to the "upwind" scheme.

The SDM can also be interpreted as a sort of *least-square method*:

Let $A = \frac{d}{dx}$ and $A^t = -\frac{d}{dx}$; then u minimizes $\|w' - f\|$ if

$$u' = Au = f \iff A^tAu = A^tf \iff -u'' = -f, \qquad \text{the continuous form.}$$

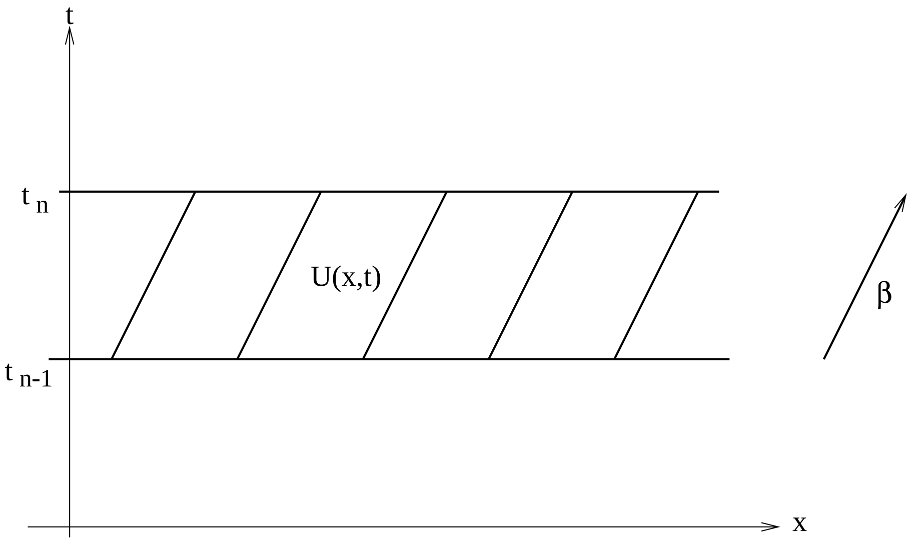
While multiplying $u' = Au = f$ by v and integrating over $(0,1)$ we have

$$\int_0^1 U'v'dx = \int_0^1 fv'dx \quad \text{(Weak form)},$$

where we replaced u' by U'.

For the time-dependent convection problem, the oriented time-space element are used:

$$\dot{u} + \beta u' - \varepsilon u'' = f.$$



Set $U(x,t)$ such that U is piecewise linear in x and piecewise constant in the $(\beta, 1)$-direction. Combine with SDM and add up some artificial viscosity, $\hat{\varepsilon}$, depending on the residual term to get for each time interval I_n:

$$\int_{I_n}\int_{\Omega}\Big[(\dot{U}+\beta U)\Big(v+\frac{\beta}{2}h\dot{v}\Big)+\hat{\varepsilon}\; U'v'\Big]\,dxdt=\int_{I_n}\int_{\Omega} f\Big(v+\frac{\beta}{2}hv'\Big)\,dxdt. \quad \square$$