Chapter 21. The power of abstraction

In chapter 8, we proved under certain assumptions the following:

Boundary value problem (BVP)$\Leftrightarrow$ Variational formulation (VF)$\Leftrightarrow$ Minimization problem (MP),

$$(BVP): \qquad \begin{cases} -(a(x)u'(x))' = f(x) & 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

(VF): Find $u(x)$, with $u(0) = u(1) = 0$, such that

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \quad \forall v \in H_0^1, \quad \text{where}$$
$$H_0^1 = \left\{ v : \int_0^1 \left(v(x)^2 + v'(x)^2 \right) dx < \infty, \quad v(0) = v(1) = 0 \right\}$$

(MP): Find $u(x)$, with $u(0) = u(1) = 0$, such that $u(x)$ minimizes the *functional* F given by

$$F(v) = \frac{1}{2} \int_0^1 v'(x)^2 dx - \int_0^1 f(x)v(x)dx$$

We can actually take instead of H_0^1, the space

$$\mathcal{H}_0^1 = \left\{ f : [0,1] \to \mathbb{R} : \int_0^1 f'(x)^2 dx < \infty, \wedge f(0) = f(1) = 0 \right\}.$$

Let now V be a *vector space* and define a *bilinear form.*

$a(\cdot,\cdot) : V \times V \to \mathbb{R}$, i.e. for $\alpha, \beta, x, y \in \mathbb{R}$ and $u, v, w \in V$, we have that

$$a(\alpha u + \beta v, w) = \alpha \cdot a(u,w) + \beta \cdot a(v,w)$$
$$a(u, xv + yw) = x \cdot a(u,v) + y \cdot a(u,w)$$

<u>Ex.</u> Let $V = \mathcal{H}_0^1$ and define

$$a(u,v) \equiv (u,v) \equiv \int_0^1 u'(x)v'(x)dx,$$

then $(\cdot,\cdot)$ is *symmetric*, i.e. $(u,v) = (v,u)$, *bilinear* (obvious), and *positive definite* in the sense that

$$(u,u) \geq 0, \quad \text{and } (u,u) = 0 \Leftrightarrow u \equiv 0.$$

Note that

$$(u,u)=\int_0^1 u'(x)^2dx=0 \Leftrightarrow u'(x)=0,$$

thus $u(x)$ is constant and since $u(0)=u(1)=0$ we have $u(x)\equiv 0$.

Definition: A linear function $L:V\to\mathbb{R}$ is called a linear form on V: If

$$L(\alpha u+\beta v)=\alpha L(u)+\beta L(v)$$

Example. Let

$$L(v)=\int_0^1 fvdx, \forall v\in\mathcal{H}_0^1,$$

Then our (VF) can be restated as follows:

Find $u\in\mathcal{H}_0^1$ such that

$$(u,v)=L(v), \forall v\in\mathcal{H}_0^1.$$

Generalizing the *abstract problem*:

Find $u\in V$, such that

$$a(u,v)=L(v), \forall v\in V.$$

Let now $\|\cdot\|_V$ be a norm corresponding to a scalar product $(\cdot,\cdot)_V$ defined on $V\times V$. Then assuming that $a(\cdot,\cdot)$ is *coercive* (*V-elliptic*), and $a(\cdot,\cdot)$ and $L(\cdot)$ are continuous: i.e., there are constants c_1, c_2 and c_2 such that:

(1) $$a(v,v)\geq c_1\|v\|_V^2, \quad \forall v\in V \qquad \text{(coercivity)}$$
(2) $$|a(u,v)|\leq c_2\|u\|_V\|v\|_V, \quad \forall u,v\in V \qquad (a \text{ is continuous})$$
(3) $$|L(v)|\leq c_3\|v\|_V, \quad \forall v\in V \qquad (L \text{ is continuous}).$$

Note. Since L is linear, we have using the relation (3) above that

$$|L(u)-L(v)|=|L(u-v)|\leq c_3\|u-v\|_V,$$

which shows that $L(u)\to L(v)$ as $u\to v$, in V. Thus L is continuous.

Similarly the assumption $|a(u,v)|\leq c_1\|u\|_V\|v\|_V$ implies that $a(\cdot,\cdot)$ is continuous in each variable.

Definition: The energy norm on V is defined by $\|v\|_a=\sqrt{a(v,v)}, \ v\in V$.

Recalling the relations (1) and (2) above, the energy norm satisfies

$$c_1\|v\|_V^2\leq a(v,v)=\|v\|_a^2\leq c_2\|v\|_V^2.$$

Therefore the energy norm $\|v\|_a$ is equivalent to the abstract $\|v\|_V$ norm.

Example For the scalar product

$$(u,v)=\int_0^1 u'(x)v'(x)dx, \quad \text{in} \quad \mathcal{H}_0^1,$$

and the norm

$$\|u\|=\sqrt{(u,u)},$$

the relations (1) and (2) are valid with $c_1=c_2\equiv 1$

(1) $(v,v)=\|v\|^2$ is an identity

(2) $|(u,v)|\leq \|u\|\|v\|$ is the Cauchy's inequality sketched below:

Proof of (2): Using the obvious inequality $2ab\leq a^2+b^2$, we have

$$2|(u,w)|\leq \|u\|^2+\|w\|^2.$$

We let $w=(u,v)\cdot v/\|v\|^2$ then

$$2|(u,w)|=2\Big|\Big(u,(u,v)\frac{v}{\|v\|^2}\Big)\Big|\leq \|u\|^2+|(u,v)|^2\frac{\|v\|^2}{\|v\|^4}$$

Thus

$$2\frac{|(u,v)|^2}{\|v\|^2}\leq \|u\|^2+|(u,v)|^2\frac{\|v\|^2}{\|v\|^4},$$

which multiplying by $\|v\|^2$, gives

$$2|(u,v)|^2\leq \|u\|^2\cdot\|v\|^2+|(u,v)|^2,$$

and hence

$$|(u,v)|^2\leq \|u\|^2\cdot\|v\|^2,$$

and the proof is complete. $\square$

Definition: A *Hilbert space* is a *complete* linear space with a scalar product.

To define complete linear space we first need to define a *Cauchy sequence* of *real* or *complex* numbers.

Definition: A sequence $\{z_k\}_{k=1}^\infty$ is a Cauchy sequence if for every $\varepsilon>0$, there is an integer $N>0$, such that $m,n>N\Rightarrow |z_m-z_n|<\varepsilon$.

Theorem 1: Every Chaucy sequence in $\mathbb{C}$ is convergent. More precisely: If $\{z_k\}_{k=1}^\infty\subset\mathbb{C}$ is a Cauchy sequence, then there is a $z\in\mathbb{C}$, such that for every $\epsilon>0$, there is an integer $M>0$, such that $m\geq M\Rightarrow |z_m-z|<\varepsilon$.

Definition: A linear space V (vector space) with the norm $\|\cdot\|$ is called *complete* if every Cauchy sequence in V is convergent. In other words: For every $\{v_k\}_{k=1}^{\infty}$ with the property that for every $\varepsilon > 0$ there is an integer $N > 0$, such that $m, n > N \Rightarrow \|v_m - v_n\| < \varepsilon$, (i.e. for every Cauchy sequence) there is a $v \in V$ such that for every $\varepsilon > 0$ there is an integer $M > 0$ such that $m \geq M \Rightarrow |v_m - v| < \varepsilon$.

Theorem 2: $\mathcal{H}_0^1 = \{f : [0,1] \to \mathbb{R} : \int_0^1 f'(x)^2 dx < \infty, \wedge f(0) = f(1) = 0\}$ is a complete Hilbert space with the norm

$$\|u\| = \sqrt{(u,u)} = \Big(\int_0^1 u'(x)^2 dx \Big)^{1/2}.$$

Poincare's inequality in 1D-case: If $u(0) = u(L) = 0$ then

$$\int_0^L u(x)^2 dx \leq C_L \cdot \int_0^L u'(x)^2 dx,$$

where C_L is a constant independent of $u(x)$ but depends on L.

Proof: Using the Cauchy Schwarz inequality we have

$$\begin{aligned} u(x) &= \int_0^x u'(y)dy \leq \int_0^x |u'(y)|dy \leq \int_0^L |u'(y)| \cdot 1 dy \\ &\leq \Big(\int_0^L u'(y)^2 dy \Big)^{1/2} \Big(\int_0^L 1^2 dy \Big)^{1/2} = \sqrt{L} \Big(\int_0^L u'(y)^2 dy \Big)^{1/2}. \end{aligned}$$

Thus

$$u(x)^2 \leq L \int_0^L u'(y)^2 dy,$$

and hence

$$\int_0^L u(x)^2 dx \leq L \int_0^L \Big(\underbrace{\int_0^L u'(y)^2 dy}_{\text{independent of } x} \Big) dx = L^2 \int_0^L u'(x)^2 dx \qquad \square$$

Exercise: Show that Poincare's inequality is not valid for $0 < x < \infty$.

Linear functionals:

- We define a *functional* ℓ as a mapping from a (linear) function space V into $\mathbb{R}$, i.e.,

$$\ell : V \to \mathbb{R}.$$

- A funcitonal ℓ is called linear if

$$\begin{aligned} \ell(u+v) &= \ell(u)+\ell(v) \quad \text{for all } u,v\in V \\ \ell(\alpha u) &= \alpha\cdot\ell(u) \quad \text{for all } u\in V \text{ and } \alpha\in\mathbb{R} \end{aligned}$$

- A functional is called bounded if there is a constant C such that

$$|\ell(u)| \leq C\cdot\|u\| \quad \text{for all } u\in V \quad (C \text{ is independnet of } u)$$

Example 1: If $f\in L^2(0,1)$, i.e. $\int_0^1 f(x)^2dx$ is bounded, then

$$\ell(v) = \int_0^1 u(x)v(x)dx$$

is a bounded linear functional.

Exercise: Show that ℓ, defined in Example 1 above is linear.

Exercise: Prove using Cauchy's and Poincare's inequalities that ℓ, defined as in Example 1, is bounded in $\mathcal{H}_0^1$.

Recalling that $(u,v) = \int_0^1 u'(x)v'(x)dx$ and $\ell(v) = \int_0^1 u(x)v(x)dx$, we may redefine our variational formulation (VF) and minimization problem (MP), from chapter 8 as (V) and (M), respectively:

(V) Find $u\in\mathcal{H}_0^1$, such that $(u,v)=\ell(v)$ for all $v\in\mathcal{H}_0^1$.

(M) Find $u\in\mathcal{H}_0^1$, such that $F(u)=\min_{v\in\mathcal{H}_0^1} F(v)$ with $F(v)=\frac{1}{2}\|v\|^2-\ell(v)$.

Now we can show that there exists (existence) a unique solution for (V) and (M).

First we note that there exists a real number σ such that $F(v)>\sigma$ for all $v\in\mathcal{H}_0^1$, (otherwise it is not possible to minimize F). Namely,

$$F(v) = \frac{1}{2}\|v\|^2 - \ell(v) \geq \frac{1}{2}\|v\|^2 - \gamma\|v\|,$$

where γ is the constant bounding ℓ, i.e. $|\ell(v)|\leq\gamma\|v\|$.

But since

$$0 \leq \frac{1}{2}(\|v\|-\gamma)^2 = \frac{1}{2}\|v\|^2 - \gamma\|v\| + \frac{1}{2}\gamma^2,$$

we have that

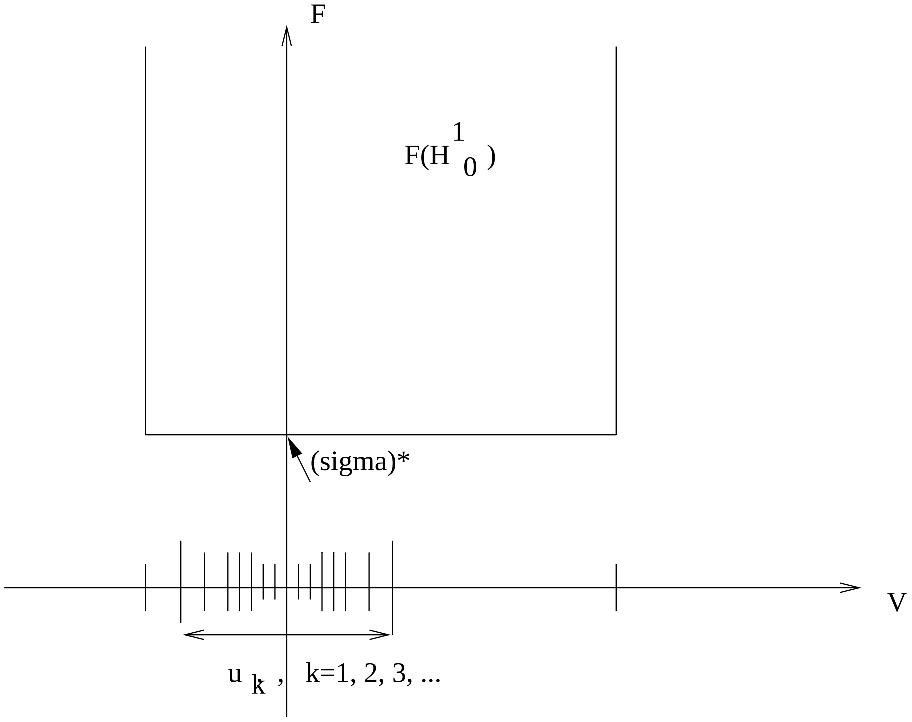
$$\frac{1}{2}\|v\|^2 - \gamma\|v\| \geq -\frac{1}{2}\gamma^2.$$

Let now σ^* be the largest real number σ such that

(1) $\quad F(v) > \sigma \quad$ for all $v \in \mathcal{H}_0^1$.

Take now a sequence of functions $\{u_k\}_{k=0}^{\infty}$, such that

(2) $\quad F(u_k) \to \sigma^*$.



and

To show that there exists (existence) a unique solution for (V) and (M) we need first to prove

(i) It is always possible to find such a sequence $\{q_k\}_{k=0}^{\infty}$, such that $F(u_k) \to \sigma^{\bullet}$ (because $\mathbb{R}$ is complete.)

(ii) The parallelogram law (elementary linear algebra).

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$

Using (ii) and the linearity of ℓ we write

$$\begin{aligned}\|u_k - u_j\|^2 &= 2\|u_k\|^2 + 2\|u_j\|^2 - \|u_k + u_j\|^2 - 4\ell(u_k) - 4\ell(u_j) + 4\ell(u+v) \\ &= 2\|u_k\|^2 - 4\ell(u_k) + 2\|u_j\|^2 - 4\ell(u_j) - \|u_k + u_j\|^2 + 4\ell(u_k + u_j) \\ &= 4F(u_k) + 4F(u_j) - 8F\Big(\frac{u_k + u_j}{2}\Big),\end{aligned}$$

where we have used the definition of $F(v) = \frac{1}{2}\|v\|^2 - \ell(v)$ with $v = u_k,\ u_j$, and $v = (u_k + u_j)/2$, respectivey. In particular by linearity of ℓ:

$$-\|u_k + u_j\|^2 + 4\ell(u_k + u_j) = -4\Big\|\frac{u_k + u_j}{2}\Big\|^2 + 8\ell\Big(\frac{u_k + u_j}{2}\Big) = -8F\Big(\frac{u_k + u_j}{2}\Big).$$

Now since $F(u_k) \to \sigma^*$ and $F(u_j) \to \sigma^*$, then

$$\|u_k - u_j\|^2 \le 4F(u_k) + 4F(u_j) - 8\sigma^* \to 0, \qquad \text{as} \qquad k, j \to \infty.$$

Thus we have shown that $\{u_k\}_{k=0}^{\infty}$ is a Cauchy sequence. Now since $\{u_k\} \subset \mathcal{H}_0^1$ and $\mathcal{H}_0^1$ is complete thus $\{u_k\}_{k=1}^{\infty}$ is convergent. Hence

$$\exists u, \quad \text{such that} \quad u = \lim_{k\to\infty} u_k.$$

By the continuity of F we get that

(3) $\qquad \lim_{k\to\infty} F(u_k) = F(u).$

By (2) and (3) $F(u) = \sigma^*$ and by (1) and the definition of σ^* we have

$$F(u) < F(v), \qquad \forall v \in \mathcal{H}_0^1.$$

This in our minimization problem (M). And since (M) $\Leftrightarrow$ (V) we conclude that:

there is a unique $u \in \mathcal{H}_0^1$ *, such that* $\underline{\ell(v) = (u, v) \quad \forall v \in \mathcal{H}_0^1}$. $\quad\square$

Summing up we have proved that:
Every bounded linear functional can be represented as a scalar product with a given function **u**. *This* **u** *is the unique solution for both (V) and (M).*

Theorem. [Riesz representation theorem.]
If V is a Hilbert space with the scalar product (u, v) and norm $\|u\| = \sqrt{\langle u, u\rangle}$, and $\ell(v)$ is a bounded linear functional on V, then there is a unique $u \in V$, such that $\quad \ell(V) = (u, v), \quad \forall v \in V.$

Lax-Milgram theorem. [A general version of Riesz theorem]
Assume that $\ell(v)$ is bounded linear and $a(u, v)$ is bilinear bounded and elliptic in V, then there is a unique $u \in V$, such that

$$a(u, v) = \ell(v), \quad \forall v \in V.$$

Recall that:

Bilinear means that $a(u, v)$ satisfies the same properties as a scalar product, however it need NOT! to be *symmetric.*

Bounded means that:

$$|a(u, v) \leq \beta \|u\| \|v\|, \quad \text{for some constant } \beta > 0.$$

Elliptic means that:

$$a(v, v) \geq \alpha \|v\|^2, \quad \text{for some } \alpha > 0.$$

<u>Note</u>

$$\text{If } a(u, v) = (u, v), \text{ then } \alpha = \beta = 1.$$