Chapter 6. Galerkin's Method

Galerkin was born in 1871 in Russia. He began doing research in engineering while he was in prison in 1906 - 1907 for his participation in the anti-tsarist revolutionary movement. His method was introduced in a paper on elasticity published in 1915. (CDE p. 127)

Galerkin's method for solving a general differential equation is based on seeking an approximate solution, which is

1. easy to differentiate and integrate
2. spanned by a set of nearly orthogonal basis functions in a finite-dimensional space.

Approximate solution

Ex. Let $u(t)$ be the solution to the *ordinary* differential equation given by $u'(t) - \lambda u(t) = 0$, and let $U(t)$ be the approximate solution spanned by the basis functions 1, t and t^2. Thus

$$U(t) = A \cdot 1 + B \cdot t + C \cdot t^2 \text{ and } U'(t) = B + 2C \cdot t.$$

Inserting $U(t)$ and $U'(t)$ in the differential equation, we get

$$\begin{aligned} &B + 2C \cdot t - \lambda(A \cdot 1 + B \cdot t + C \cdot t^2) = 0, \text{ and thus} \\ &-\lambda C t^2 + (2C - \lambda B)t + B - \lambda A = 0. \end{aligned}$$

This is a simple algebraic equation, however we need three different equations to calculate A, B and C.

The Galerkin method using the *Galerkin orthogonality* property of the approximate solution $U(t)$ avoids this complexity.

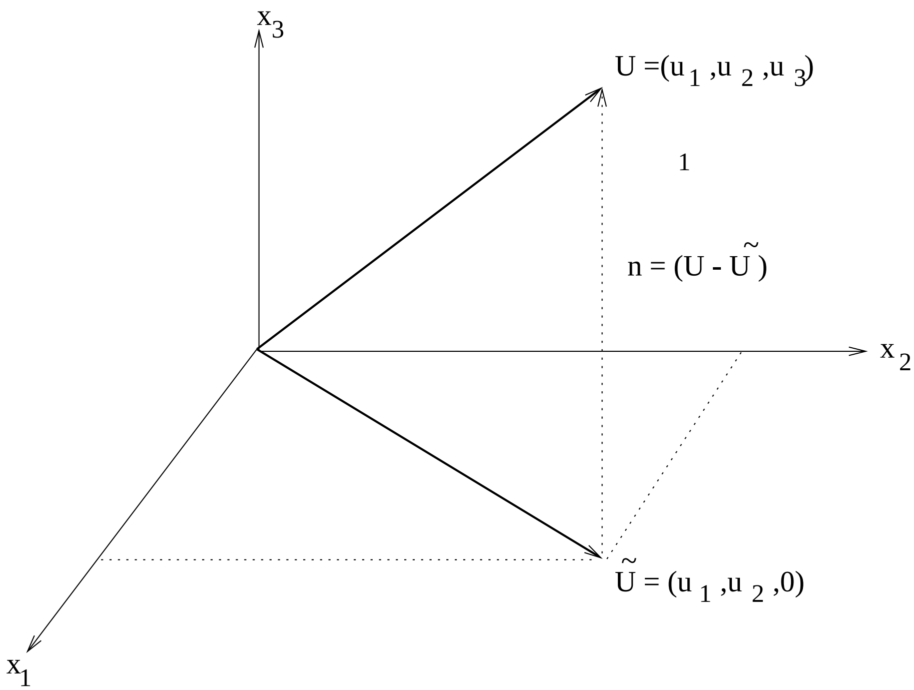
Galerkin's method and orthogonal projection

Projection in $\mathbb{R}^2$:

Let $u = (u_1, u_2, u_3)$ and assume that for some reasons we only have u_1 and u_2 available. Letting $x = (x_1, x_2, x_3) \in \mathbb{R}^3$,

the objective, then is to find $\tilde{U} \in \{x : x_3 = 0\}$ such that $(U - \tilde{U})$ is as small as possible.

For orthogonal projection; $z \cdot n = 0$, for all $z \in \{x : x \cdot n = 0, x_3 = 0\}$.



Obviously in this case $\tilde{U} = (u_1, u_2, 0)$ and we have $(U - \tilde{U}) \perp \tilde{U}$.

Note! If $\mathbf{u} \in \mathbb{R}^n$, $\mathbf{u} = (u_1, u_2, \ldots, u_{n-1}, u_n)$, and

$\mathbf{u}_m = (u_1, u_2, \ldots, u_m, u_{m+1} = 0, \ldots, u_n = 0)$, then the Euclidian distance:

$|\mathbf{u} - \mathbf{u}_m| = \sqrt{u_{m+1}^2 + u_{m+2}^2 + \ldots + u_n^2} \to 0$ as $m \to n$.

Nearly orthogonal basis functions

Definition: A set of functions or vectors V build a *linear space* if $\forall u, v \in V$ and $\alpha \in \mathbb{R}$, we have that

$$\begin{array}{rl} \text{(i)} & u + \alpha v \in V \\ \text{(ii)} & u + v = v + u \\ \text{(iii)} & \exists(-u) \text{ such that } u + (-u) = 0 \end{array}$$

Definition: W is a *scalar product* space if W is a linear space and there is a real valued *scalar product operator*, $< \cdot, \cdot >$, defined on $W \times W$, such that that

$$\begin{array}{ll} \text{(i)} & < u, v > = < v, u >, \qquad \forall u, v \in W \qquad \text{(symmerty)} \\ \text{(ii)} & < u + \alpha v, w > = < u, w > + \alpha < v, w >, \\ & \quad \forall u, v, w \in W, \qquad \alpha \in \mathbb{R}, \quad \text{(bilinearity)} \end{array}$$

Definition: A usual scalar product for two real valued functions $u(x)$ and $v(x)$ is defined by

$$\langle u, v\rangle = \int_0^T u(x)v(x)dx,$$

Definition: $u(x)$ and $v(x)$ are orthogonal if $\langle u, v\rangle = 0$.

Definition: A norm associated with this scalar product is defined by

$$\|u\| = \sqrt{\langle u, u\rangle} = \langle u, u\rangle^{\frac{1}{2}} = \Big(\int_0^T |u(x)|^2 dx\Big)^{\frac{1}{2}}$$

and is called the L_2 norm of $u(x)$.

Note! The difference between a vector space and a function space in $\mathbb{R}^2$

Ex. Vector space	Ex. Function space
Basis $x = (1,0)$ and $y = (0,1)$	$f(t) = t$ and $g(t) = t^2, t \in [0,1]$
Norm $\|x\| = \langle x, x\rangle^{\frac{1}{2}} = \sqrt{x_1^2 + x_2^2}$	$\|u\| = \langle u, u\rangle^{\frac{1}{2}} = \Big(\int_0^1 |u(t)|^2 dt\Big)^{\frac{1}{2}}$
	$\langle f(t), g(t)\rangle = \int_0^1 f(t)g(t)dt$
$\langle x, y\rangle = \langle (1,0),(0,1)\rangle = 1\cdot 0 + 0 \cdot 1 = 0$	$\langle t, t^2\rangle = \int_0^1 t\cdot t^2 dt = \Big[\frac{t^4}{4}\Big]_0^1 = \frac{1}{4} \neq 0$
Then we can conclude that $x \perp y$,	but t and t^2 are not orthogonal.

Here we recall one of the most useful inequalities, *Cauchy-Schwartz inequality*:

$$|\langle u, v\rangle| \leq \|u\| \cdot \|v\|$$

Some usual spaces

a. $v \in C^n$ if v and all its partial derivatives of order $\leq n$ are continuous.
   Thus C^0 denotes the set of continuous functions
   Ex. $C^k([0,T])$ is the space of all functions having derivatives of order $\leq k$ to be continuous on $[0,T]$.
   Ex. Let $x \in \Omega \subseteq R^n; v(t,x) \in C^1(\mathbb{R}^+, C^2(\Omega))$, i.e.

$$\frac{\partial u}{\partial t} \text{ and } \frac{\partial^2 u}{\partial x_i \partial x_j} \quad i,j = 1,\ldots,n \text{ are continuous.}$$

b. $P^q(a,b) = \{$The space of polynomials in x of degree $\leq q, a \leq x \leq b\}$.

A possible basis for $P^q(0,1)$ would be $\{x^j\}_{j=0}^q = \{1, x, x^2, x^3, \ldots, x^q\}$.

The dimension of P^q is therefore $q+1$.

Ex. The Taylor polynomial of degree q of a function $u(x)$ at x_0:

$$u(x_0) + u'(x_0)(x - x_0) + \frac{1}{2}u''(x_0)(x - x_0)^2 + \ldots + \frac{1}{q!}D^{(q)}u(x_0)(x - x_0)^q$$

Let $V^{(q)}(0,1) = \{\underbrace{x^0}_{=1}, x^1, x^2, \ldots, x^q\} = \{1, x, x^2, \ldots, x^q\}, 0 \leq x \leq 1$ and

$V_0^{(q)}(0,1) = \{v : v \in V^{(q)}, v(0) = v(1) = 0\}$, then $V_0^{(q)} \subset V^{(q)} \subset P^q(0,1)$.

c. Legendre polynomials are given by

$$P_k(x) = (-1)^k \frac{d^k}{dx^k}(x^k(1-x)^k) \text{ or } P_n(x) = \frac{2}{2^n n!}\frac{d^n}{dx^n}(x^2 - 1)^n$$

Ex. $P_0(x) = 1 \quad P_1(x) = x \quad P_2(x) = \frac{3}{2}x^2 - \frac{1}{2} \quad P_3(x) = \frac{5}{2}x^3 - \frac{3}{2}x$

d. Trigonometric polynomials.

$$T^q = \left\{ f(x) = \sum_{k=0}^{q} \left(a_k \cos\left(\frac{2\pi}{T}kx\right) + \beta_k \sin\left(\frac{2\pi}{T}kx\right)\right)\right\}$$

e. Lagrange bases $\{\lambda_i\}_{i=0}^q \in P^q(a,b)$ associated to the distinct $(q+1)$ points $\xi_0 < \xi_1 < \ldots < \xi_q$ in (a,b) determined by the requirement that $\lambda_i(\xi_j) = 1$ if $i = j$, and 0 otherwise.

$$\lambda_i(x) = \prod_{k \neq i} \frac{x - \xi_k}{\xi_i - \xi_k}$$

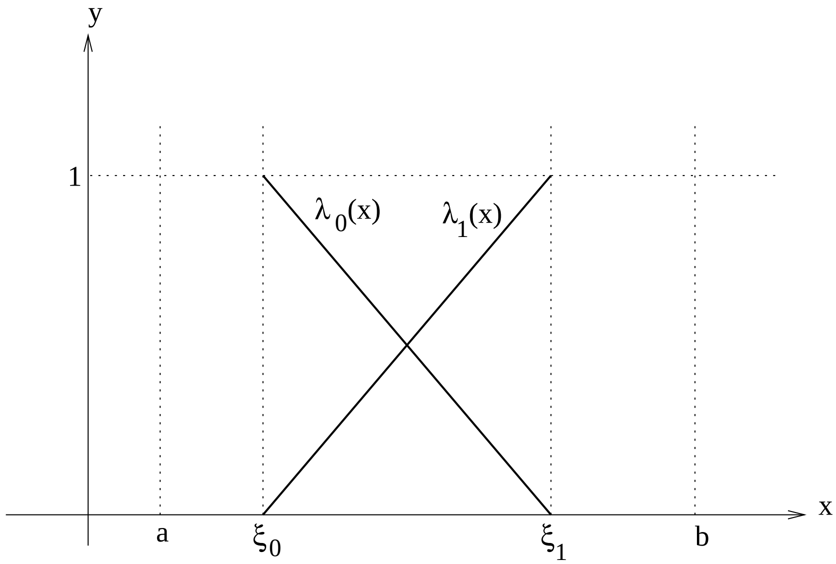
The polynomial $p \in P^q(a,b)$ that has the value $p_i = p(\xi_i)$ at the nodes $x = \xi_i$ for $i = 0, 1, \ldots, q$ expressed in terms of the corresponding Lagrange basis is given by

$$p(x) = p_0\lambda_0(x) + p_1\lambda_1(x) + \ldots + p_q\lambda_q(x).$$

Note! For every node $x = \xi_i$ we have associated a base function $\lambda_i(x), i = 0, 1, \ldots, q$, thus if $p \in P^q(a,b)$, then we have $q+1$ nodes and $(q+1)$ basis functions.

Ex. Linear Lagrange basis functions for $q = 1$ are

$$\lambda_0(x) = (x - \xi_1)/(\xi_0 - \xi_1) \text{ and } \lambda_1(x) = (x - \xi_0)/(\xi_1 - \xi_0)$$



f. Polynomial interpolant $\pi_q f \in P^q(a,b)$ of a continuous function $f(x)$ on $[a,b]$:

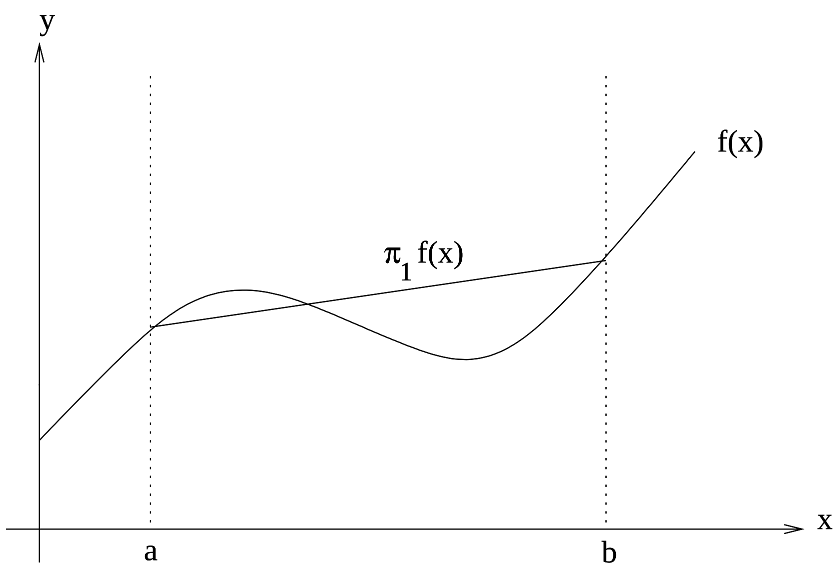
Choose distinct interpolation nodes $a = \xi_0 < \xi_1 < \ldots < \xi_q = b$.

$\pi_q f \in P^q(a,b)$ interpolates $f(x)$ at the nodes $\{\xi_i\}$ for $i = 0, \ldots, q$.

Note! There are $(q+1)$ nodes for which $\pi_q f(\xi_i) = f(\xi_i)$.

Now Lagrange's formula gives

$$\pi_q f(x) = f(\xi_0)\lambda_0(x) + f(\xi_1)\lambda_1(x) + \ldots + (f(\xi_q)\lambda_q(x) \text{ for } a \leq x \leq b$$



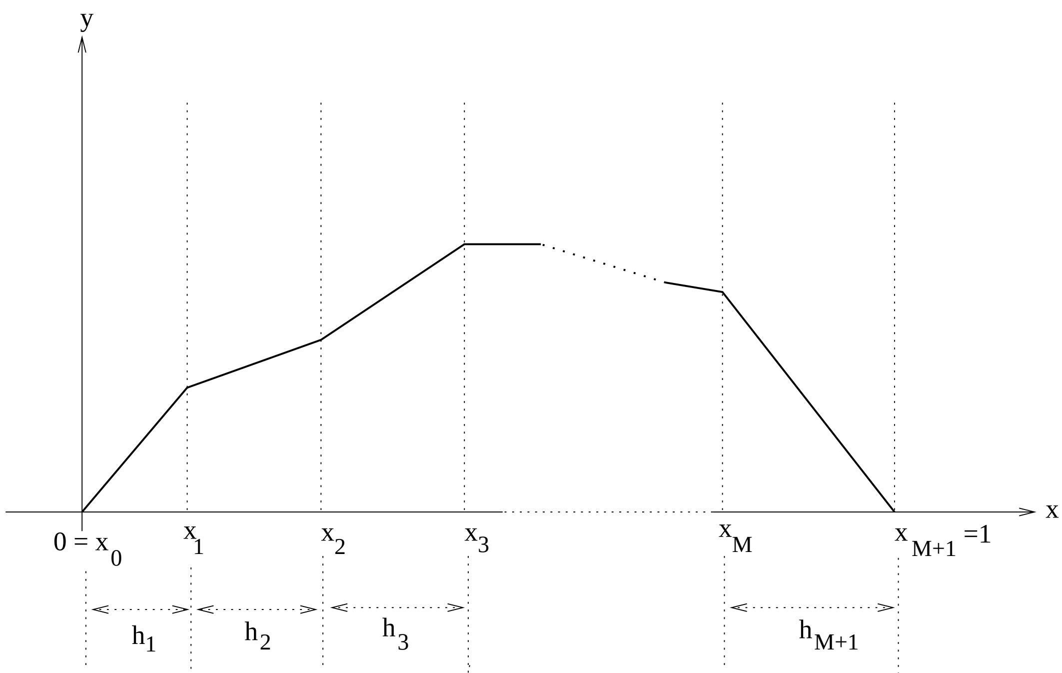
Ex. For $q = 1$ there are $(q+1) = 2$ nodes, $f(a)$ and $f(b)$, and 2 bases, $\lambda_0(x), \lambda_1(x)$. We have $\xi_0 = a$ and $\xi_1 = b$, then

$$\lambda_i(x) = \prod_{j \neq i} \frac{x - \xi_j}{\xi_i - \xi_j} \text{ gives } \lambda_0(x) = \frac{x-b}{a-b} \text{ and } \lambda_1(x) = \frac{x-a}{b-a} \text{ and}$$

we get the polynomial interpolant

$$\pi_1 f(x) = f(a)\frac{x-b}{a-b} + f(b)\frac{x-a}{b-a}$$

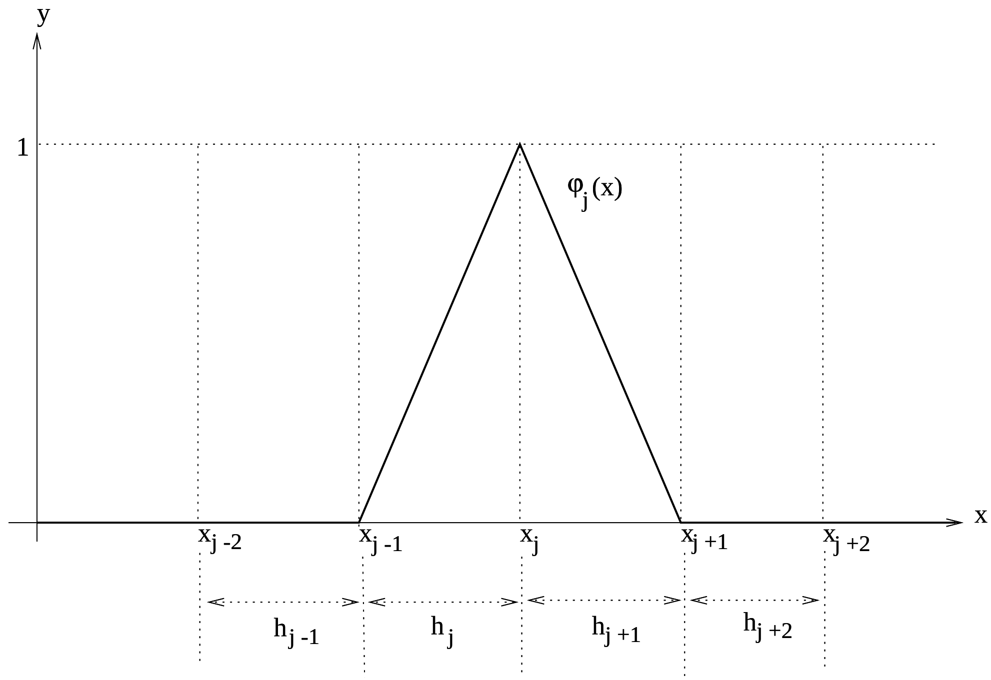
g. $V_h^{(q)} = \{v : v$ is continuous piecewise linear function on $T_h\}, V_h^{(q)} \subset P^q(0,1)$, where $Th : 0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = 1, q = M+1$, with $h_j = x_j - x_{j-1}$, is a partition of $(0,1)$ into $(M+1)$ subintervals.



Note! $\overset{\circ}{V}{}_h^{(q)} = \{v : v \in \circ V_h^{(q)}, v(0) = v(1) = 0\}$.

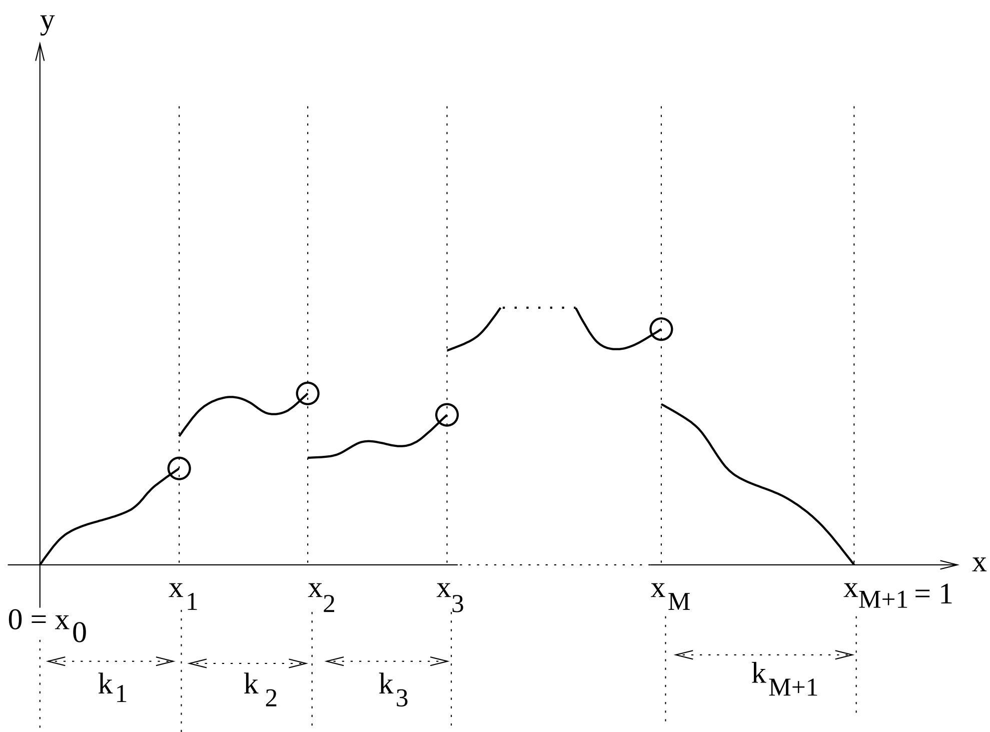
One usual basis for V_h is $\varphi_j(x)$:

$$\varphi_i(x) = \begin{cases} \frac{x_{i+1}-x}{h_{i+1}} & x_i \leq x \leq x_{i+1} \\ \frac{x - x_{x_i - 1}}{h_i} & x_{i-1} \leq x \leq x_i \\ 0 & x \leq x_{i-1} \qquad x_{i+1} \leq x \end{cases}$$



h. $W_k^{(q-1)} \subset P^q(0,1)$

$W_k^{(q-1)} = \{w : w$ discontinuous piecewise polynoms of degree $(q-1)$ on $T_k\}$.



i. $H^s \subseteq C^s(0,T)$

$$H^s = \left\{ f : \|f\| + \sum_{k\leq s} \left\| \frac{\partial^k f}{\partial t^k} \right\| < \infty \right\}$$

Examples of IVP

(1) An initial value problem, (IVP), in population dynamics:

$$\begin{cases} \dot{u}(t) = \lambda \cdot u(t) \quad 0 < t < 1 \\ u(0) = u_0 \end{cases} \quad \text{where}$$

$\dot{u}(t) = \frac{du}{dt}$, λ is a positive constant.

This equation has the increasing, analytic solution $u(t) = u_0 \cdot e^{\lambda \cdot t}$, which would blow up as $t \to \infty, (\lambda > 0)$.

In *general* we have $\dot{u}(t) = F(u,t)$, where $u(t) \in R^n$ and $t \in R^+$, thus

$$u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ \dots \\ u_n(t) \end{pmatrix} = (u_1(t), u_2(t), \dots, u_n(t))^T \text{ and } F : R^n \times R^+ \to R^n$$

(2) (PDE) $\begin{cases} -\Delta u(x) + ab \cdot \nabla u(x) = f \quad x \in \Omega \\ u(x) = 0 \quad x \in \partial\Omega \end{cases}$, where

$$b\nabla_x(u) = (b_1, b_2, \dots, b_n) \begin{pmatrix} u_{x_1} \\ u_{x_2} \\ \dots \\ u_{x_n} \end{pmatrix}$$

(3) Heat equation with Neumann boundary condition:

$$(\text{PDE}) \quad \begin{cases} \frac{\partial u}{\partial t} = \Delta u \quad x \in \Omega \\ \frac{\partial u}{\partial n} = 0 \quad x \in \partial\Omega \end{cases}$$

Numerical solutions of (IVP)

(a) A *finite difference* method.

Approximate with explicit forward Euler method.

$$(1) \qquad \begin{cases} \frac{u(t_{k+1})-u(t_k)}{t_{k+1}-t_k} = \lambda \cdot u(t_k) \\ u(0) = u_0 \end{cases}, \text{ where } \dot{u}(t) \approx \frac{u(t_{k+1}) - u(t_k)}{t_{k+1} - t_k}$$

There are corresponding finite difference methods for PDE's.

(b) *Galerkin's metod.* A finite element method for approximating (IVP).

Let $U(t)$ be an approximation of the real solution $u(t)$ of the equation (1), then

$$\dot{u}(t) - \lambda \cdot u(t) = 0 \text{ and}$$
$$\dot{U}(t) - \lambda \cdot U(t) \neq 0$$

<u>Definition:</u> If $U(t)$ is an approximation of $u(t)$, then

$$R(U(t)) = \dot{U}(t) - \lambda \cdot U(t)$$

is called the residual *error* of $U(t)$.

We have $V^{(q)} = \{\xi_0, \xi_1 t, \xi_2 t^2, \ldots, \xi_q t^q\}$ and $V_0^{(q)} = \{\xi_1 t, \xi_2 t^2, \ldots, \xi_q t^q\}$.

Multiply the equation (1) by a function $v(t) \in V_0^{(q)}$ and integrate!

$\int_0^T u'(t)v(t)dt = \lambda \cdot \int_0^T u(t) \cdot v(t)dt, \quad \forall v(t) \in V_0^{(q)} = \{\xi_1 t, \xi_2 t^2, \ldots, \xi_q t^q\}$, then

$\int_0^T (u'(t) - \lambda \cdot u(t))v(t)dt = 0$

Now we want to find an approximate solution $U(t)$ in the *trial space*,

$$V^{(q)} = \{\xi_0, \xi_1 t, \xi_2 t^2, \ldots, \xi_q t^q\}, \xi_0 = v(0), \xi_k \in \mathbb{R}, 0 \leq k \leq q$$

Note! If $v(t) \in V^{(q)}$, then $v(0) = \xi_0 + 0 + \ldots 0 = \xi_0$

If $v(t) \in V_0^{(q)}$, then $v(0) = 0 + 0 + \ldots + 0 = 0$

As above the residual $R(U(t))$ is orthogonal to the *test function space*,

$$V_0^{(q)} = \{v(t) \in V^{(q)} : v(0) = 0\} = \{\xi_1 t, \xi_2 t^2, \dots, \xi_q t^q\},$$

Note! $\quad V_0^{(q)} \subseteq V^{(q)}$ and $R(U(t)) \perp v(t); \forall v(t) \in V^{(q)}$.

In our case the real solution belongs to $C((0,T))$, or better to H^s which is a subspace of $C((0,T))$.

We look for a solution $U(t)$ in a finite dimensional subspace e.g. $V^{(q)}$.

The approximate differential equation is now

$$\begin{cases} \dot{U}(t) = \lambda \cdot U(t) \quad 0 < t < 1 \\ U(0) = u_0 \end{cases}$$

1. Multiply the differential equation by a function $v(t)$ from the test function space. Since $R(U(t)) \perp v(t)$ and according to the definition we have $R(U(t)) = \dot{U}(t) - \lambda \cdot U(t)$, thus

$$\int_0^1 (\dot{U}(t) - \lambda \cdot U(t))v(t)dt = \int_0^1 R(U(t))v(t)dt = \langle R(U(t)), v(t)\rangle = 0,\ \forall v(t) \in V_0^{(q)}$$

   Then the *Galerkin method is formulated* as follows:

   Given $u(t)$, find the approximate solution $U(t) \in V^{(q)}$, such that

$$(2)\dots \int_0^1 R(U(t))v(t)dt = \int_0^1 (U'(t) - \lambda \cdot U(t))v(t)dt = 0,\ \forall v(t) \in V_0^{(q)}$$

2. If $U(t) = \sum_{k=0}^q \xi_k t^k$, then $\dot{U}(t) = \sum_{k=1}^q k\xi_k t^{k-1}$ and $v_j(t) = t^j, j = 1, 2, \dots, q$.

   Since $v(t) \in V_0^{(q)}$ we have $v_0(t) = 0, v_1(t) = t, v_2(t) = t^2, \dots, v_q(t) = t^q$, which inserting in (2) implies that

$$\int_0^1 \Big(\sum_{k=1}^q k\xi_k t^{k-1} - \lambda \sum_{k=0}^q \xi_k t^k\Big) \cdot t^j dt = 0, \quad j = 1, 2, \dots, q$$

   This equation we can rewrite as

$$\int_0^1 \Big(\sum_{k=1}^q (k\xi_k t^{k+j-1} - \lambda \cdot \xi_k t^{k+j}) - \lambda\xi_0 \cdot t^j\Big) dt = 0$$

Integrate! ξ_k and k are constants independent of t.

$$\sum_{k=1}^{q} \xi_k \Big[k \cdot \frac{t^{k+1}}{k+j} - \lambda \frac{t^{j+k+1}}{j+k+1}\Big]_{t=0}^{t=1} - \Big[\lambda \cdot \xi_0 \frac{t^{j+1}}{j+1}\Big]_{t=0}^{t=1} = 0, \quad \text{then}$$

$$\sum_{k=1}^{q} \Big(\frac{k}{k+j} - \frac{\lambda}{k+j+1}\Big)\xi_k = \frac{\lambda}{j+1} \cdot \xi_0 \quad j = 1, 2, \ldots, q$$

Ex. Let $\lambda = 1, \xi_0 = 1$ and $q = 2$

$j = 1$

$$\Big(\frac{1}{1+1} - \frac{1}{1+1+1}\Big) \cdot \xi_1 + \Big(\frac{2}{2+1} - \frac{1}{2+1+1}\Big)\xi_2 = \frac{1}{1+1} \cdot 1 \text{ gives } \frac{1}{6} \cdot \xi_1 + \frac{5}{12} \cdot \xi_2 = \frac{1}{2}$$

$j = 2$

$$\Big(\frac{1}{1+2} - \frac{1}{1+2+1}\Big) \cdot \xi_1 + \Big(\frac{2}{2+2} - \frac{1}{2+2+1}\Big)\xi_2 = \frac{1}{2+1} \cdot 1 \text{ gives } \frac{1}{12} \cdot \xi_1 + \frac{3}{10} \cdot \xi_2 = \frac{1}{3},$$

then we have the equation system

$$\begin{cases} 2\xi_1 + 5\xi_2 = 6 \\ 5\xi_1 + 18\xi_2 = 20 \end{cases}, \text{ which gives } \xi_1 = 0.22\ldots \quad \xi_2 = 1.11\ldots$$

Now let the approximate solution be $U(t) = 1 + 0,22 \cdot t + 1,11 \cdot t^2$, then

$\dot{U}(t) = 0,22 + 2,22 \cdot t$.

Note! The *residual error*, $R(U(t))$, of $U(t)$ for this example is

$$R(U(t)) = \dot{U}(t) - \lambda \cdot Ut) = \dot{U}(t) - U(t) = 0,22 + 2,22t - 1 - 0,22t - 1,11t^2,$$

thus

$$R(U(t)) = -0,88 + 2 \cdot t - 1,11 \cdot t - 1,11 \cdot t^2 \quad \text{(We want } R(U(t)) \cong 0).$$

Hence $a_{jk} = \dfrac{k}{k+j} - \dfrac{\lambda}{k+j+1}$, although invertible, is *ill-conditioned*, mostly because $\{t^j\}_{j=1}^{q}$ does not form an orthogonal basis.

Instead the use of Legendre OG-polynomials would make the problem well conditoned.

Heat equation. The finite element method
(CDE pp. 113-114)

Heat equations are separated in **a.** stationary heat equations, $\dot{u} = \frac{du}{dt} = 0$

b. time dependent heat equation $\dot{u} = \frac{du}{dt} \neq 0$

Ex. (PDE) Stationary (time independent) heat equation in 1D

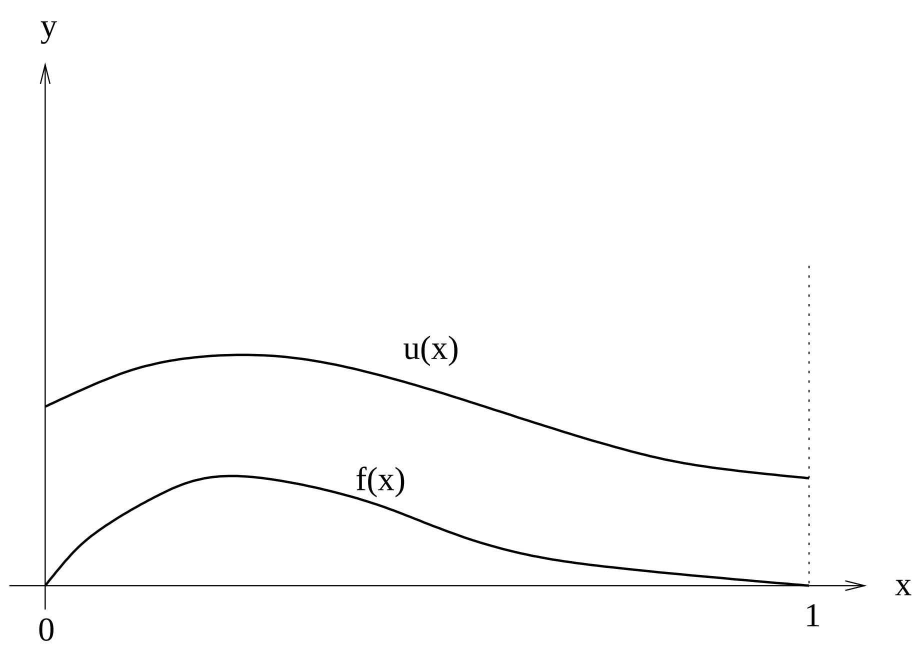
Notation:

$u(x)$ is the temperature $x \in (0,1)$

$q(x)$ is the heat flux in the direction of the positive x-axis

$f(x)$ is the heat source

$a(x)$ is the heat conductivity coefficient



(i) **Conservation of energy:**

Note! In 1D case there is only one heat direction, along the x-axis!

Heat flux through end points x_1 and x_2, i.e. the heat produced in (x_1, x_2) per unit time:

$$q(x_2) - q(x_1) = \int_{x_1}^{x_2} f(x)dx \text{ thus } f(x) = q'(x) \quad x \in (0,1) \ldots (1)$$

Fourier's Law:

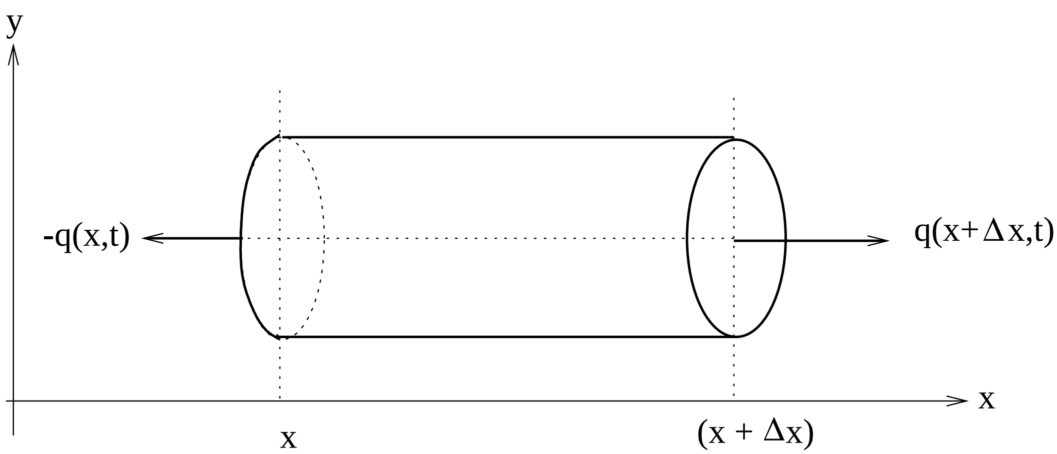
Heat flows from warm regions to cold is proportional to the temperature gradient $a(x)$, and is given by constitutive equation for heat flow.

$$q(x) = -a(x) \cdot u'(x) \text{ then } q'(x) = -(a(x) \cdot u'(x))' \ldots (2)$$

(1) and (2) give $(a(x) \cdot u'(x))' = f(x)$, which is the *stationary heat equation in 1D.*

<u>Ex.</u> Time dependent heat equation in 1D.

Energy balance $\dot{u} = \dfrac{du}{dt}$



The heat produced by the heat source along x axis and in an interial of length Δx is $f(x) \cdot \Delta x$.

The heeat flow through the end points x and $(x + \Delta x)$ is $q(x + \Delta x, t) - q(x, t)$.

Then $$\dot{u} \cdot \Delta x = f(x) \cdot \Delta x - [q(x + \Delta x, t) - q(x, t)]$$

Divide by Δx and let $\Delta x \to 0$, then

$$\dot{u} = f(x) - \lim_{\Delta x \to 0} \frac{q(x + \Delta x) - q(x)}{\Delta x} = f(x) - q'(x),$$

but $q'(x) = -(a(x) \cdot u'(x))'$ and we have

$$\dot{u} - (a(x) \cdot u'(x))' = f(x) \quad 0 < x < 1,$$

which is *the time dependent heat eequation* in the x-direction.

The Galerkin method on the stationary heat equation in 1-D:
$\dot{u} = \frac{du}{dt} = 0$

$$\begin{cases} -\frac{d}{dx}\Big(a(x) \cdot \frac{d}{dx}u(x)\Big) = f(x) \quad 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

Let $a(x) = 1$, then we have

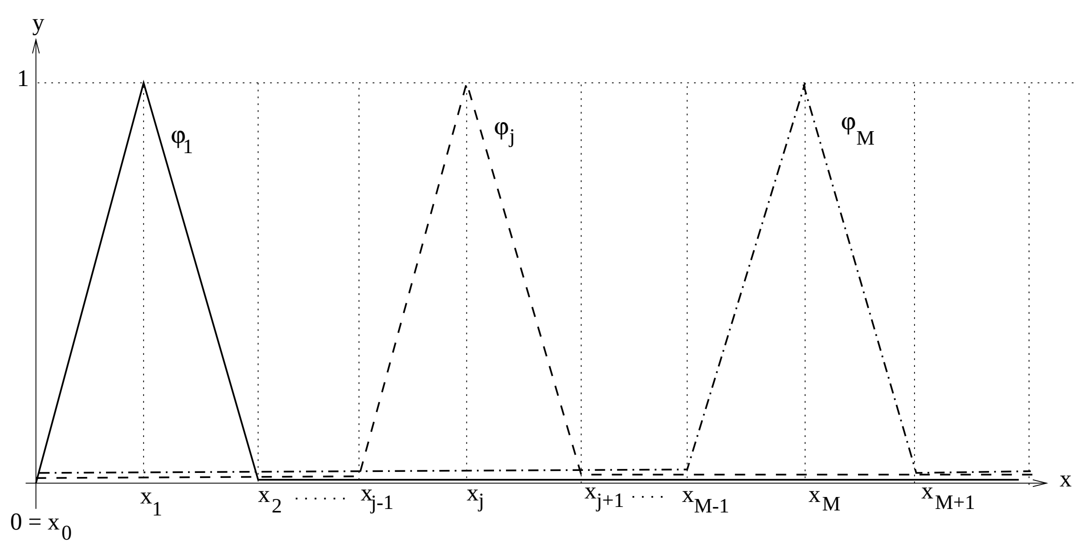
$$\begin{cases} -u''(x) = f(x) \quad 0 < x < 1 \\ u(0) = u(1) = 0 \end{cases}$$

Let

i. $T_h : 0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = 1$ be a partition of $(0,1), h_j = x_j - x_{j-1}$

ii. $V_h^0 = \{v : v$ continuous, piecewise linear functions on T_h, with $v(0) = v(1) = 0\}$

iii. $\{\varphi_j\}_{j=1}^M$ be bases functions for V_h:



Now the *Galerkin method* for this equation is formulated as follows:

Find the approximate solution $U(x) \in V_h^0$ such that

$$(3) \ldots \int_0^1 (-U''(x) - f(x))v(x)dx = 0 \quad \forall v(x) \in V_h^0$$

Observe that if $U(x) \in V_h^0$, then $U(x)''$ is either equal to zero or is not a well-defined equation and the equation (3) does not make sense, unless $f(x) \equiv 0$, but then $u(x) \equiv 0$ and we have the trivial case.

However, if we consider instead the equation after partial integration we get

$$-\int_0^1 U''(x)v(x)dx = \int_0^1 U'(x)v'(x)dx - [U'(x)v(x)]_0^1$$

and since $v(0) = v(1) = 0$ for $v(x) \in V_h$ we get

$$-\int_0^1 U''(x)v(x)dx = \int_0^1 U'(x)v'(x)dx$$

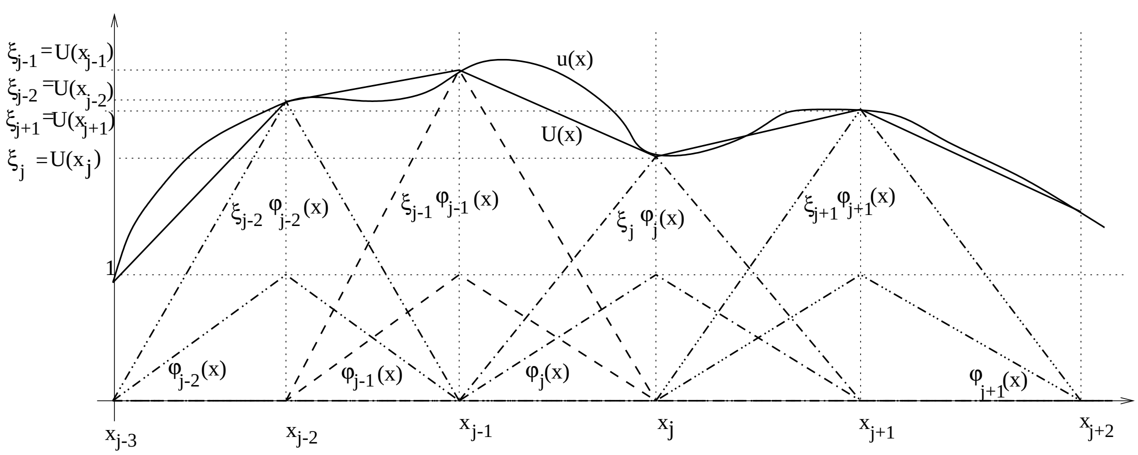
Now for $U(x), v(x) \in V_h, U'(x)$ and $v'(x)$ are well defined (except at the nodes) and the equation ($\diamondsuit$) has a meaning.

The Galerkin finite element method (FEM) is now reduced to:
(CDE pp. 115-120)

Find $U(x) \in V_h^0$ such that

$$(4) \ldots \int_0^1 U'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \text{ for all } v(x) \in V_h^0$$

We shal determine $\xi_j = u(x_j)$ *the approximate* values of $u(x)$ at the node points, x_j.



Then using bases functions $\varphi_j(x)$, we may write

$$U(x) = \sum_{j=1}^{M} \xi_j \cdot \varphi_j(x) \text{ and } U'(x) = \sum_{j=1}^{M} \xi_j \varphi'(x)$$

Now we can write

$$(4)\ldots \int_0^1 U'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \text{ as}$$
$$\sum_{j=1}^{M} \xi_j \int_0^1 \varphi_j' \cdot v'(x)dx = \int_0^1 f(x)v(x)dx \quad \forall v(x) \in V_h^0$$

Note! Since every $v(x) \in V_h^0$ is a linear combination of the basis functions $\varphi_j(x)$, it suffices to try with $v(x) = \varphi_k(x)$, for $k = 1, 2, \ldots, M$.

That is to find ξ_j (constants), $1 \leq j \leq M$ such that

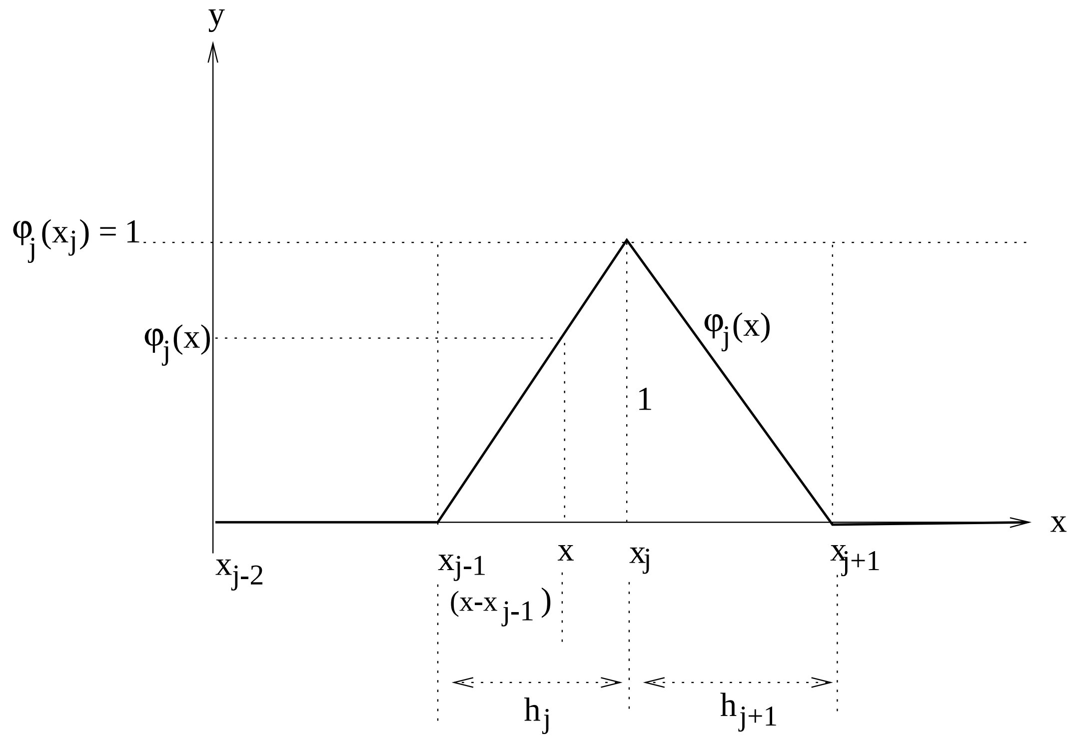
$$(1)\ldots \sum_{j=1}^{M} \underbrace{\left(\int_0^1 \varphi_j'(x) \cdot \varphi_k'(x)dx\right)}_{\substack{a_j \\ \text{Stiffness matrix}}} \cdot \xi_j = \underbrace{\int_0^1 f(x)\varphi_k(x)dx}_{\substack{b_k \\ \text{Loud vector}}} \text{ for } k = 1, 2, \ldots, M.$$

The equation ($\heartsuit$) we can rewrite as $\mathbf{A}\xi = \mathbf{b}$, where

$$\mathbf{A} = \{a_{j,k}\}_{j,k=1'}^{M} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_M \end{pmatrix} \quad \text{and} \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \ldots \\ \xi_M \end{pmatrix}$$

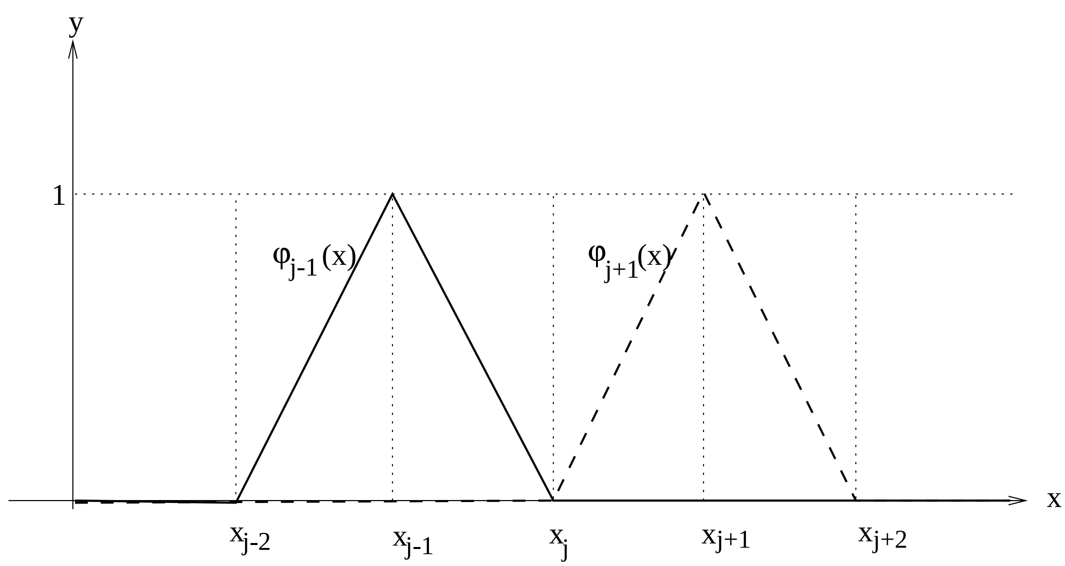
To calculate the stiffness matrix $\mathbf{A}$ we first determine $\varphi_j'(x)$:

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i} & x_{i-1} \leq x \leq x_i \\ \frac{x_{i+1} - x}{h_{i+1}} & x_i \leq x \leq x_{i+1} \\ 0 & \text{else} \end{cases} \text{ then } \varphi_i'(x) = \begin{cases} \frac{1}{h_i} & x_{i-1} \leq x \leq x_i \\ -\frac{1}{h_{i+1}} & x_i \leq x \leq x_{i+1} \\ 0 & \text{else} \end{cases}$$

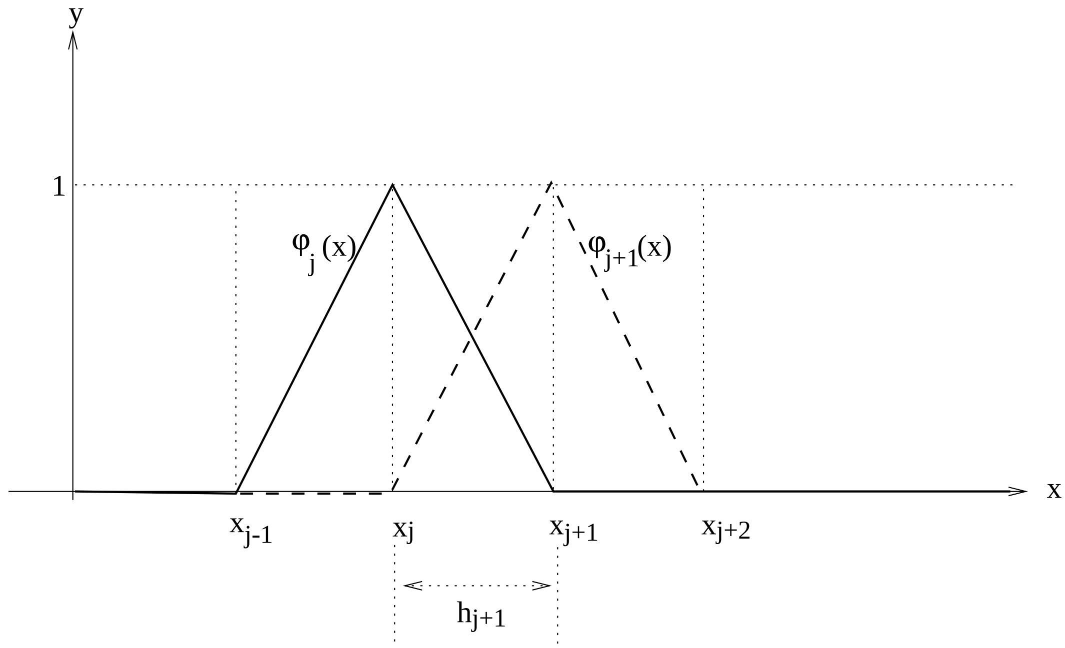


Stiffness matrix A:

For $\underline{|i-j|>1}$, we have $a_{jk} = \int_0^1 \varphi_j'(x) \cdot \varphi_k'(x)dx = 0$,



since for $|i-j|>1$ we have that $\varphi_i(x)$ and $\varphi_j(x)$ have non-overlapping supports.



$$a_{ii} = \int_{x_{i-1}}^{x_i} \left(\frac{1}{h_i}\right)^2 dx + \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right)^2 dx = \frac{\overbrace{x_i - x_{i-1}}^{h_i}}{h_i^2} + \frac{\overbrace{x_{i+1} - x_i}^{h_{i+1}}}{h_{i+1}^2} = \frac{1}{h_i} + \frac{1}{h_{i+1}}$$

$\underline{j = i+1}$

$$a_{i,i+1} = \int_{x_i}^{x_{i+1}} \left(-\frac{1}{h_{i+1}}\right) \cdot \frac{1}{h_{i+1}} dx = -\frac{x_{i+1} - x_i}{h_{i+1}^2} = -\frac{1}{h_{i+1}}$$

Changing i to $(i-1)$, we get $a_{i-1,i} = -\frac{1}{h_{i+1}}$

To *summarize*, we have

$$\begin{cases} a_{ij} = 0 & \text{if } |i-j| > 1 \\ a_{ii} = \frac{1}{h_i} + \frac{1}{h_{i+1}} & i = 1, 2, \ldots, M \\ a_{i-1,i} = -\frac{1}{h_i} & i = 2, 3, \ldots, M \end{cases}$$

Thus by symmetry

$$\mathbf{A} = \begin{bmatrix} \frac{1}{h_1}+\frac{1}{h_2} & -\frac{1}{h_2} & 0 & \dots & 0 \\ -\frac{1}{h_2} & \frac{1}{h_2}+\frac{1}{h_3} & -\frac{1}{h_3} & 0 & 0 \\ 0 & \dots & \dots & \dots & 0 \\ \dots & 0 & \dots & \dots & -\frac{1}{h_M} \\ 0 & \dots & 0 & -\frac{1}{h_M} & \frac{1}{h_M}+\frac{1}{h_{M+1}} \end{bmatrix}$$

With *uniform mesh*, i.e. $h_i = h$ we get $\mathbf{A} = \frac{1}{h} \cdot \begin{vmatrix} 2 & -1 & 0 & \dots & \dots & 0 \\ -1 & 2 & -1 & 0 & \dots & \dots \\ 0 & -1 & 2 & -1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & 0 \\ \dots & \dots & 0 & -1 & 2 & -1 \\ 0 & \dots & \dots & 0 & -1 & 2 \end{vmatrix}$

Here are some properties for the matrix $\mathbf{A}$:

- $\mathbf{A}$ is a sparse, tridiagonal and symmetric matrix.
- This may be interpreted that the basis are "nearly" orthogonal.

Definition: The matrix $\mathbf{A}$ is positive definite if

$$\forall \eta \in R^M, \eta \neq 0, \eta^T \mathbf{A} \eta > 0 \text{ i.e. } \sum_{i,j=1}^{M} \eta_i \cdot A \cdot \eta_j > 0$$

Proposition: If the square matrix $\mathbf{A}$ is positive definite then

i. $\mathbf{A}^{-1}\exists$ "$\mathbf{A}$ is invertible"
ii. $\mathbf{A}\xi = b$ has a unique solution

Proof:

(i) Suppose $\mathbf{Ax} = \mathbf{0}$ then $\mathbf{x}^T\mathbf{Ax} = \mathbf{0}$, but A is positive definite, then $\mathbf{x} \equiv \mathbf{0}$ and $\mathbf{A}$ has full Range and we can conclude that $\mathbf{A}$ is invertible.

(ii) Since $\mathbf{A}$ is invertible $\mathbf{A}\xi = \mathbf{b}$ has a unique solution $\xi = \mathbf{A}^{-1}\mathbf{b}$ □

However it is a bad idea to invert a matrix to solve the linear equation system.

Corollary: For $M = 2$. $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, U(x, y) = \begin{pmatrix} x \\ y \end{pmatrix}$, then

$$U^T AU = (x, y) \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (x, y) \begin{pmatrix} 2x - y \\ -x + 2y \end{pmatrix} = 2x^2 - xy - xy + 2y^2 =$$
$$= x^2 + y^2 + x^2 - 2xy + y^2 = x^2 + y^2 + (x - y)^2 \geq 0,$$

then $\mathbf{A}$ is positive definite.

$U^T AU = 0$ only if $x = y = 0$ i.e. $U = 0$.

Loud vector b:

We have $b_i = \int_0^1 f(x)\varphi_i(x)dx$, then $b_i = \int_{x_{i-1}}^{x_i} f(x)\frac{x - x_{i-1}}{h_i}dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1} - x}{h_{i+1}}dx$

Conclusion:

1. We need to approximate functions by polynomials agreeing with the functional values at certain points (nodes): Interpolations. Chapter 5.
2. We need to integrate or approximate integrals over subintervals and then sum: Gauss quadrature rules. Chapter 5.
3. We need to solve linear systems of equations.
   Gauss - elimination, Gauss-Seidel, Gauss-Jacobi. Chapter 7.

From chapter 6 you at least need to know

Galerkin orthogonality

Definition: Linear space
Scalar product space
Scalar product for functions
Ortogonal functions
Norm for functions
Residual error
Trial space
Test function space
Uniform mesh

Caucy-Schwartz inequality

Spaces $C^n(a,b), P^q(a,b), V^{(q)}, V_0^{(q)}, V_h^{(q)}, V_h^{0(q)}, W_k^{(q-1)}$

Lagrange basis and polynom

Polynomial interpolant $\pi_q f$

Formulation for the Galerkin method

Formulation for the Galerkin finite element method (FEM)

Basis $\varphi_i(x)$ for V_h

Calculate stiffness matrix and loud vector matrix