Chapter 5. Polynomial Interpolation

(CDE pp. 43 - 58, 73 - 84)

Consider $P^q(a,b)$; the vector space of all polynomials of degree $\leq q$ on the interval (a,b), and the basis functions $1, x, x^2, \ldots, x^q$.

Lagrange basis (Cardinal functions): This is the set of polynomials $\{\lambda_i\}_{i=0}^q \subset P^q(a,b)$ associated to the $(q+1)$ distinct points, $a = x_0 < x_1 < \ldots < x_q = b$, in (a,b) and determined by the requirement that $\lambda_i(x_j) = 1$ if $i = j$ and 0 otherwise, i.e,

$$\lambda_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_q)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_q)}, \quad x \in (a,b).$$

Note that $\lambda_i(x) = \prod_{j \neq i} \Big(\frac{x-x_j}{x_i-x_j}\Big)$ does not contain the singular factor $\frac{x-x_i}{x_i-x_i}$, and

$$\lambda_i(x_j) = \frac{(x_j-x_0)(x_j-x_1)\ldots\overbrace{(x_j-x_{i-1})(x_j-x_{i+1})}^{\text{note}}\ldots(x_j-x_q)}{(x_i-x_0)(x_i-x_1)\ldots\underbrace{(x_i-x_{i-1})(x_i-x_{i+1})}_{\text{note}}\ldots(x_i-x_q)} = \delta_{ij}.$$

Thus for $i = 1, 2, \ldots, q$, the $\lambda_i(x)$ are polynomials of degree q, on (a,b), satisfying

$$\lambda_i(x_j) = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}$$

Ex. Let $q = 2$, then we have $a = x_0 < x_1 < x_2 = b$, and for instance

$$i = 1, j = 2 \Rightarrow \delta_{12} = \lambda_1(x_2) = \frac{(x_2-x_0)(x_2-x_2)}{(x_1-x_0)(x_1-x_2)} = 0$$

$$i = j = 1 \Rightarrow \delta_{11} = \lambda_1(x_1) = \frac{(x_1-x_0)(x_1-x_2)}{(x_1-x_0)(x_1-x_2)} = 1.$$

A polynomial $p \in P^q(a,b)$ that has the values $p_i = p(x_i)$ at the nodes xi_i for $i = 0, 1, \ldots, q$, can be expressed in terms of the corresponding Lagrange basis as

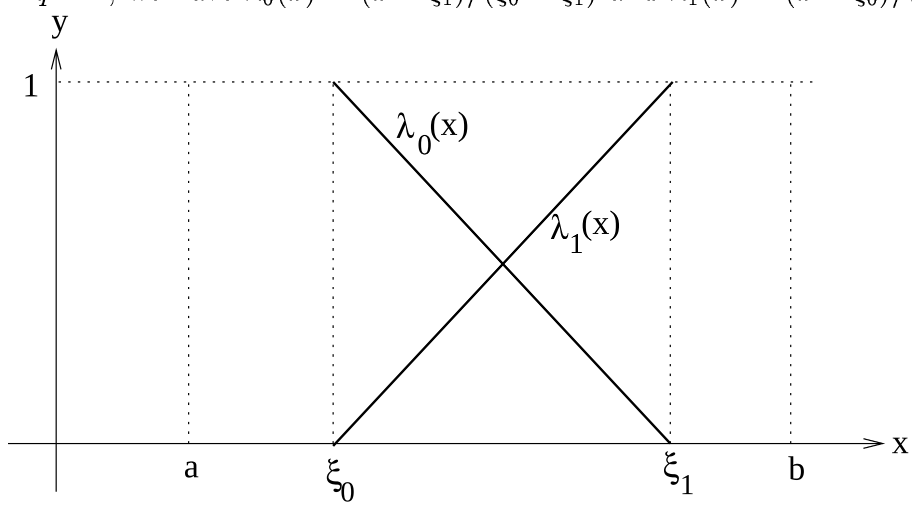
$$p(x) = p_0\lambda_0(x) + p_1\lambda_1(x) + \ldots + p_q\lambda_q(x).$$

Note that $p(x_i) = p_0\lambda_0(x_i) + p_1\lambda_1(x_i) + \ldots p_i\lambda_i(x_i) + \ldots + p_q\lambda_q(x_i)$, and since

$$\lambda_i(x_j) = \begin{cases} 1 & i = j \\ 0 & \text{else} \end{cases}, \text{ thus } p(x_i) = p_i$$

We may construct Lagrange basis for arbitrary subintervals $(\xi_0, \xi_1) \subset (a, b)$, as well:

Ex. If $q = 1$, we have $\lambda_0(x) = (x - \xi_1)/(\xi_0 - \xi_1)$ and $\lambda_1(x) = (x - \xi_0)/(\xi_1 - \xi_0)$



Linear Lagrange basis functions, $q = 1$

Let us also recall the *Taylor polynomial* of degree q for the function $u(x)$ about x_0.

$$f(x) = T_q f(x_0) + R_q f(x_0),$$

where

$$T_q f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + \ldots + \frac{1}{q!}f^{(q)}(x_0)(x - x_0)^q,$$

is the *Taylor interpolation polynomial* of degree q, approximating the function $f(x)$ about $x = x_0$ and

$$R_q f(x) = \frac{1}{(q+1)!}f^{(q)}(\xi)(x - x_0)^{q+1},$$

where ξ is a point between x_0 and x, is the *remainder* term.

Correspondingly, we may define the *Lagrange interpolation polynomial* $\pi_q f$ with $\pi_q f \in P^q(a, b)$, where $f(x)$ is continuous on $[a, b]$. Choose distinct *interpolation nodes*

$$a \leq \xi_0 < \xi_1 < \ldots < \xi_q \leq b,$$

then $\pi_q f \in P^q(a,b)$ interpolates $f(x)$ at the nodes $\xi_i,\ i = 0,\dots,q$, if

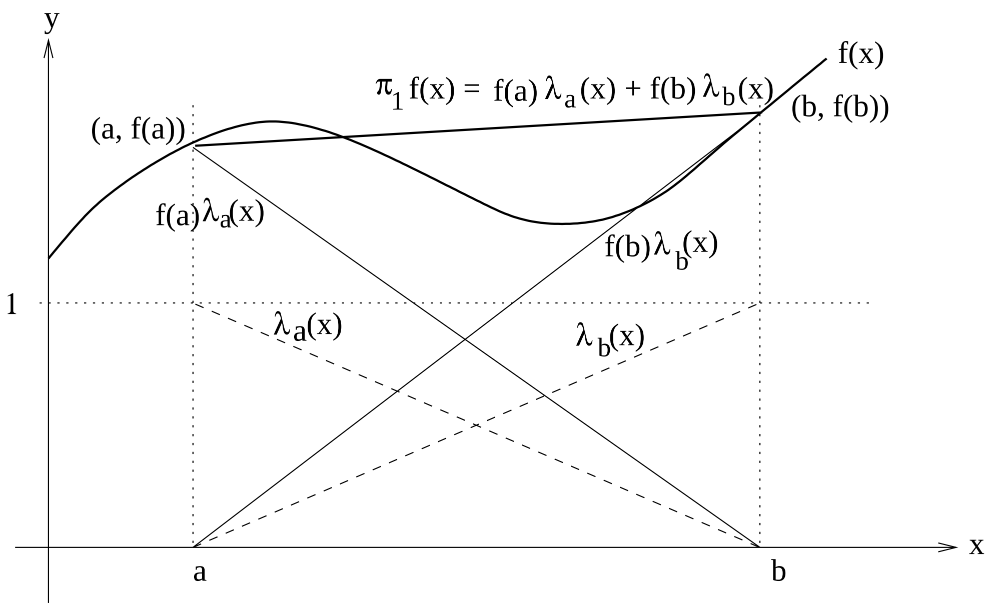
$$\pi_q f(\xi_i) = f(\xi_i).$$

Now the Lagrange's formula $p(x) = p_0\lambda_0(x) + p_1\lambda_1(x) + \ldots + p_q\lambda_q(x)$ for $\pi_q f(x)$ reads as follows:

$$\pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) + \ldots + f(\xi_q)\lambda_q(x), \quad a \le x \le b.$$

Ex. For $q = 1$, and considering the whole interval we have only the nodes a and b. We recall that $\lambda_a(x) = \dfrac{x-b}{a-b}$ and $\lambda_b(x) = \dfrac{x-a}{b-a}$, thus

$$\pi_1 f(x) = f(a)\lambda_a(x) + f(b)\lambda_b(x).$$



The linear interpolant $\pi_1 f(x)$ of a function $f(x)$

Note that $\lambda_a(x) + \lambda_b(x) = 1, \forall x \in [a,b]$.

Theorem 5.1; Interpolation errors

Taylor: $$|f(x) - T_q f(x_0)| = R_q(f) \le \frac{1}{(q+1)!}(x-x_0)^{q+1} \cdot |\max_{x\in[a,b]} f^{(q+1)}(x)|$$

The Taylor interplation error is of degree $q+1$ near $x = x_0$

Lagrange: $$|f(x) - \pi_q f(x)| \le \frac{1}{(q+1)!}\prod_{i=0}^{q}(x-x_i) \cdot \max_{a\le x\le b}|f^{(q+1)}(x)|$$

The Lagrange interpolation error is of degree 1 at each of the points $x_0, x_1, \ldots, x_q$.

Proof. (CDE pp.79 - 81)

The *Taylor part* is well known.

For the *Lagrange* interpolation error we note that at the nodes x_i we have

$f(x_i) - \pi_q f(x_i) = 0, \quad$ for $i = 0, 1, \ldots, q, \quad$ thus

$f(x) - \pi_q f(x) = (x - x_0)(x - x_1)\ldots(x - x_q)g(x)$ for some $g(x), x \in [a, b]$

To determine $g(x)$, we define a function φ and use the *generalized Rolle's theorem*:

$$\varphi(t) := f(t) - \pi_q f(t) - (t - x_0)(t - x_1)\ldots(t - x_q)g(x)$$

Note that $g(x)$ is independent of t! But $\varphi(t) = 0$ in the nodes, $x_i,\ i = 0, \ldots q$ as well as on $t = x$, i.e. $\varphi(x_0) = \varphi(x_1) = \ldots = \varphi(x_q) = \varphi(x) = 0$. Thus $\varphi(t)$ has $(q + 2)$ roots in the interval $[a, b]$.

Using linear interpolation, we can derive the following interpolation error approximating the derivative of f:

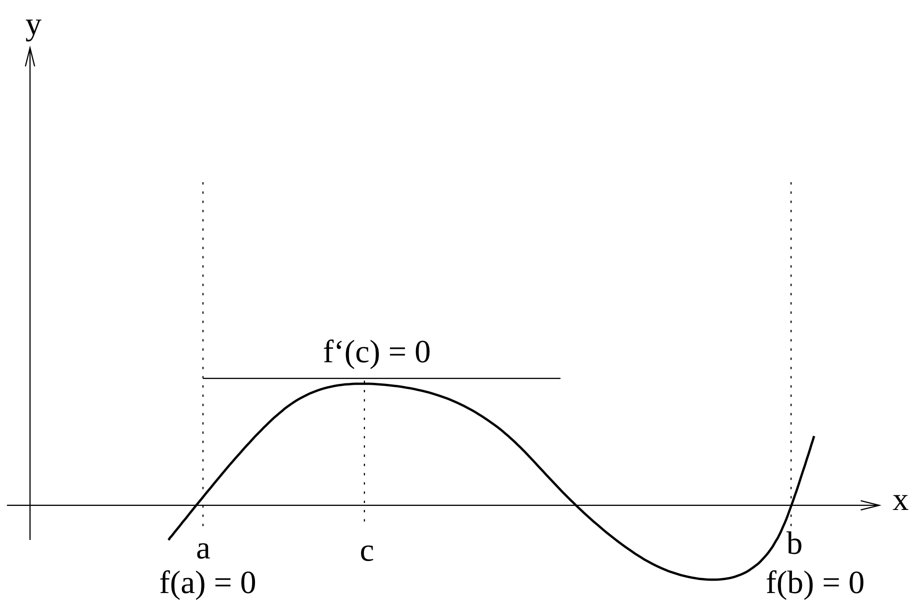
Theorem 5.2: Let $\xi_0 \le x \le \xi_1$ then

$$|f'(x) - (\pi_1 f)' f| \le \frac{(x - \xi_0)^2 + (x - \xi_1)^2}{2(\xi_1 - \xi_0)} \max_{[a,b]} |f''|.$$

The proof is straightforward, and therefore omited.

Rolle's theorem: If $f(x)$ is a continuous function on the closed, finite interval $[a, b]$ and differentiable on the open interval $(a, b), f(a) = f(b) = 0$, then there exists a point c in the open interval (a, b) such that

$$\frac{f(b) - f(a)}{b - a} = f'(c) = 0$$



Generalized Rolle's theorem:

If a continuous function $\varphi(x)$ has $(q+2)$ roots, $x_0, x_1, \ldots, x_q, x$, in a closed interval $[a, b]$, then there is a point ξ in the interval generated by $x_0, x_1, \ldots, x_q, x$, such that $\varphi^{(q+1)}(\xi) = 0$.

Differentiating $\varphi(t) = f(t) - \pi_q f(t) - (t-x_0)(t-x_1)\ldots(t-x_q)g(x)$, $(q+1)$ times with respect to t gives

$$\varphi^{(q+1)}(t) = f^{(q+1)}(t) - 0 - (q+1)!g(x)$$

because $\deg(\pi_q f(x)) = q$ and $(t-x_0)(t-x_1)\ldots(t-x_q) = t^{q+1} + \alpha t^q + \ldots$, (for some constant α), and $g(x)$ is independent of t.

Thus $0 = \varphi^{(q+1)}(\xi) = f^{(q+1)}(\xi) - (q+1)!g(x)$ and we have $g(x) = \dfrac{f^{(q+1)}(\xi)}{(q+1)!}$.

Hence the error in Lagrange interpolation is

$$E(x) = f(x) - \pi_q f(x) = \frac{f^{(q+1)}(\xi)}{(q+1)!} \prod_{i=0}^{q} (x - x_i)$$

Definitions: We assume that f is a real valued function such that integrals as well as the max on the left hand sides below are well-defined.

L_p**-norm** $\quad \|f(x)\|_{L^p(a,b)} = \left(\int_a^b |f(x)|^p dx \right)^{\frac{1}{p}} \quad 1 \le p < \infty$

L_∞**-norm** $\quad \|f(x)\|_{L^\infty(a,b)} = \max_{x \in [a,b]} |f(x)|$

Vector norm: Let $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ be two column, vectors we define their *scalar product* by

$$< x, y >= \mathbf{x}^T \mathbf{y} = x_1 y_1 + \cdots + x_n y_n,$$

and the vector norm $\|\mathbf{x}\|$ as

$$\|x\| = \sqrt{\langle x, x \rangle} = \sqrt{x_1^2 + \cdots + x_n^2}.$$

Theorem 5.3. For $q = 1$, i.e. only 2 interpolation nodes (the end-points of the interval), there are interpolation constants, c_i, independent of the function $f(x)$ and the interval (a, b) such that (CDE pp.79 - 84)

(1) $\|\pi_1 f - f\|_{L^\infty(a,b)} \leq c_i (b-a)^2 \|f''\|_{L^\infty(a,b)}$

(2) $\|\pi_1 f - f\|_{L^\infty(a,b)} \leq c_i (b-a) \|f'\|_{L^\infty(a,b)}$

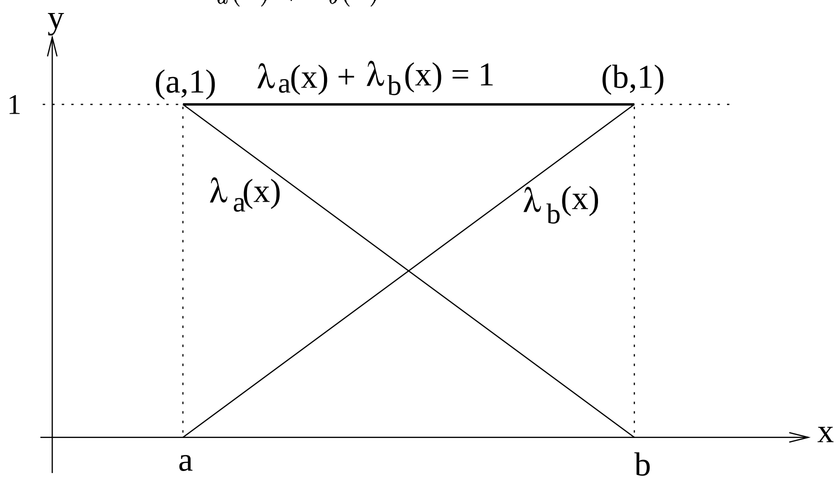
(3) $\|(\pi_1 f)' - f'\|_{L^\infty(a,b)} \leq c_i (b-a) \|f''\|_{L^\infty(a,b)}$

Proof:

$\underline{q=1} \Rightarrow$ $\pi_1 f(x)$ is a linear function. Consider a single interval $a \leq x \leq b$. Every linear function on $[a,b]$ can be written as a linear combination of

$$\lambda_a(x) \text{ and } \lambda_b(x), \text{ where } \lambda_a(x) = \frac{x-b}{a-b} \text{ and } \lambda_b(x) = \frac{x-a}{b-a}.$$

We have that $\lambda_a(x) + \lambda_b(x) = 1$.



Now $\pi_1 f(x) = f(a)\lambda_a(x) + f(b)\lambda_b(x)$.

By the Taylor's expansion for $f(a)$ and $f(b)$ about x we can write

$f(a) = f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\eta_a), \eta_a \in [a,x]$

$f(b) = f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b), \eta_b \in [x,b]$,

thus

$$\begin{aligned}\pi_1 f(x) =& [f(x) + (a-x)f'(x) + \frac{1}{2}(a-x)^2 f''(\eta_a)]\lambda_a(x) + \\ &+ [f(x) + (b-x)f'(x) + \frac{1}{2}(b-x)^2 f''(\eta_b)]\lambda_b(x)\end{aligned}$$

Rearranging the terms and using the fact that $(a-x)\lambda_a(x)+(b-x)\lambda_b(x)=0$ we get

$$\pi_1 f(x) = f(x)[\lambda_a(x)+\lambda_b(x)] + f'(x)[(a-x)\lambda_a(x)+(b-x)\lambda_b(x)]+ \\ +\frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x) + \frac{1}{2}(b-x)^2 f''(\eta_b)\lambda_b(x) = \\ = f(x) + \frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x) + \frac{1}{2},(b-x)^2 f''(\eta_b)\lambda_b(x)$$

we conclude that

$$|\pi_1 f(x) - f(x)| = |\frac{1}{2}(a-x)^2 f''(\eta_a)\lambda_a(x) + \frac{1}{2}(b-x)^2 f''(\eta_b)\lambda_b(x)|,$$

but

i. $a \le x \le b \Rightarrow (a-x)^2 \le (a-b)^2$

ii. $\lambda_a(x) \le 1$ and $\lambda_b(x) \le 1$ (according to the definition)

iii. $f''(\eta_a) \le \|f''(x)\|_{L^\infty(a,b)}$ and $f''(\eta_b) \le \|f''(x)\|_{L^\infty(a,b)}$

Thus

$$|\pi_1 f(x) - f(x)| \le \tfrac{1}{2}(a-b)^2 \cdot 1 \cdot |f''(x)\|_{L^\infty(a,b)} + \tfrac{1}{2}(a-b)^2 \cdot 1 \cdot \|f''(x)\|_{L^\infty(a,b)}$$

and hence

$$|\pi_1 f(x) - f(x)| \le (a-b)^2 \|f''(x)\|_{L^\infty(a,b)} \quad \text{with} \quad c_i = 1. \quad \square$$

The other two are proved similary!

We generalize theorem 5.3 to an arbitrary number of interpolation internals 9subdivisions):

Theorem 5.4.

Let $0 = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = 1$ be a partition of $[0,1]$ and $h = |x_{j+1} - x_j|,\ j = 0, 1, \ldots, n.$

Let $\pi_h v(x)$ be the piecewise linear interpolation of $v(x)$. Then there are interpolation constants c_i such that

(1) $\quad \|\pi_h v - v\|_{L_p} \le c_i \|h^2 v''\|_{L_p} \qquad p = 1, 2, \ldots, \infty$

(2) $\quad \|(\pi_h v)' - v'\|_{L_p} \le c_i \|h v''\|_{L_p}$

(3) $\quad \|\pi_h v - v\|_{L_p} \le c_i \|h v'\|_{L_p}$

For $p = \infty$ this is just theorem 5.3, applied to each subinterval.

Remark. For a uniform mesh we have h constant and therefore in this case h can be written outside the norms above.

The proof is also a generalization of the proof of theorem 5.3.

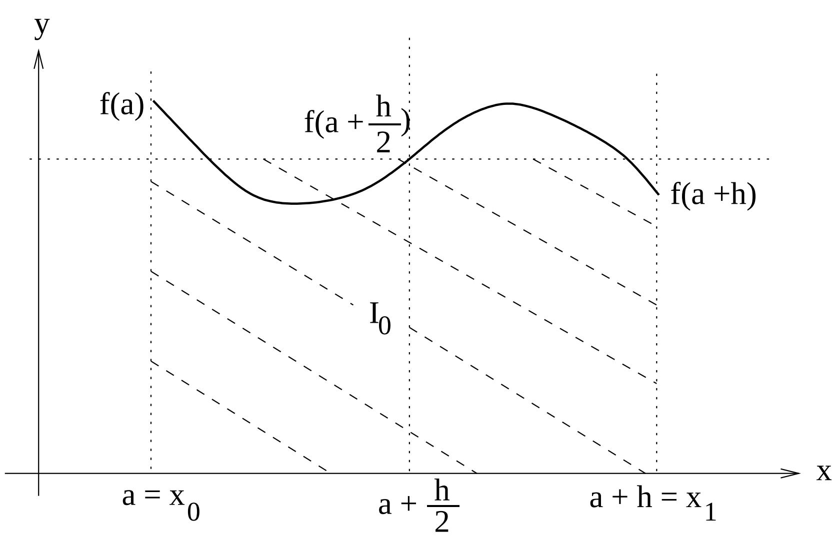
Numerical integration methods
(CDE pp. 95 - 100)

We want to approximate the integral $I = \int_a^b f(x)dx$ where, on each subinterval, we approximate f using piecewise polynomials of degree d. We denote the approximate value by I_d.

(1) *Midpoint rule:* (approximating f by constants on each subinterval)

Let $a = x_0 < x_1, x_2 < \ldots < x_n < x_{n+1} = b$ be a uniform partition of $[a, b]$ and $h = |x_{j+1} - x_j|, j = 0, 1, \ldots, n$. Then in the first interval and using the value of f at the MIDPOINT we get

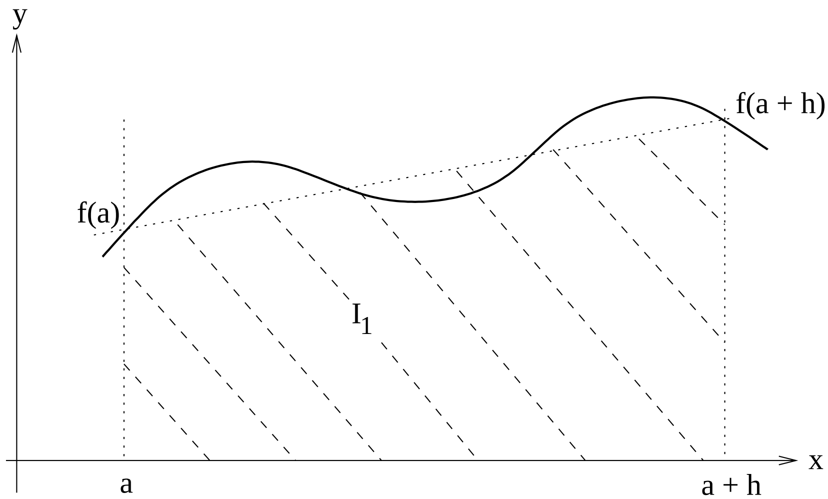
$$I \sim I_0 = h \cdot f(a + \frac{h}{2}).$$



(2) *Trapezoidal rule*: (approximating f by linear functions on each subinterval)

Analogously, this time using the values of f at the two end-points we are approximating the integral by the area of the TRAPEZOIDAL domain in figur below (for simplicity we have assumed $f \geq 0$).

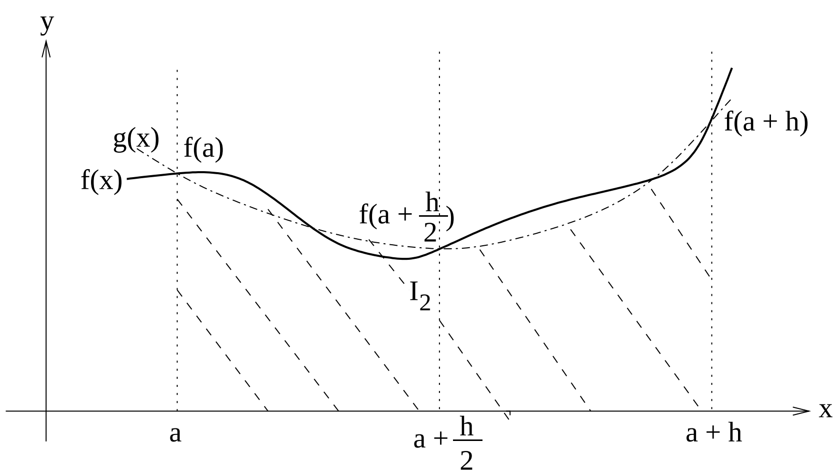
$$I \sim I_1 = h \cdot f(a) + \frac{h[f(a+h) - f(a)]}{2} = h\frac{f(a) + f(a+h)}{2}.$$



(3) *Simpson's rule*: (approximating f by quadratic functions on each subinterval)

This corresponds to a quadrature rule based on piecewise quadratic interpolation using the endpoints and midpoin of each sub-interval.

Let $g(x) = Ax^2 + Bx + C$, then $I \sim I_2 = \int_a^{a+h} g(x)dx$.



Since g interpolates f and the interpolation points are: $(a, f(a)), (a+\frac{h}{2}, f(a+\frac{h}{2}))$ and $(a+h, f(a+h))$, i.e. the graph of g passes through these 3 points and thus their coordinates must satisfy in the equation for $g(x)$. Hence we have

$$\begin{cases} f(a) = Aa^2 + Ba + C \\ f\left(a + \dfrac{h}{2}\right) = A\left(a + \dfrac{h}{2}\right)^2 + B(a + \dfrac{h}{2}) + C \\ f(a+h) = A(a+h)^2 + B(a+h) + C. \end{cases}$$

To solve this equation system, we can use the matrix form

$$\begin{pmatrix} a^2 & a & 1 \\ \left(a+\frac{h}{2}\right)^2 & a+\frac{h}{2} & 1 \\ (a+h)^2 & a+h & 1 \end{pmatrix} \begin{pmatrix} A \\ B \\ C \end{pmatrix} = \begin{pmatrix} f(a) \\ f\left(a+\frac{h}{2}\right) \\ f(a+h) \end{pmatrix}.$$

Remark. The rules (1), (2) and (3) use values of the function at *equally spaced* points.

(4) *Gauss quadrature rule.* This is to choose the points of evolution in an *optimal* manner, not at equally spaced points. We demonstrate this rule through an example viz:

Problem: Choose the nodes $x_i \in [a, b]$, and coefficients c_i, $1 \le i \le n$ minimizing the error

$$\int_a^b f(x)dx - \sum_{i=1}^n c_i f(x_i) \text{ for an } \textit{arbitrary} \text{ function } f(x).$$

Solution. There are $2n$ parameters consisting of n nodes x_i and n coefficients c_i. Optimal choice of these parameters produces the quadrature rule which, exactly, determines $2n$ parameters. Thus is *exact* for *polynomials* of *degree* $\le 2n-1$.

<u>Ex.</u> Let $n = 2$ and $[a, b] = [-1, 1]$.
Then the coefficients are c_1 and c_2 and the nodes are x_1 and x_2.

Thus

$$\int_{-1}^1 f(x)dx \approx c_1 f(x_1) + c_2 f(x_2),$$

is exact for polynomials of degree ≤ 3. The basis for polynomials of degree 3 are $1, x, x^2$ and x^3. Hence for all functions f of the form $f(x) = Ax^3 + Bx^2 + Cx + D$ the approximation above is indeed an equality. Thus to determine the coefficients c_1, c_2 and the nodes x_1, x_2, it suffices to use the above approximation as equality when f is replaced by the basis functions $1, x, x^2$ and x^3:

$$\int_{-1}^{1} 1dx = c_1 + c_2 \text{ and we get } [x]_{-1}^{1} = 2 = c_1 + c_2$$
$$\int_{-1}^{1} xdx = c_1 \cdot x_1 + c_2 \cdot x_2 \text{ and } \left[\frac{x^2}{2}\right]_{-1}^{1} = 0 = c_1 \cdot x_1 + c_2 \cdot x_2$$
$$\int_{-1}^{1} x^2dx = c_1 \cdot x_1^2 + c_2 \cdot x_2^2 \text{ and } \left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3} = c_1 \cdot x_1^2 + c_2 \cdot x_2^2$$
$$\int_{-1}^{1} x^3dx = c_1 \cdot x_1^3 + c_2 \cdot x_2^3 \text{ and } \left[\frac{x^4}{4}\right]_{-1}^{1} = 0 = c_1 \cdot x_1^3 + c_2 \cdot x_2^3$$

Summing up:

$$\begin{cases} c_1 + c_2 = 2 \\ c_1x_1 + c_2x_2 = 0 \\ c_1x_1^2 + c_2x_2^2 = \frac{2}{3} \\ c_1x_1^3 + c_2x_2^3 = 0 \end{cases} \text{ this } 4 \times 4 \text{ system of equations gives } \begin{cases} c_1 = 1 \\ c_2 = 1 \\ x_1 = -\frac{\sqrt{3}}{3} \\ x_2 = \frac{\sqrt{3}}{3} \end{cases}$$

Thus the approximation

$$\int_{-1}^{1} f(x)dx \approx c_1f(x_1) + c_2f(x_2) = f\Big(-\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big)$$

is exact for polynomials of degree ≤ 3.

Ex. Let $f(x) = 3x^2 + 2x + 1$

Then $\int_{-1}^{1}(3x^2 + 2x + 1)dx = [x^3 + x^2 + x]_{-1}^{1} = 4$ and

$$f\Big(-\frac{\sqrt{3}}{3}\Big) + f\Big(\frac{\sqrt{3}}{3}\Big) = 3 \cdot \frac{3}{9} - 2 \cdot \frac{\sqrt{3}}{3} + 1 + 3 \cdot \frac{3}{9} + 2 \cdot \frac{\sqrt{3}}{3} + 1 = 4,$$

which is the exact solution of the integral.

Generalized Gauss quadrature. To generalize Gauss quadrature rule Legendre polynomials are used:

Choose $\{P_n\}_{n=0}^{\infty}$ such that

(1) For each n, P_n is a polynomial of degree n.

(2) $P_n \perp P_m$ if $m < n \Leftrightarrow \int_{-1}^{1} P_n(x)P_m(x)dx = 0$

The *Legendre* polynomial formula:

$$P_k(x) = (-1)^k \frac{d^k}{dx^k}(x^k(1-x)^k) \text{ or } P_n(x) = \frac{2}{2^n n! dx^n}(x^2-1)^n$$

i.e. $P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

The roots of these polynomials are *distinct*, *symmetric* and *correct choices* as *quadrature* points, i.e. they are giving the points $x_i, 1 \leq i \leq n$, as the roots of the n-th Legendre polynomial. ($p_0 = 1$ is an exception).

Ex. Roots to the Legendre polynoms is *quadrature* points:

$P_1(x) = x = 0$

$P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} = 0$ gives $x_{1,2} = \pm\frac{\sqrt{3}}{3}$
(compare with the result above)

$P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x = 0$ gives $x_1 = 0, x_{2,3} = \pm\sqrt{\frac{3}{5}}$

Recall $\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} c_i f(x_i)$ is *exact* for polynomials of degree $\leq 2n-1$.

Theorem 5.5. (Not in CDE)

Suppose that $x_i, i = 1, 2, \ldots, n$, are roots of n-th Legendre polynomial P_n and that

$$c_i = \int_{-1}^{1} \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) dx, \text{ where } \prod_{\substack{j=1 \\ j \neq i}}^{n} \left(\frac{x - x_j}{x_i - x_j}\right) \text{ is the Lagrange basis.}$$

If $f(x)$ is a polynomial of degree $< 2n$, then $\int_{-1}^{1} f(x)dx \equiv \sum_{i=1}^{n} c_i f(x_i)$.

Proof: Consider a polynomial $R(x)$ of degree $< n$. Rewrite $R(x)$ as $(n-1)$ Lagrange polynomials with nodes at the roots of the n-th Legendre polynomial P_n. This representation of $R(x)$ is exact, since the error is

$$E(x) = \frac{1}{n!}(x - x_1)(x - x_2)\ldots(x - x_n)R^{(n)}(\xi), \quad \text{where } R^{(n)}(\xi) \equiv 0.$$

Further we have $R(x) = \sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \Big(\frac{x - x_j}{x_i - x_j}\Big) R(x_i)$, so that

$$\int_{-1}^{1} R(x)dx = \int_{-1}^{1} \Big[\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \Big(\frac{x - x_j}{x_i - x_j}\Big) R(x_i)\Big] dx = \sum_{i=1} \underbrace{\Big[\int_{-1}^{1} \prod_{\substack{j=1 \\ j\neq i}}^{n} \Big(\frac{x - x_j}{x_i - x_j}\Big) dx\Big]}_{=c_i} R(x_i).$$

Moreover

(i) $$\int_{-1}^{1} R(x)dx = \sum_{i=1}^{n} c_i R(x_i)$$

Now consider a polynomial, $P(x)$, of degree $< 2n$. Divide $P(x)$ by the n-th Legendre polynomial $P_n(x)$.

$$P(x) = Q(x) \times P_n(x) + R(x)$$

we have that

$$\deg P(x) < 2n, \quad \deg Q(x) < n, \quad \deg P_n(x) = n, \quad \deg r(x) < n.$$

and

$$\int_{-1}^{1} P(x)dx = \int_{-1}^{1} Q(x)P_n(x)dx + \int_{-1}^{1} R(x)dx$$

Since $Q(x) \perp P_n(x), \forall Q(x)$ with degree $< n$, then $\int_{-1}^{1} Q(x)P_n(x)dx = 0$, and we get

$$\int_{-1}^{1} P(x)dx = \int_{-1}^{1} R(x)dx$$

Then x_i's are roots of $P_n(x)$ and thus $P_n(x_i) = 0$ and we can use (ii) to write

$$P(x_i) = Q(x_i)P_n(x_i) + R(x_i) \text{ and we get } P(x_i) = R(x_i)$$

Now we have $\int_{-1}^{1} P(x)dx = \int_{-1}^{1} R(x)dx = (i) = \sum_{i=1}^{n} c_i R(x_i) = \sum_{i=1}^{n} c_i P(x_i)$.

Summing up: $\int_{-1}^{1} P(x)dx = \sum_{i=1}^{n} c_i P(x_i)$. $\square$

From Chapter 5 you at least need to know:

Lagrange interpolation

Taylor interpolation error
Lagrange interpolation error

Definitions:	Lebesgue p-norms Lebesgue max-norm
Theorem 5.3.	Estimate of errors in lebesgue-norms w.r.t. Lagrange interpolation
Theorem 5.4.	Estimate of errors in Lebesgue-norms w.r.t. piecewise linear Interpolation

Gauss quadrature rule