Chapter 8. Error estimates for FEM in 1D (Two-points BVPs)

1. Dirichlet problem:

Consider a horizontal elastic bar occupying the interval $I := [0, 1]$.
Let $u(x)$ denote the displacement at a point $x \in I$, and $a(x)$ be the modulus of elasticity.

Consider the boundary value problem:

$$(\text{BVP})_1 \qquad \begin{cases} -\Big(a(x)u'(x)\Big)' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

We assume that $a(x)$ is pieceiwse continuous in $(0, 1)$, bounded for $0 \le x \le 1$ and $a(x) > 0$ for $0 \le x \le 1$.

Let $v(x)$ and $v'(x), x \in I$, be square integrable functions, that is: $v,\ v' \in L^2(0, 1)$, and set

$$H_0^1 = \Big\{ v(x) : \int_0^1 (v(x)^2 + v'(x)^2) dx < \infty;\ \ v(0) = v(1) = 0 \Big\}.$$

The variational formulation for $(\text{BVP})_1$ is obtained by multiplying the equation by a function $v(x) \in H_0^1(0, 1)$ and integrating over $(0, 1)$:

$$-\int_0^1 [a(x)u'(x)]'v(x)dx = \int_0^1 f(x)v(x)dx.$$

By partial integration we get

$$-\Big[a(x)u'(x)v(x)\Big]_0^1 + \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx.$$

Now since $v(0) = v(1) = 0$ we have

$$\int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx.$$

So that the *variational formulation* for the given equation is:

Find $u(x) \in H_0^1$ such that

$$(\text{VF})_1 \qquad \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \quad \forall v(x) \in H_0^1$$

Corollary: u satisfies $(\mathrm{BVP})_1 \Leftrightarrow u$ satisfies $(\mathrm{VF})_1$.

Proof: ($\Rightarrow$) For simplicity we let $a(x) = 1$, then $(\mathrm{BVP})_1$ would be

$$\begin{cases} -u''(x) = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

Integrating by parts and using $v(0) = v(1) = 0$ we get now

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \quad \forall v(x) \in H_0^1.$$

Thus, for $a(x) = 1$, the solution $u(x)$ for the $(\mathrm{BVP})_1$ satisfies $(\mathrm{VF})_1$.

($\Leftarrow$) Consider $(\mathrm{VF})_1$ in the form $-\int_0^1 [a(x)u'(x)]'v(x)dx = \int_0^1 f(x)v(x)dx$, which can also be written as

$$\text{(1)} \qquad \int_0^1 \Big[-\Big(a(x)u'(x)\Big)' - f(x)\Big] v(x)dx = 0, \qquad \forall v(x) \in H_0^1$$

We *claim* that this gives

$$-\Big(a(x)u'(x)\Big)' - f(x) \equiv 0, \qquad \forall x \in (0,1).$$

Suppose that our claim is <u>not true!</u> Then there exists a $\xi \in (0,1)$, such that

$$-\Big(a(\xi)u'(\xi)\Big)' - f(\xi) \neq 0,$$

where we may assume without loss of generality that

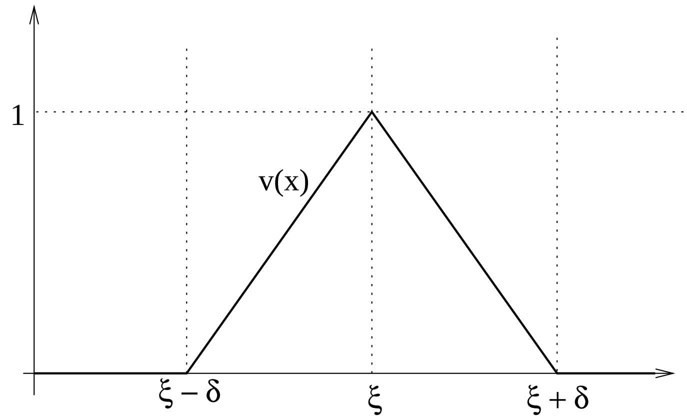
$$-\Big(a(\xi)u'(\xi)\Big)' - f(\xi) > 0 \quad (\text{or } < 0).$$

Assuming that $f \in C(0,1)$ and $a \in C^1(0,1)$, by continuity $\exists \delta > 0$ such that

$$-\Big(a(x)u'(x)\Big)' - f(x) > 0, \qquad \text{for} \quad x \in (\xi - \delta, \xi + \delta).$$

Take $v(x)$ in (1) as a hat function, $v(x) \neq 0$ on $(\xi - \delta, \xi + \delta)$, (see figure below):

Then we have $v(x) \in H_0^1$ and $\int_0^1 \underbrace{\Big[-\Big(a(x)u'(x)\Big)' - f(x)\Big]}_{>0} \underbrace{v(x)}_{>0}\, dx > 0$, which contradicts (1), thus our claim is true and the proof is complete. $\square$



Conclusion:

i) If both f and a are continuous and a is differentiable, i.e. $f \in C(0,1)$ and $a \in C^1(0,1)$, then (BVP) and (VF) have the same solution.

ii) If $a(x)$ is discontinuous, then (BVP) is not always well-defined but (VF) has meaning. Therefore (VF) covers a larger set of data than (BVP).

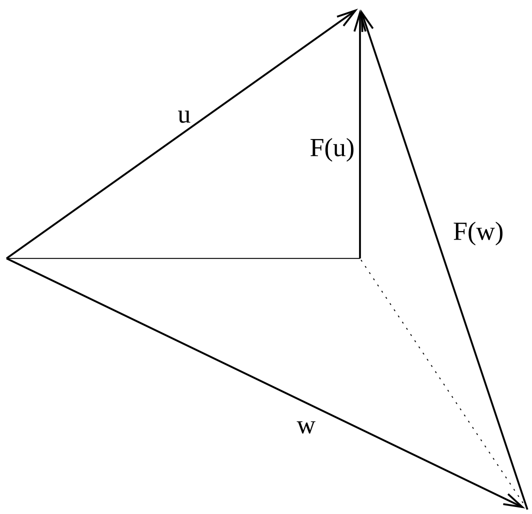
Minimization problem (MP). For the $(\text{BVP})_1$ above:

$$\begin{cases} -\Big(a(x)u](x)\Big)' = f(x), \quad 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

we formulate a *minimization problem* (MP) as:

Find $u \in H_0^1$ such that $F(u) \leq F(w)$, $\forall w \in H_0^1$, where $F(w)$ is the total energy of $w(x)$ given by

$$F(w) = \underbrace{\frac{1}{2}\int_0^1 a(w')^2 dx}_{\text{Internal energy}} - \underbrace{\int_0^1 f w dx}_{\text{Load potential}}$$



Corollary: $(MP) \Leftrightarrow (VF)$ i.e.

$$F(u) \le F(w), \forall w \in H_0^1 \Leftrightarrow \int_0^1 au'v'dx = \int_0^1 fvdx, \quad \forall v \in H_0^1.$$

Proof: ($\Leftarrow$) For $w \in H_0^1$, let $v = w - u$, $(w = v + u)$, then $v \in H_0^1$ and

$$\begin{aligned} F(w) = F(u+v) &= \frac{1}{2}\int_0^1 a\Big((u+v)'\Big)^2 dx - \int_0^1 f(u+v)dx = \\ &= \underbrace{\frac{1}{2}\int_0^1 2au'v'dx}_{(i)} + \underbrace{\frac{1}{2}\int_0^1 a(u')^2 dx}_{(ii)} + \frac{1}{2}\int_0^1 a(v')^2 dx \\ &\quad - \underbrace{\int_0^1 fudx}_{(iii)} - \underbrace{\int_0^1 fvdx}_{(iv)} \end{aligned}$$

but $(i) + (iv) = 0$, since by $(\mathrm{VF})_1$, $\int_0^1 au'v'dx = \int_0^1 fvdx$. Further, by definition of F we have $(ii) + (iii) = F(u)$. Thus

$$F(w) = F(u) + \frac{1}{2}\int_0^1 a(x)(v'(x))^2 dx,$$

and since $a(x) > 0$ we have $F(w) > F(u)$. $\square$

($\Rightarrow$) Let now $F(u) \le F(w)$ and set $g(\varepsilon, w) = F(u + \varepsilon v)$, then g has a *minimum* at $\varepsilon = 0$. But

$$g(\varepsilon, w) = F(u + \varepsilon v) = \frac{1}{2}\int_0^1 a\Big((u+\varepsilon v)'\Big)^2 dx - \int_0^1 f(u+\varepsilon v)dx =$$

$$= \frac{1}{2}\int_0^1 a(u')^2 + a\varepsilon^2(v')^2 + 2a\varepsilon u'v'\}dx - \int_0^1 fudx - \varepsilon\int_0^1 fvdx.$$

Now we compute the derivative $g'_\varepsilon(\varepsilon, w)$.

Note that $\int_0^1 fudx$ and $\int_0^1 a(u')^2 dx$ are independent of ε, therefore

$$g'_\varepsilon(\varepsilon, w) = \frac{1}{2}\{2a\varepsilon(v')^2 + 2au'v'\}dx - \int_0^1 fvdx$$

Minimum corresponds to $\varepsilon = 0$, where $g'_\varepsilon\Big|_{(\varepsilon=0)} = 0$, i.e.

$$\int_0^1 au'v'dx - \int_0^1 fvdx = 0.$$

Thus we conclude that $F(u) \le F(w) \Rightarrow \int_0^1 au'v'dx = \int_0^1 fvdx$, which is $(\mathrm{VF})_1$.

□

2. A mixed Boundary Value Problem

Note that changing the boundary conditions requires a change in the varitional formulation. Consider the eqution:

$$(\mathrm{BVP})_2 \qquad \begin{cases} -\Big(a(x)u'(x)\Big)' = f(x), & 0 < x < 1 \\ u(0) = 0, & a(1)u'(1) = g_1 \neq 0. \end{cases}$$

(In the $(\mathrm{BVP})_1$ the boundary conditions are $u(0) = u(1) = 0$)

We multiply the equation by a suitable function $v(x)$, $(v(0) = 0)$, and integrate to obtain

$$-\int_0^1 [a(x)u'(x)]'v(x)dx = \int_0^1 f(x)v(x)dx,$$

by partial integration we get that

$$-[a(x)u'(x)v(x)]_0^1 + \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx$$

but $v(0) = 0$ and $\underbrace{a(1)u'(1)}_{g_1} v(1) = g_1 v(1)$ so that

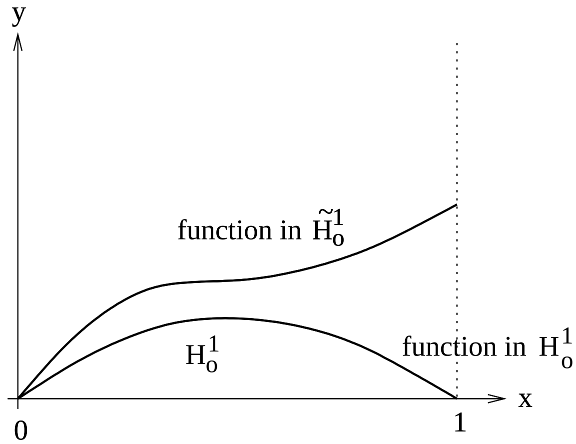
$$(2) \qquad \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx + g_1 v(1), \quad \forall v \in \tilde{H}_0^1,$$

where

$$\tilde{H}_0^1 = \{v(x) : \int_0^1 [v(x)^2 + v'(x)^2]dx < \infty, \text{ such that } v(0) = 0\}.$$

Recall that

$$H_0^1 = \{v(x) : \int_0^1 [v(x)^2 + v'(x)^2]dx < \infty, \text{ such that} v(0) = v(1) = 0\}.$$



Then (2) gives the variational formulation: Find $u \in \tilde{H}_0^1$ such that

$$(\text{VF})_2 \qquad \int_0^1 a(x)u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx + g_1 v(1), \quad \forall v \in \tilde{H}_0^1$$

Corollary: $(\text{BVP})_2 \Leftrightarrow (\text{VF})_2$

$(\Rightarrow)$ Trivial (Just as the formalism above)

$(\Leftarrow)$ To prove that a solution of the variational problem $(\text{VF})_2$ is also a solution of the two-point boundary value problem $(\text{BVP})_2$ we have to show:

(i) the solution satisfies the differential equation

(ii) the solution satisfies the boundary conditions

Integrating by parts, we have

$$\int_0^1 a(x)u'(x)v'(x)dx = [a(x)u'(x)v(x)]_0^1 - \int_0^1 [a(x)u'(x)]'v(x)\ dx$$

Since $v(0) = 0$, we get

$$\int_0^1 a(x)u'(x)v'(x)dx = a(1)u'(1)v(1) - \int_0^1 [a(x)u'(x)]'vdx$$

Thus (2), i.e. the variational formulation $(\text{VF})_2$ can be written as

$$(\text{VF})_3 \qquad -\int_0^1 [a(x)u'(x)]'vdx + a(1)u'(1)v(1) = \int_0^1 f(x)v(x)dx + g_1 v(1)$$

$(\text{VF})_3$ is valid for every $v(x) \in \tilde{H}_0^1(0,1)$, so that we may first choose $v(x)$ as in the Dirichlet problem: $-(au')' = f, v(1) = 0$ then we get

$$(3) \qquad -\int_0^1 [a(x)u'(x)]'vdx = \int_0^1 f(x)v(x)dx, \qquad \forall v(x) \in H_0^1$$

Now as in the previous case (3) gives the differential equation, thus claim (i) is through.

Also because of (3), $(\text{VF})_3$ is reduced to $g_1 v(1) = a(1)u'(1)v(1)$, which chosing $v(1) \neq 0$, e.g. , $v(1) = 1$, gives that $g_1 = a(1)u'(1)$ and the proof is complete.

□

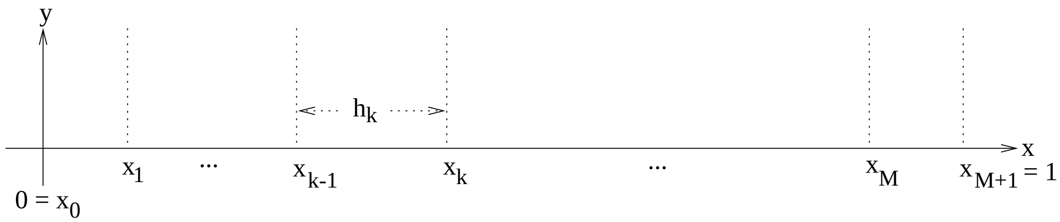
Comments:

i) Dirichlet boundary conditions: (essential B.C.) *Strongly imposed.*
   Inforced explicitly to the trial and test functions in (VF).

ii) Neumann and Robin Boundary conditions; (natural B.C.)
   Are automatically satisfied in (VF), therefore are *weakly imposed.*

The finite element method. (FEM)

Let $T_h = (0 = x_0 < x_1 < \ldots < x_M < x_{M+1} = 1)$ be a partition of $0 \leq x \leq 1$ into subintervals $I_k = [x_{k-1}, x_k]$ and $h_k = x_k - x_{k-1}$



Define a piecewise constant function $h(x) = x_k - x_{k-1} = h_k$ for $x \in I_k$. Let now $V_h^{(0)} = \{v : v(x)$ is continuous and linear on each subinterval, $v(0) = v(1) = 0\}$. Note $V_h^{(0)}$ is a subspace of

$$H_0^1 = \{v(x) : \int_0^1 [v(x)^2 + v'(x)^2]dx < \infty, \text{ such that } v(0) = v(1) = 0\}.$$

A *finite element formulation* of the Dirichlet problem (BVP) is now given by:

Find $U_h \in V_h^{(0)}$ such that

$$\text{(FEM)} \qquad \int_0^1 a(x)U_h'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \quad \forall v \in V_h^{(0)}$$

Now the purpose is to make *estimate of error* arising in approximating the solution for BVP by the functions in $V_h^{(0)}$.

Definition of some norms:

(1) L^p-norm $\quad \|v\|_{L^p} = \left(\int_0^1 |v(x)|^p dx\right)^{\frac{1}{p}}$

(2) L^∞-norm $\quad \|v\|_{L^\infty} = \sup_{x\in[0,1]} |v(x)|$

(3) Weighted L^2-norm $\quad \|v\|_a = \left(\int_0^1 a(x)|v(x)|^2 dx\right)^{\frac{1}{2}}$

(4) Energy-norm $\quad \|v\|_E = \left(\int_0^1 a(x)|v'(x)|^2 dx\right)^{\frac{1}{2}}$

Note! $\quad \|v\|_E = \|v'\|_a$

$\|v\|_E$ describes the "elastic energy" for an elastic string modeled for the Dirichlet (BVP) problem.

Error estimates in the energy norm

Theorem 8.1. Let $u(x)$ be a solution of the Dirichlet (BVP) and $U_h(x)$ is a solution of (FEM), given below:

(BVP)
$$\begin{cases} -[a(x)u'(x)]' = f(x), & 0 < x < 1 \\ u(0) = 0 \quad a(1)u'(1) = g_1 \neq 0, \end{cases}$$

or

$$\begin{cases} -[a(x)u'(x)] = f(x), & 0 < x < 1 \\ u(0) = 0 \quad u(1) = 0. \end{cases}$$

(FEM)
$$\int_0^1 a(x)U_h'(x)v'(x)dx = \int_0^1 f(x)v(x)dx, \qquad \forall v \in V_h^{(0)}, U_h(x) \in V_h^{(0)}$$

Then

$$\|u - U_h\|_E \leq \|u - v\|_E, \forall v(x) \in V_h^{(0)}$$

Note! This means that the finite element solution $U_h \in V_h^{(0)}$ is the best approximation of the solution u by functions in $V_h^{(0)}$.

Proof: Take an arbitrary $v \in V_h^{(0)}$, then

(4)
$$\begin{aligned} \|u - U_h\|_E^2 &= \int_0^1 a(x)(u' - U_h')^2(x)dx \\ &= \int_0^1 a(x)\Big(u'(x) - U_h'(x)\Big)\Big(u'(x) \underbrace{-v'(x) + v'(x)}_{=0} - U_h'(x)\Big)dx \\ &= \int_0^1 a(x)\Big(u'(x) - U_h'(x)\Big)\Big(u'(x) - v'(x)\Big)dx \\ &\quad + \int_0^1 a(x)\Big(u'(x) - U_h'(x)\Big)\Big(v'(x) - U_h'(x)\Big)\, dx \end{aligned}$$

Now since $v - U_h \in V_n^{(0)} \subset H_0^1$, we have the variational formulation

$$\int_0^1 a(x)u'(x)\Big(v'(x) - U_h'(x)\Big)dx = \int_0^1 f\Big(v(x) - U_h(x)\Big),$$

with its finite element counterpart

$$\int_0^1 a(x)U_h'(x)\Big(v'(x) - U_h'(x)\Big)dx = \int_0^1 f\Big(v(x) - U_h(x)\Big).$$

Subtracting these two relations the last line of the estimate (4) above vanishes so that we have

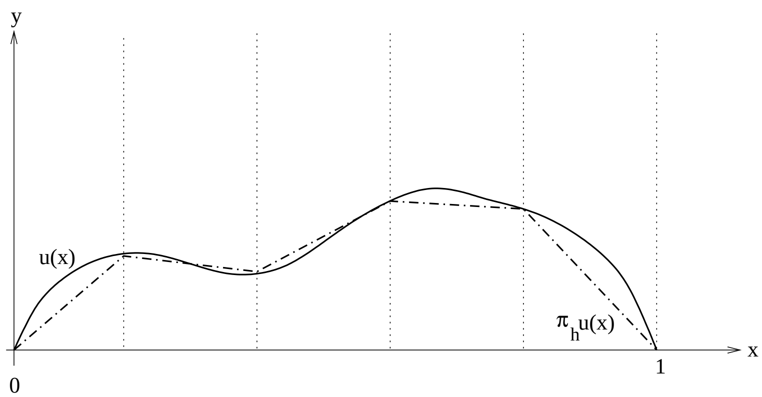
$$\begin{aligned}\|u-U_h\|_E^2 &= \int_0^1 a(x)[u'(x)-U_h'(x)][u'(x)-v'(x)]dx\\ &= \int_0^1 a(x)^{\frac12}[u'(x)-U_h'(x)]a(x)^{\frac12}[u'(x)-v'(x)]dx\\ &\le \Big(\int_0^1 a(x)[u'(x)-U_h'(x)]^2dx\Big)^{\frac12}\Big(\int_0^1 a(x)[u'(x)-v'(x)]^2dx\Big)^{\frac12}\\ &= \|u-U_h\|_E\cdot\|u-v\|_E,\end{aligned}$$

where, in the last estimate, we used Cauchy Schwarz inequality. Thus

$$\|u-U_h\|_E \le \|u-v\|_E,$$

and the proof is complete $\square$

Next step is to show that there exists a function $v(x)\in V_h^{(0)}$ such that $\|u-v\|_E$ is not "too large". The function that we shall study is $v(x)=\pi_h u(x)$: the *piecewise linear interpolant of* $u(x)$.



Let us recall an earlier interpolation error estimate in L_p-norm:

Theorem 8.2.

(i) Let $0 = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} = 1$ be a partition of $[0,1]$ and $h = (x_{j+1} - x_j), j = 0, 1, \ldots, n$

(ii) Let $\pi_h v(x)$ be the piecewise linear interpolant of $v(x)$.

Then there is an interpolation constant c_i such that

$$\|\pi_h v - v\|_{L_p} \le c_i \|h^2 v''\|_{L_p} \quad 1 \le p \le \infty \tag{5}$$

$$\|(\pi_h v)' - v'\|_{L_p} \le c_i \|h v''\|_{L_p} \tag{6}$$

$$\|\pi_h v - v\|_{L_p} \le c_i \|h v'\|_{L_p}. \tag{7}$$

An apriori error estimate

An *apriori error estimate* depends on the *exact solution* $u(x)$ and NOT on the approximate solution $U_h(x)$. In such estimates the error analyses are performed theoretically and before computations.

Theorem 8.3. Let u and U_h be the solutions of the Dirichlet problem (BVP) and the finite element problem (FEM), respectively. Then there exists an interpolation constant C_i , depending only on $a(x)$, such that

$$\|u - U_h\|_E \le C_i \|h^2 u''\|_a.$$

Proof: According to the theorem 8.1 we have

$$\|u - U_h\|_E \le \|u - v\|_E, \forall v \in V_h^{(0)}.$$

But since $\pi_h u(x) \in V_h^{(0)}$, then

$$\begin{aligned}\|u - U_h\|_E &\le \|u - \pi_h u\|_E = \|u' - (\pi_h u)'\|_a \\ &\le C_i \|h^2 u''\|_a = C_i \Big(\int_0^1 a(x) h^2(x) u''(x)^2 \, dx \Big)^{1/2},\end{aligned}$$

where in the last inequality above we use theorem 8.2. □

Now if the objective is to divide (0,1) into a fixed, finite, number of subintervals, then one can use the proof of theorem 8.3: to obtain an optimal (a best possible) partition of (0,1); in the sense that: whenever $a(x)u''(x)$ gets large we compensate by making $h(x)$ smaller.

This, however, requires that the *exact solution $u(x)$ is known.*

2. An a posteriori error estimate

Now we want to study *"a posteriori" error analysis*, where instead of the unknown value of $u(x)$, we use *the known values of the approximate solution* to estimate the error.

This means that the error analysis performed *after the computation* is completed.

Notation: We shall denote the error by $e(x)$, i.e., $e(x) = u(x) - U_h(x)$. Then $e \in H_0^1$.

Below we derive an a posteriori error estimate of (BVP):

$$\begin{cases} -[a(x)u'(x)]' = f(x), & 0 < x < 1, \\ u(0) = u(1) = 0. \end{cases}$$

The definition of the *energy norm* gives

$$\|e(x)\|_E^2 = \int_0^1 a(x)(e'(x))^2 dx = \int_0^1 a(x)(u'(x) - U_h'(x))e'(x)dx =$$
$$= \int_0^1 a(x)u'(x)e'(x)dx - \int_0^1 a(x)U_h'(x)e'(x)dx$$

But (VF) gives that

$$\int_0^1 a(x)u'(x)e'(x)dx = \int_0^1 f(x)e(x)dx.$$

Thus we get

$$\|e(x)\|_E^2 = \int_0^1 f(x)e(x)dx - \int_0^1 a(x)U_h'(x)e'(x)dx$$

Now in the integrals above we add and subtract $\pi_h e(x)$ and $\pi_h e'(x)$ respectively, where $\pi_h e(x)$ is the *interpolant of the error.* Then

$$\|e(x)\|_E^2 = \int_0^1 f(x)[e(x) - \pi_h e(x)]dx + \underbrace{\int_0^1 f(x)\pi_h e(x)dx}_{(i)}$$
$$- \int_0^1 a(x)U_h'(x)[e'(x) - \pi_h e'(x)]dx - \underbrace{\int_0^1 a(x)U_h'(x)\pi_h e'(x)dx}_{(ii)}.$$

Now since $U_h(x)$ is a solution of the (FEM) and $\pi_h e(x) \in V_h^{(0)}$ we have

$$\int_0^1 a(x)U_h'(x)\pi_h e'(x)dx = \int_0^1 f(x)\pi_h e(x)dx \Longrightarrow -(ii) + (i) = 0.$$

Hence

$$\begin{aligned}
\|e(x)\|_E^2 &= \int_0^1 f(x)[e(x) - \pi_h e(x)]dx - \underbrace{\int_0^1 a(x)U_h'(x)[e'(x) - \pi_h e'(x)]dx}_{(iii)} \\
&= \int_0^1 f(x)[e(x) - \pi_h e(x)]dx - \sum_{k=1}^{M} \int_{x_{k-1}}^{x_k} a(x)U_h'(x)(e'(x) - (\pi_h e'(x))dx
\end{aligned}$$

Now, we integrate by parts over each subinterval (x_{k-1}, x_k):

$$\begin{aligned}
&- \int_{x_{k-1}}^{x_k} \underbrace{a(x)U_h'(x)}_{g(x)} \underbrace{(e'(x) - \pi_h e'(x))}_{F'(x)} dx = [\text{P.I.}] = \\
&= -\Big[\underbrace{a(x)U_h'(x)}_{g(x)} \underbrace{(e(x) - \pi_h e(x))}_{F(x)} \Big]_{x_{k-1}}^{x_k} + \int_{x_{k-1}}^{x_k} \underbrace{(a(x)U_h'(x))'}_{g'(x)} \underbrace{(e(x) - \pi_h e(x))}_{F(x)} dx
\end{aligned}$$

Since $e(x_k) = \pi_h e(x_k), k = 0, 1 \ldots, M$, where x_ks are the interpolation nodes so that $F(x_k) = F(x_{k-1}) = 0$, and thus

$$- \int_{x_{k-1}}^{x_k} a(x)U_h'(x)(e'(x) - \pi_h e'(x))dx = \int_{x_{k-1}}^{x_k} \Big(a(x)U_h'(x)\Big)'(e(x) - \pi_h e(x))dx.$$

Hence summering over k, we get

$$- \int_0^1 a(x)U_h'(x)[e'(x) - \pi_h e'(x)]dx = \int_0^1 [a(x)U_h'(x)]'(e(x) - \pi_h e(x))dx,$$

and therefore

$$\begin{aligned}
\|e(x)\|_E^2 &= \int_0^1 f(x)[e(x) - \pi_h e(x)]dx + \int_0^1 [a(x)U_h'(x)]'(e(x) - \pi_h e(x))dx \\
&= \int_0^1 \{f(x) + [a(x)U_h'(x)]'\}(e(x) - \pi_h e(x))dx,
\end{aligned}$$

Let now $R(U_h(x)) = f(x) + (a(x)U_h'(x))'$, where $R(U_h(x))$ is the *residual error*, which is a well-defined function except in the set $\{x_k\}$, since $(a(x_k)U_x'(x_k))'$ are not defined.

Now we can rewrite the above estimate as:

$$\begin{aligned}
\|e(x)\|_E^2 &= \int_0^1 R(U_h(x))(e(x)-\pi_h e(x))dx = \\
&= \int_0^1 \frac{1}{\sqrt{a(x)}} h(x)R(U_h(x)) \cdot \sqrt{a(x)}\Big(\frac{e(x)-\pi_h e(x))}{h(x)}\Big)\, dx \\
&\leq \Big(\int_0^1 \frac{1}{a(x)} h^2(x)R^2(U_h(x))dx\Big)^{\frac{1}{2}} \Big(\int_0^1 a(x)\Big(\frac{(e(x)-\pi_h e(x))}{h(x)}\Big)^2 dx\Big)^{\frac{1}{2}},
\end{aligned}$$

where we have used Cauchy Schwarz inequality. Now recalling the definition of the weighted L^2-norm we have,

$$\underbrace{\Big\|\frac{(e(x)-\pi_h e(x))}{h(x)}\Big\|_a}_{(v)} = \Big(\int_0^1 a(x)\Big(\frac{(e(x)-\pi_h e(x))}{h(x)}\Big)^2 dx\Big)^{\frac{1}{2}}.$$

To estimate (v) we use the interpolation estimate (3) for $e(x)$ in a subinterval, then

$$\Big\|\frac{(e(x)-\pi_h e(x))}{h(x)}\Big\|_a \leq C_i \cdot \|e'(x)\|_a = C_i \cdot \|e(x)\|_E.$$

Thus

$$\|e(x)\|_E^2 \leq \Big(\int_0^1 \frac{1}{a(x)} h^2(x)R^2(U_h(x))dx\Big)^{\frac{1}{2}} \cdot C_i \cdot \|e(x)\|_E,$$

and hence we have:

Theorem 8.4. There is an interpolation constant c_i depending only on $a(x)$ such that the finite element approximation $U_h(x)$ satisfies

$$\|e(x)\|_E \leq c_i\Big(\int_0^1 \frac{1}{a(x)} h^2(x)R^2(U_h(x))dx\Big)^{\frac{1}{2}}$$

Now we have an error estimate which uses the *approximate* solution and which can be used for mesh-refinements, [changing the length of the interval $h(x)$ in the regions (subintervals) which is necessary.]

The idea is: Assume that one seeks an error bound

$$\|e(x)\|_E \leq \text{TOL} \quad \text{(errortolerance)}.$$

Use following steps:

(i) Make an initial partition of the interval

(ii) Compute the corresponding FEM solution $U_h(x)$ and residual $R(U_h(x))$.

(iii) If $\|e(x)\|_E > \text{TOL}$ refine the mesh in the places for which $\frac{1}{a(x)}R^2(U_h(x))$ is large and perform (ii) and (iii) again.

From Chapter 8 you at least need to know:

Definitions: H_0^1 and $\tilde{H}_0^1$
Weighted L^2-norm
Energy-norm
Error $e(x)$

Variational formulation (VF)
Formulate minimization problem (MP)
Finite element formulation (FEM)

Corollary: $u(x)$ satisfies (MP) $\Leftrightarrow$ $u(x)$ satisfies (VF).
Corollary: $u(x)$ satisfies $(\text{BVP})_1 \Leftrightarrow u(x)$ satisfies $(\text{VF})_1$

Theorem 8.1: Let $u(x)$ be a solution of the Dirichlet (BVP) and $U_h(x)$ is a solution of

(FEM), then $\|u - U_h\|_E \leq \|u - v\|_E, \forall v(x) \in V_h^{(0)}$

Theorem 8.3: (A priori estimate)
Theorem 8.4: (A posteriori estimate)