Chapter 7. Solving Linear Algebraic Systems
(CDE pp. 129 -)

How to solve the linear system of equations $Ax = b \Leftrightarrow x = A^{-1}b$

Direct methods. (CDE pp. 136 - 138)

$$Ax = b \equiv \sum_{j=1}^{n} a_{ij}x_j = b_i, \quad i, = 1, 2, \ldots, n \text{ or } \begin{cases} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2 \\ \ldots \\ a_{21}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n \end{cases}$$

$$\text{where } A := \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} & b_n \end{pmatrix} \text{ is the coefficient matrix.}$$

Note:

(1) It is a bad idea to calculate A^{-1} and then multiply by b.

(2) If A is an upper (or lower) triangulare, i.e. $a_{ij} = 0$ for $i > j$ (or $i < j$), and A is invertible, then we can solve x using the *back substitution method*:

$$\begin{cases} a_{11}x_1+ & a_{12}x_2+ & \ldots & +a_{1n}x_n & = b_1 \\ 0+ & a_{22}x_2+ & \ldots & +a_{2n}x_n & = b_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0+ & \ldots & +0 & +a_{nn}x_n & = b_n \end{cases}, \text{ then } \begin{cases} x_1 = \dfrac{b_1 - a_{12}x_2 - \ldots - a_{1n}x_n}{a_{11}} \\ \ldots \\ x_{n-1} = \dfrac{b_{n-1} - a_{n-1,n}x_n}{a_{n-1,n-1}} \\ x_n = \dfrac{b_n}{a_{nn}} \end{cases}$$

The number of multiplications to solve x_n are zero and the number of divisions is one.

To solve x_{n-1} we need one multiplication and one division.

To solve x_1 we need $(n-1)$ multiplication and one division, thus

- # multiplications $= 1 + 2 + \ldots + (n-1) = \frac{n(n-1)}{2} = \frac{n^2}{2} + Q(n)$, $Q(n)$ is a remainder of order n.
- # divisions $= n$.

Gaussian elimination

A linear system is not changed under following *elementary row operations*:

(i) interchanging two equations

(ii) adding a multiple of one equation to another

(iii) multiplying an equation by a nonzero constant

Definition:

$$U = \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \text{ is an } \textit{upper} \text{ triangular } 3 \times 3 \text{ matrix.}$$

$$D = \begin{pmatrix} a & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & f \end{pmatrix} \text{ is a } \textit{diagonal} \ 3 \times 3 \text{ matrix.}$$

$$L = \begin{pmatrix} a & 0 & 0 \\ g & d & 0 \\ h & i & f \end{pmatrix} \text{ is a } \textit{lower} \text{ triangular } 3 \times 3 \text{ matrix.}$$

The Gauss *elimination procedure* relay on the elementary row operations and converts the coefficient matrix of the linear equation system to an *upper* triangular matrix.

To this end, we start from the first row of the coefficient matrix of the equation system and using elementary row operations eliminate the elements $a_{i1}, i > 1$, under a_{11} (make $a_{i1} = 0$).

The equation system corresponding to this newly obtained matrix $\tilde{A}$ with elements $\tilde{a}_{ij}, \tilde{a}_{i1} = 0, i > 1$, has the same solution as the original one. We repeat the same procedure of the elementary row operations to eliminate the elements $a_{i2}, i > 2$, from the matrix $\tilde{A}$.

Continuing in this way, we thus obtain an upper triangular matrix U with corresponding equation system equivalent to the original system (has the same solution).

We illustrate this procedure through an *example*:

Solve the equation system:

$$\begin{cases} 2x_1 + \ \ x_2 + \ \ x_3 = 2 \\ 4x_1 - \ \ x_2 + 3x_3 = 0 \\ 2x_1 + 6x_2 - 2x_3 = 10 \end{cases}, \text{ the coefficient matrix is } \left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & -1 & 3 & 0 \\ 2 & 6 & -2 & 10 \end{array}\right), \text{ where}$$

$$\begin{cases} a_{11} = 2 \\ a_{21} = 4 \\ a_{31} = 2 \end{cases}. \text{ We use the multipliers } m_{i1}, i > 1, \begin{cases} m_{21} = \dfrac{a_{21}}{a_{11}} = \dfrac{4}{2} = 2 \\ m_{31} = \dfrac{a_{31}}{a_{11}} = \dfrac{2}{2} = 1 \end{cases}$$

Multiply the first row by m_{21} and then subtract it from row 2 and replace the result in row 2:

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 4 & -1 & 3 & 0 \\ 2 & 6 & -2 & 10 \end{array}\right) \begin{array}{c} \cdot(-2) \\ \\ \\ \end{array}, \text{ then } \left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & -3 & 1 & -4 \\ 2 & 6 & -2 & 10 \end{array}\right)$$

Similarly, we multiply the first row by $m_{31} = 1$ and subtract it from row 3 to get

$$\left(\begin{array}{ccc|c} 2 & 1 & 1 & 2 \\ 0 & -3 & 1 & -4 \\ 0 & 5 & -3 & 8 \end{array}\right). \text{ Now we have } \begin{cases} \tilde{a}_{22} = -3 \\ \tilde{a}_{32} = 5 \end{cases} \text{ and } \tilde{A} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 5 & -3 \end{pmatrix}.$$

Now let $m_{32} = \frac{5}{-3}$, then multiply the second row in $\tilde{A}$ by m_{32} and subtract it from row 3.

Then $\begin{pmatrix} 2 & 1 & 1 & | & 2 \\ 0 & -3 & 1 & | & -4 \\ 0 & 0 & -\frac{4}{3} & | & \frac{4}{3} \end{pmatrix}$, where $U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix}$ is a upper triangular matrix.

Now we get the equivalent equationsystem $\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ -3x_2 + x_3 = -4 \\ -\frac{4}{3}x_3 = \frac{4}{3} \end{cases}$ with the solution $\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$, which, as we can see is also the solution of the original equations.

Definition. We define the lower triangular matrices:

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -m_{21} & 1 & 0 \\ -m_{31} & 0 & 1 \end{pmatrix}, L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -m_{32} & 1 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$$

L_1, L_2 and L_3 are *unite* lower triangular 3×3-matrix's, with the property that

$$L = (L_2L_1)^{-1} = L_1^{-1}L_2^{-1} \text{ and } A = LU.$$

LU factorization of the matrix A

We generalize the above procedure fron 3×3 system of equations to $n \times n$ and we have then $A = LU$, where L is a unite lower triangular matrix and U is an upper triangular matrix obtained from A by Gauss elimination.
(CDE pp. 138 - 140)

To solve the system $Ax = b$ we let now $y = Ux$, and first solve $Ly = b$ by forward substitution (from the first row to the last) and obtain the vector y, then using y as the known right hand side finally we solve $Ux = y$ by backward substitution (from the last row to the first) and get the solution x.

In our example we have $m_{21} = 2, m_{31} = 1$ and $m_{32} = -\frac{5}{3}$, then

$$L_1 = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}, L_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{3} & 1 \end{pmatrix} \text{ and } L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{5}{3} & 1 \end{pmatrix}$$

Now we get $L_1 A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 4 & -1 & 3 \\ 2 & 6 & -2 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 5 & -3 \end{pmatrix} = \tilde{A}$.

This corresponds the first two elementary row operations in Gaussian elimination

$$L_2 L_1 A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{3} & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 5 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} = U.$$

This corresponds to the last (third) elementary row operation performed in our example.

Claim: $(L_{n-1} L_{n-2} \dots L_1)^{-1} = L$ and for $n = 3$ we have $(L_2 L_1)^{-1} = L$ where $L = \begin{pmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{pmatrix}$ where m_{ij} are the multipliers defined above.

Thus $Ax = b \Longleftrightarrow (LU)x = b \Leftrightarrow L(Ux) = b$.

As we outlined we let $y = Ux$ and first solve $Ly = b$ to obtain y. Then with such obtained y as the right hand side we solve x from $Ux = y$.

We illustrate this procedure through our *example*:

$Ly = b$

In our example we have that

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{5}{3} & 1 \end{pmatrix} \text{ and } b = \begin{pmatrix} 2 \\ 0 \\ 10 \end{pmatrix}$$

Thus we get the system

$$Ly = b \Leftrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & -\frac{5}{3} & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ 10 \end{pmatrix}, \text{ i.e.,} \begin{cases} y_1 = 2 \\ 2y_1 + y_2 = 0 \\ y_1 - \frac{5}{3}y_2 + y_3 = 10 \end{cases}$$

Now using forward substitution we get $\begin{cases} y_1 = 2 \\ y_2 = -4 \\ y_3 = \frac{4}{3} \end{cases}$

As for $\underline{Ux = y}$ we have $U = \begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix}$ and $y = \begin{pmatrix} 2 \\ -4 \\ \frac{4}{3} \end{pmatrix}$, then

$$\begin{pmatrix} 2 & 1 & 1 \\ 0 & -3 & 1 \\ 0 & 0 & -\frac{4}{3} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ -4 \\ \frac{4}{3} \end{pmatrix}$$

and we get the system of equations $\begin{cases} 2x_1 + x_2 + x_3 = 2 \\ -3x_2 + x_3 = -4 \\ -\frac{4}{3}x_3 = \frac{4}{3} \end{cases}$

Now using backward substitution as before we get the solution

$$\begin{cases} x_1 = 1 \\ x_2 = 1 \\ x_3 = -1 \end{cases}$$

Cholesky's method: (CDE pp. 146 - 147)

Theorem. Let A be a symmetric matrix, $(a_{ij} = a_{ji})$, then the following statements are equivalent:

(1) A is positive definite.

(2) The eigenvalues of A are positive.

(3) *Sylvesters criterion* $\det(\Delta_k) > 0$ for $k = 1, 2, \ldots, n$, where $\Delta_k = \begin{pmatrix} a_{11} & \ldots & a_{1k} \\ \ldots & \ldots & \ldots \\ a_{k1} & \ldots & a_{kk} \end{pmatrix}$

(4) $A = LL^T$ where L is lower triangular and has positive diagonal elements. (Cholesky's factorization)

We do not give a proof of this theorem. The interested reader is referred to literature in algebra and matrix theory.

Iterative methods (CDE pp. 151 -)

Jacobi iteration

Instead of solving $Ax = b$ directly, consider iterative solution methods based on computing a sequence of approximations $x^{(k)}, k = 1, 2, \ldots$ such that

$$\lim_{k\to\infty} x^{(k)} = x \text{ or } \lim_{k\to\infty} \|x^{(k)} - x\| = 0 \text{ for some norm.}$$

$$Ax = b \Leftrightarrow \begin{cases} a_{11}x_1 + & a_{12}x_2 & \ldots & +a_{1n}x_n & = b_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1,1}x_1 + & \ldots & \ldots & +a_{n-1,n}x_n & = b_{n-1} \\ a_{n1}x_1 + & \ldots & \ldots & +a_{nn}x_n & = b_n \end{cases}$$

Assume that $a_{ii} \neq 0$, then

$$\begin{cases} x_1 = -\frac{1}{a_{11}}[a_{12}x_2 + a_{13}x_3 + \ldots + a_{1n}x_n - b_1] \\ x_{n-1} = -\frac{1}{a_{n-1,n-1}}[a_{n-1,1}x_1 + a_{n-1,2}x_2 + \ldots + a_{n-1,n}x_n - b_{n-1}] \\ x_n = -\frac{1}{a_{nn}}[a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{n,n}x_n - b_n] \end{cases}$$

Given an initial approximation of the solution $= x^{(0)} = \begin{pmatrix} x_1^{(0)} = c_1 \\ x_2^{(0)} = c_2 \\ \ldots \\ x_n^{(0)} = c_n \end{pmatrix}$

the iteration steps are given by

$$\begin{cases} x_1^{(k+1)} = -\dfrac{1}{a_{11}}[a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \ldots + a_{1n}x_n^{(k)} - b_1] \\ \\ x_2^{(k+1)} = -\dfrac{1}{a_{22}}[a_{21}x_1^{(k)} + a_{23}x_3^{(k)} + \ldots + a_{2n}x_n^{(k)} - b_2] \\ \qquad\qquad\qquad\qquad \ldots \\ x_n^{(k+1)} = -\dfrac{1}{a_{nn}}[a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \ldots + a_{n,n-1}x_{n-1}^{(k)} - b_n] \end{cases}$$

Or in compact form: *Jacobi coordinates*

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Leftrightarrow a_{ii}x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j + b_i, \text{ then } a_{ii}x_i^{(k+1)} = -\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j^{(k)} + b_i$$

Convergence criterion:

Jacobi gives convergence to the exact solution if A is *diagonally dominant*.

$$|a_{ii}| > \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \quad i = 1, 2, \ldots, n$$

Ex. $A = \begin{pmatrix} 4 & 2 & 1 \\ 1 & 5 & 1 \\ 0 & 1 & 3 \end{pmatrix}$ is diagonally dominant. (Check it!)

Note, the Jacobi method needs less operations than Gauss elimination.

Ex. Solve $Ax = b$ where $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ and $b = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

A is diagonally dominant.

Now consider $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, i.e. the linear equation system

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 = 1 \end{cases}.$$

Choose zero initial values for x_1 and x_2, i.e. $x_1^{(0)} = 0$ and $x_2^{(0)} = 0$ and build the Jacobi iteration system $\begin{cases} 2x_1^{(k+1)} = x_2^{(k)} + 1 \\ 2x_2^{(k+1)} = x_1^{(k)} + 1 \end{cases}$, where k is the iteration step.

Then we have $\begin{cases} 2x_1^{(1)} = x_2^{(0)} + 1 \\ 2x_2^{(1)} = x_1^{(0)} + 1 \end{cases}$, with the solution $\begin{cases} x_1^{(1)} = \frac{1}{2} \\ x_2^{(1)} = \frac{1}{2} \end{cases}$.

In the next iteration step $\begin{cases} 2x_1^{(2)} = x_2^{(1)} + 1 \\ 2x_2^{(1)} = x_1^{(1)} + 1 \end{cases}$ we get $\begin{cases} 2x_1^{(2)} = \frac{1}{2} + 1 \\ 2x_2^{(2)} = \frac{1}{2} \end{cases} \Rightarrow \begin{cases} x_1^{(2)} = \frac{3}{4} \\ x_2^{(2)} = \frac{3}{4} \end{cases}$

Continuing we have obviously $\lim_{k\to\infty} x_1^{(k)} = x_1$ and $\lim_{k\to\infty} x_2^{(k)}$, where $x_1 = x_2 = 1$.

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	$\frac{1}{2}$	$\frac{1}{2}$
2	$\frac{3}{4}$	$\frac{3}{4}$
3	$\frac{7}{8}$	$\frac{7}{8}$

Now if we use $\|e_k\|_\infty = \max_{i=1,2} |x_i^{(k)} - x_i|$, then

$$\|e_1\|_\infty = \max(|x_1^{(1)} - x_1|, |x_2^{(1)} - x_2|) = \max\left(\left|\frac{1}{2} - 1\right|, \left|\frac{1}{2} - 1\right|\right) = \frac{1}{2}$$

$$\|e_2\|_\infty = \max(|x_1^{(2)} - x_1|, |x_2^{(2)} - x_2|) = \max\left(\left|\frac{3}{4} - 1\right|, \left|\frac{3}{4} - 1\right|\right) = \frac{1}{4}$$

$$\|e_3\|_\infty = \max(|x_1^{(3)} - x_1|, x_2^{(3)} - x_2|) = \max\left(\left|\frac{7}{8} - 1\right|, \left|\frac{7}{8} - 1\right|\right) = \frac{1}{8}$$

In this way $\|e_{k+1}\|_\infty = \frac{1}{2}\|e_k\|_\infty$, where e_k is the error for step $k, k \geq 0$.

Gauss-Seidel Method

Give an initial approximation of the solution $x = \begin{pmatrix} x_1^{(0)} = c_1 \\ x_2^{(0)} = c_2 \\ \dots \\ x_n^{(0)} = c_n \end{pmatrix}$,

then the iteration steps are given by

$$\begin{cases} x_1^{(k+1)} = -\dfrac{1}{a_{11}}[a_{12}x_2^{(k)} + a_{13}x_3^{(k)} + \dots + a_{1n}x_n^{(k)} - b_1] \\ x_2^{(k+1)} = -\dfrac{1}{a_{22}}[a_{21}x_1^{(k+1)} + a_{23}x_3^{(k)} + \dots + a_{2n}x_n^{(k)} - b_2] \\ \dots \\ x_{n-1}^{(k+1)} = -\dfrac{1}{a_{n-1,n-1}}[a_{(n-1),1}x_1^{(k+1)} + \dots + a_{(n-1),n-2}x_{n-2}^{(k+1)} + a_{(n-1),n}x_n^{(k)} - b_{n-1}] \\ x_n^{(k+1)} = -\dfrac{1}{a_{nn}}[a_{n1}x_1^{(k+1)} + a_{n2}x_2^{(k+1)} + \dots + a_{n,n-1}x_{n-1}^{(k+1)} - b_n] \end{cases}$$

Or in a compact way in *Gauss-Seidel coordinates*.

$$\sum_{j=1}^{n} a_{ij}x_j = b_i \Leftrightarrow \sum_{j=1}^{i} a_{ij}x_j + \sum_{j=1+1}^{n} a_{ij}x_j = b_i \Leftrightarrow \sum_{j=1}^{i} a_{ij}x_j = -\sum_{j=i+1}^{n} a_{ij}x_j + b_i,$$

and therefore $\displaystyle\sum_{j=1}^{i} a_{ij}x_j^{(k+1)} = -\sum_{j=i+1}^{n} x_j^{(k)} + b_i$

Now we have $a_{ii}x_i^{(k+1)} = -\displaystyle\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} + b_i$.

<u>Ex.</u> We consider the same example as above. $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

The Jacobi iteration system is $\begin{cases} 2x_1^{(k+1)} = x_2^{(k)} + 1 \\ 2x_2^{(k+1)} = x_1^{(k)} + 1 \end{cases}$.

The *Gauss-seidel* iteration system is $\begin{cases} 2x_1^{(k+1)} = x_2^{(k)} + 1 \\ 2x_2^{(k+1)} = \underbrace{x_1^{(k+1)}}_{\text{note!}} + 1 \end{cases}$.

Choose the same initial values for x_1 and x_2, i.e. $x_1^{(0)} = 0$, and $x_2^{(0)} = 0$, then $2x_1^{(1)} = x_2^{(0)} + 1$ and we have $x_1^{(1)} = \frac{1}{2}$.

Next equation $2x_2^{(1)} = \underbrace{x_1^{(1)}}_{\text{note!}} + 1$ gives $2x_2^{(1)} = \underbrace{\frac{1}{2}}_{\text{note!}} + 1$ and $x_2^{(1)} = \frac{3}{4}$.

The first few iteration steps would give:

k	$x_1^{(k)}$	$x_2^{(k)}$
0	0	0
1	$\frac{1}{2}$	$\frac{3}{4}$
2	$\frac{7}{8}$	$\frac{15}{16}$
3	$\frac{31}{32}$	$\frac{63}{64}$

Obviously $\lim_{k\to\infty} x_1^{(k)} = \lim_{k\to\infty} x_2^{(k)} = 1$.

Now if we use $\|e_k\|_\infty = \max_{i=1,2} |x_i^{(k)} - x_i|$, then

$$\|e_1\|_\infty = \max(|x_1^{(1)} - x_1|, |x_2^{(1)} - x_2|) = \max\left(\left|\frac{1}{2} - 1\right|, \left|\frac{3}{4} - 1\right|\right) = \max\left(\frac{1}{2}, \frac{1}{4}\right) = \frac{1}{2}$$

$\|e_2\|_\infty = \max\left(\left|\frac{7}{8} - 1\right|, \left|\frac{15}{16} - 1\right|\right) = \max\left(\frac{1}{8}, \frac{1}{16}\right) = \frac{1}{8}$ and $\|e_3\|_\infty = \max\left(\frac{1}{32}, \frac{1}{64}\right) = \frac{1}{32}$

and this gives that $\|e_{k+1}\|_\infty = \frac{1}{4}\|e_k\|_\infty$, where e_k is the error for step k.

Thus we can conclude that the Gauss-Seidel method converges faster than the Jacobi method.

S.O.R. Successive over-relaxation method.

S.O.R. is a modified Gauss-Seidel iteration.

The iteration is $x_i^{(k+1)} = (1-\omega)x_i^{(k)} + \frac{\omega}{a_{ii}}\left[b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)}\right]$

if $\omega > 1$ it is an over-Relaxation and if $0 < \omega < 1$, it is an under-Relaxation.

Relaxation coordinates

$$a_{ii}x_i^{(k+1)} = a_{ii}x_i^{(k)} - \omega\Big(\sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} + \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} - b_i\Big)$$

Abstraction of iterative methods

In our procedures $Ax = b$ and $x = Bx+C$ are equivalent linear equation systems, where B is the iteration matrix and $x_{k+1} = Bx_k + C$.

Potential advantages of iteration methods over direct methods

(1) Faster (depends on B, accuracy is required)

(2) Less memory is required (Sparsity of A can be preserved.)

Questions:

(1) For a given A, what is a good choice for B?

(2) When does $x_k \to x$?

(3) What is the rate of convergence?

The error at step k is $e_k = x_k - x$ and that of step $(k+1)$ is $e_{k+1} = x_{k+1} - x$.

Then we have $e_{k+1} = x_{k+1} - x = (Bx_k + C) - (Bx - C) = B \cdot \underbrace{(x_k - x)}_{e_k} = Be_k$.

Iterating, we have $e_k = Be_{k-1} = B \cdot B \cdot e_{k-2} = B \cdot B \cdot B \cdot e_{k-3} = B^4 \cdot e_{k-4} = \ldots = B^k \cdot e_{k-k} = B^k \cdot e_0$.

Thus we have shown that $e_k = B^k \cdot e_0$.

Let $L = \begin{pmatrix} 0 & \ldots & \ldots & 0 \\ a_{21} & 0 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & \ldots & a_{n,n-1} & 0 \end{pmatrix}$, $U = \begin{pmatrix} 0 & a_{12} & \ldots & a_{1n} \\ \ldots & \ldots & \ldots & \ldots \\ \ldots & \ldots & 0 & a_{n-1,n} \\ 0 & \ldots & \ldots & 0 \end{pmatrix}$ and

$D = \begin{pmatrix} a_{11} & 0 & \ldots & 0 \\ 0 & a_{22} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \\ 0 & \ldots & 0 & a_{nn} \end{pmatrix}$, then $A = L + D + U$, which is a *splitting* of A.

Now we can rewrite $Ax = b$ as $(D + D + U)x = b$ then $Dx = -(L + U)x + b$.

Jacobi's method

$Dx_{k+1} = -(L + U)x_k + b \Rightarrow B_J = -D^{-1}(L + U)$, where B_J is the *Jacobi's iteration matrix.*

Ex. Write the linear system in the matrix form $x = B_J x + C$!

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 2x_2 = 1 \end{cases} \Rightarrow \begin{cases} x_1 = \dfrac{1}{2}x_2 + \dfrac{1}{2} \\ \\ x_2 = \dfrac{1}{2}x_1 + \dfrac{1}{2} \end{cases}$$ and written in matrix form

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix},$$ where

$$x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, B_J = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \end{pmatrix}.$$

Ex. Determine the same B_J by the formula $B_J = -D^{-1}(L + U)$,

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}, L = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}, U = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

According to the definition is $D \cdot D^{-1} = 1$, thus $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

and $D^{-1} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$.

Then we have $B_J = -D^{-1}(L+U) = \begin{pmatrix} -\dfrac{1}{2} & 0 \\ 0 & -\dfrac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \dfrac{1}{2} \\ \dfrac{1}{2} & 0 \end{pmatrix}$

Gauss-Seidel's method

As above $Ax = b$, thus $(L+D+U)x = b$ but now we choose $(L+D)x = -Ux+b$.

And similarly we have $(L+D)x_{k+1} = -Ux_k + b$ and then $B_{GS} = -(L+D)^{-1}U$, where B_{GS} is *Gauss-Seidel's iteration matrix.*

Relaxation

Gauss-Seidel gives $(L+D)x = -Ux + b$,

thus $Dx_{k+1} = Dx_k - [Lx_{k+1} + (D+U)x_k - b]$.

Relaxation $Dx_{k+1} = Dx_k - \omega[Lx_{k+1} + (D+U)x_k - b]$, where ω is the Relaxation parameter, $\omega = 1$ gives the Gauss-seidel iteration.

Now we have

$(\omega L + D)x_{k+1} = [(1-\omega)D - \omega U]x_k + \omega b$, thus $B_\omega = (\omega L + D)^{-1}[(1-\omega)D - \omega U]$ where B_ω is the *Relaxation iteration matrix.*

From Chapter 7 you at least need to know:

Gaussian elimination

Factorization of matrices

Jacob iteration

Gauss-Seidel iteration

Definitions:
- Multiplier
- Upper and lower triangular matrix
- Diagonal matrix
- Unit matrix