An Introduction to the Finite Element Method (FEM) for Differential Equations Part II: Problems in $\mathbb{R}^d$, $(d > 1)$.

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Chapter 10

Approximation in several dimensions

10.1 Introduction

This chapter is, mainly, devoted to piecewise linear finite element approximation in two dimensions. To this end, we shall prove a simple version of the Green's formula useful in weak formulations. We shall demonstrate the finite element structure through a canonical example, introduce some common general finite element spaces and finally extend the interpolation concept to higher dimensions. The study of one-dimensional problems in the previous chapters can be extended to higher dimensional domains in $\mathbb{R}^n$, $n \geq 2$. Then, the *mathematical calculus*, being in several variables, becomes somewhat involved. On the other hand, the two and three dimensional cases concern the most relevant models both from physical point of views as well as application aspects. In the sequel we assume that the reader is familiar with the calculus of several variables. A general problem to study is the convection-diffusion-absorption equation

$$\begin{cases} -\Delta u(\mathbf{x}) + \beta(\mathbf{x}) \cdot \nabla u(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega \subset \mathbb{R}^n, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{10.1.1}$$

where $n \geq 2$, $\beta(\mathbf{x})$ and $a(\mathbf{x})$ are convection and absorption coefficients, respectively and $f(\mathbf{x})$ is a source term. The discretization procedure, e.g., approximating with piecewise polynomials and deriving the approximation error in certain norm, would require extension of the interpolation estimates from the intervals in $\mathbb{R}$ (the $1d$ case) to higher dimensional domains in $\mathbb{R}^n$, $n \geq 2$. Our focus in this chapter is on the problems in two space dimensions. Then, to compute the double integrals which appear, e.g., in the weak formulations, the frequently used partial integration in the one-dimensional case is now replaced by the *Green's formula* below.

Lemma 10.1 (Green's formula)**.** *Let Ω be an open bounded subset of $\mathbb{R}^2$, with a piecewise smooth boundary and $u \in C^2(\Omega)$ and $v \in C^1(\Omega)$, then*

$$\int_\Omega (\Delta u) v \, dx \, dy = \int_{\partial\Omega} (\nabla u \cdot \mathbf{n}) v \, ds - \int_\Omega \nabla u \cdot \nabla v \, dx \, dy, \tag{10.1.2}$$

where $\mathbf{n} := \mathbf{n}(x, y)$ is the outward unit normal at the boundary point $\mathbf{x} = (x, y) \in \partial\Omega$ and ds is a curve element on the boundary $\partial\Omega$.

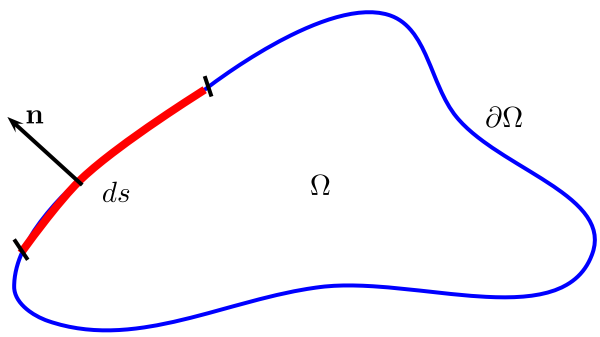


Figure 10.1: A smooth domain Ω with an outward unit normal $\mathbf{n}$

Proof. Since we shall work with polygonal domains, we give a simple proof when Ω is a rectangle, see the figure below. (The classical proof for a general domain Ω can be found in any textbook in the calculus of several variables). Then, using integration by parts we have

$$\begin{aligned}
\iint_\Omega \frac{\partial^2 u}{\partial x^2} v \, dxdy &= \int_0^b \int_0^a \frac{\partial^2 u}{\partial x^2}(x, y)\, v(x, y)\, dxdy \\
&= \int_0^b \Big(\Big[\frac{\partial u}{\partial x}(x, y)\, v(x, y) \Big]_{x=0}^a - \int_0^a \frac{\partial u}{\partial x}(x, y)\, \frac{\partial v}{\partial x}(x, y) dx \Big) dy \\
&= \int_0^b \Big(\frac{\partial u}{\partial x}(a, y)\, v(a, y) - \frac{\partial u}{\partial x}(0, y)\, v(0, y) \Big)\, dy \\
&\quad - \iint_\Omega \frac{\partial u}{\partial x}(x, y)\, \frac{\partial v}{\partial x}(x, y)\, dxdy.
\end{aligned}$$

Now identifying $\mathbf{n}$ on $\partial\Omega$, viz.

$$\begin{aligned}
&\text{on } \Gamma_1 : \mathbf{n}(a, y) = (1, 0), \\
&\text{on } \Gamma_2 : \mathbf{n}(x, b) = (0, 1), \\
&\text{on } \Gamma_3 : \mathbf{n}(0, y) = (-1, 0), \\
&\text{on } \Gamma_4 : \mathbf{n}(x, 0) = (0, -1),
\end{aligned}$$

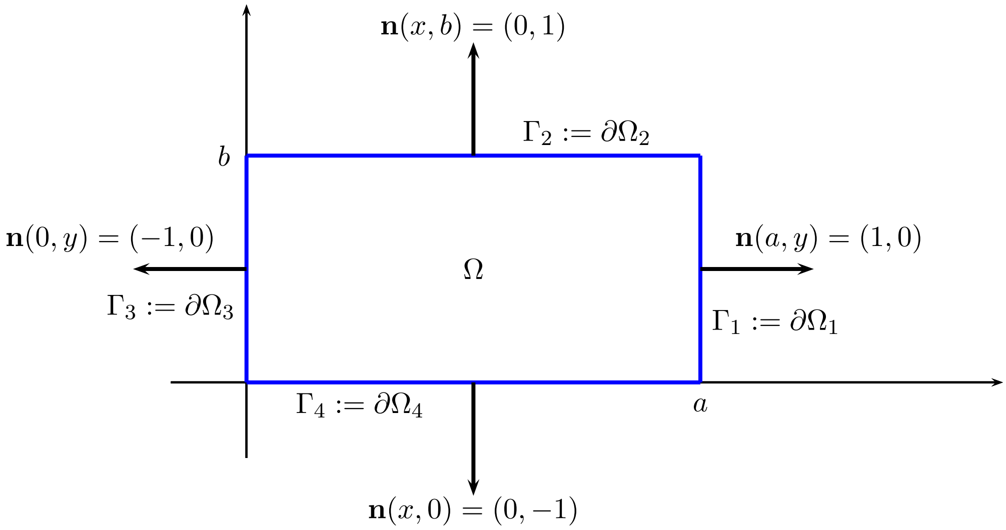


Figure 10.2: A rectangular domain with outward unit normals to its sides

the first integral on the right hand side above can be written as

$$\Big(\int_{\Gamma_1}+\int_{\Gamma_3}\Big)\Big(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big)(x,y)\cdot\mathbf{n}(x,y)v(x,y)ds.$$

Hence

$$\iint_\Omega \frac{\partial^2 u}{\partial x^2}v\,dxdy=\int_{\Gamma_1\cup\Gamma_3}\Big(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big)\cdot\mathbf{n}(x,y)v(x,y)ds-\iint_\Omega \frac{\partial u}{\partial x}\frac{\partial v}{\partial x}dxdy.$$

Similarly, in the y-direction we get

$$\iint_\Omega \frac{\partial^2 u}{\partial y^2}vdxdy=\int_{\Gamma_2\cup\Gamma_4}\Big(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\Big)\cdot\mathbf{n}(x,y)v(x,y)ds-\iint_\Omega \frac{\partial u}{\partial y}\frac{\partial v}{\partial y}dxdy.$$

Adding up these two relations the proof is complete. □

10.2 Piecewise linear approximation in 2 D

In this section we introduce the principle ideas of piecewise linear polynomial approximations of the solutions for differential equations in two dimensional polygonal domains. Hence we consider partitions (*meshes*) without any concern about curved boundaries. At the end of this chapter, in connection with a formal description of the finite element procedure, we shall briefly discuss some common extensions of the linear approximations to higher order polynomials both in triangular and quadrilateral (as well as tetrahedrons in the three dimensional case) elements. See Tables 9.1-9.4

10.2.1 Basis functions for the piecewise linears in 2 D

Recall that in the one-dimensional case a function which is linear on a interval is uniquely determined by its values at two points (e.g. the endpoints of the interval), since there is only one straight line connecting two points.

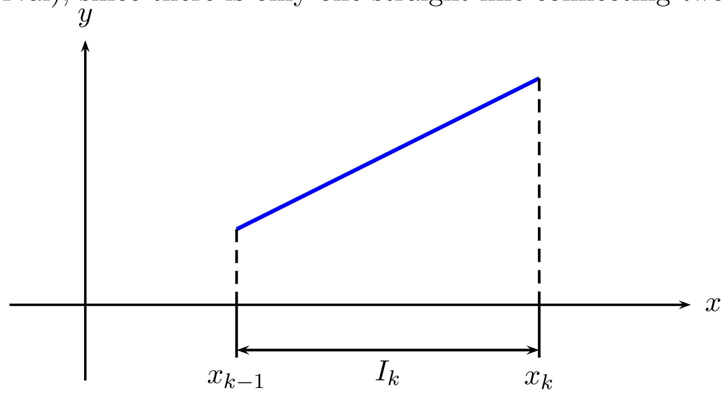


Figure 10.3: A linear function on a subinterval $I_k = (x_{k-1}, x_k)$.

Similarly a plane in $\mathbb{R}^3$ is uniquely determined by three points which are not lying on the same straight line. Therefore, for piecewise linear approximation of a function u defined on a two dimensional polygonal domain $\Omega_p \subset \mathbb{R}^2$, i.e. $u : \Omega_p \to \mathbb{R}$, it is natural to make partitions of Ω_p into triangular elements and let the sides of the triangles correspond to the endpoints of the intervals in the one-dimensional case.

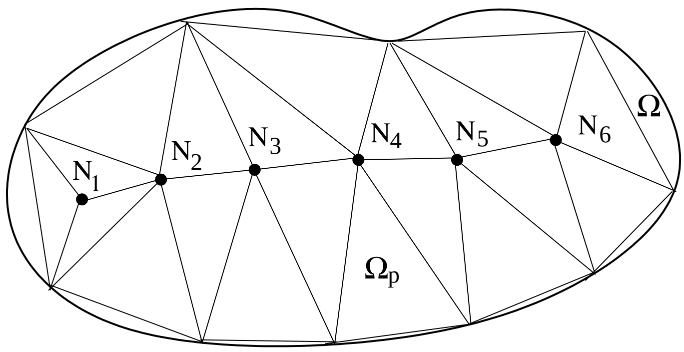


Figure 10.4: Example of triangulation of a 2D domain

The Figure 10.4 illustrates a “partitioning”, i.e., *triangulation* of a domain Ω with curved boundary where the partitioning concerns only the polygonal domain Ω_p generated by Ω. Here we have 6 internal nodes N_i, $1 \le i \le 6$ and Ω_p is the large *polygonal* domain inside Ω, which is triangulated.

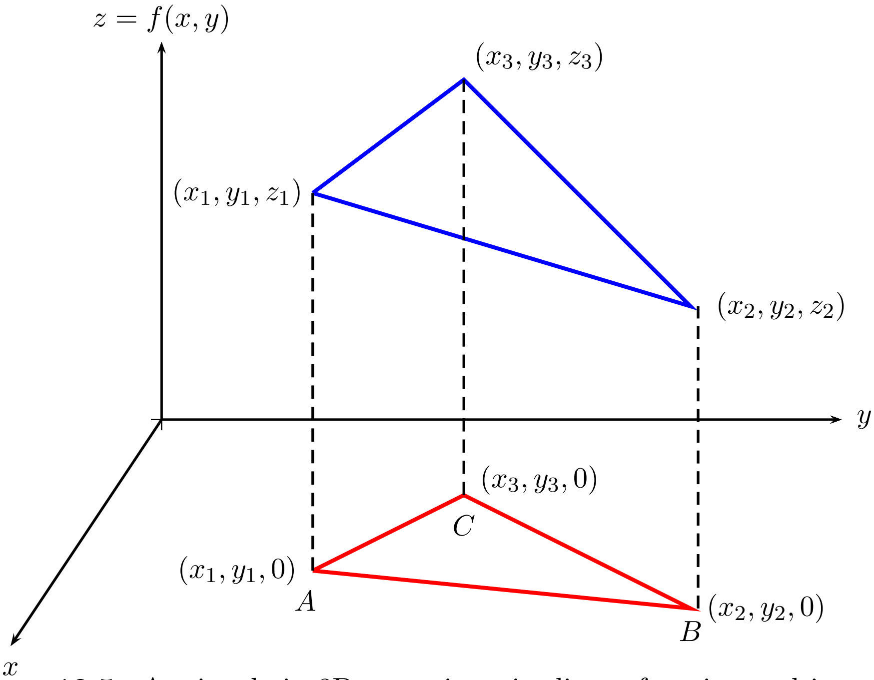


Figure 10.5: A triangle in 3D as a piecewise linear function and its projection in 2D.

Figure 10.5 shows a piecewise linear function on a single triangle (element) which is determined by its values, z_i, $i = 1, 2, 3$, at the vertices of the triangle ABC. Below we define the concepts of *subdivision* in any spatial dimension and *triangulation* in the $2D$ case.

Definition 10.1. *A subdivision of a computational domain Ω is a finite collection of open sets $\{K_i\}$ such that*

a) $K_i \bigcap K_j = \emptyset$

b) $\bigcup_i \bar{K}_i = \bar{\Omega}$.

Definition 10.2. *A triangulation of a polygonal domain $\Omega \subset \mathbb{R}^2$ is a subdivision of Ω consisting of triangles with the property that no vertex of any triangle lies in the interior of an edge of another triangle.*

Returning to Figure 10.4, for every linear function U on Ω_p,

$$U(\mathbf{x}) = U_1\varphi_1(\mathbf{x}) + U_1\varphi_2(\mathbf{x}) + \ldots + U_6\varphi_6(\mathbf{x}), \tag{10.2.1}$$

where $U_i = U(N_i),\ i = 1, 2, \ldots, 6$, are numerical values (nodal values) of U at the nodes N_i. φ_i are the basis functions with $\varphi_i(N_i) = 1$, $\varphi_i(N_j) = 0$ for $j \neq i$ and $\varphi_i(\mathbf{x})$ is linear in $\mathbf{x}$ in every triangle/element.

Note that with the Dirichlet boundary condition, as in the one-dimensional case, the test functions use $\varphi_i(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega_p$. Hence, for a Dirichlet boundary value problem, to determine the approximate solution U, in Ω_p, is reduced to find the *nodal values* $U_1, U_2, \ldots, U_6$, obtained from the corresponding discrete variational formulation.

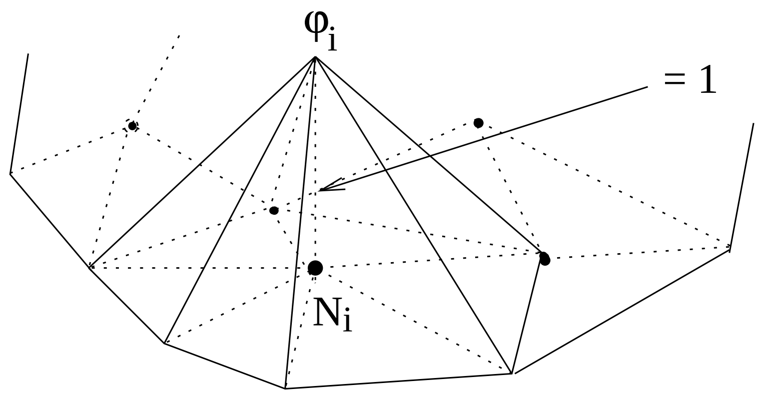


Figure 10.6: A linear basis function in a triangulation in 2D

Example 10.1. *Let* $\Omega = \{(x, y) : 0 < x < 4,\ 0 < y < 3\}$ *and make a piecewise linear FEM discretization of the following boundary value problem:*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{10.2.2}$$

Solution. Using Green's formula, the variational formulation for the problem (10.2.2) reads as follows: Find the function $u \in H_0^1(\Omega)$ such that

$$\iint_\Omega (\nabla u \cdot \nabla v)\,dxdy = \iint_\Omega fv\,dxdy, \qquad \forall v \in H_0^1(\Omega). \tag{10.2.3}$$

Recall that $H_0^1(\Omega)$ is the usual Hilbert space

$$H_0^1(\Omega) := \{v : v \in L_2(\Omega),\ |\nabla v| \in L_2(\Omega), \text{ and } v = 0, \text{ on } \Gamma := \partial\Omega\}.$$

Now we make a test function space of piecewise linears. To this end we triangulate Ω as in the Figure 10.7 and let $V_h^0 \subset V^0 := H_0^1$ be the space of continuous piecewise linears on the triangulation, $\mathcal{T}_h$ of Ω in Figure 10.7.

$V_h^0 = \{v \in \mathcal{C}(\Omega) : v \text{ is continuous, linear on each triangle and } v = 0 \text{ on } \partial\Omega\}$.

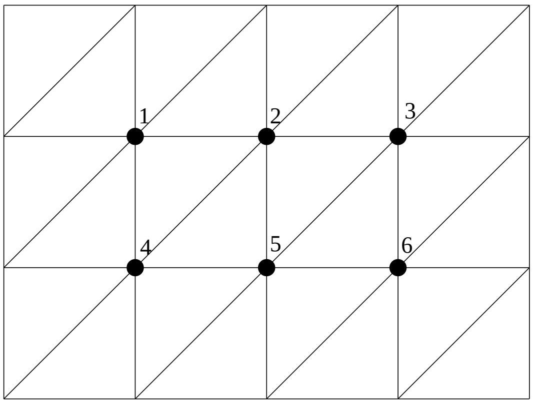
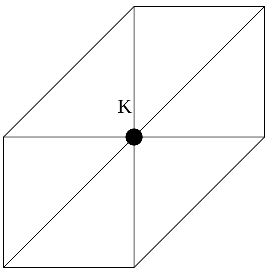


Figure 10.7: Uniform triangulation of Ω with $h = 1$

A discrete solution u_h in this scheme is uniquely determined by its values (not the nodal values for u!) at the vertices of the triangles and 0 at the boundary of the domain. In our example we have only 6 inner vertices of interest. Now as in the one-dimensional case we construct continuous piecewise linear basis functions (6 of them in this particular case), with values 1 at one of the (interior) nodes and zero at all other neighboring nodes (both interior and boundary nodes). Then we get the two-dimensional correspondence to the hat-functions φ_j, called *tent functions* as shown in the Figure 10.6.



Then, the finite element method (FEM) for (10.2.2) reads as follows: find $u_h \in V_h^0$ such that

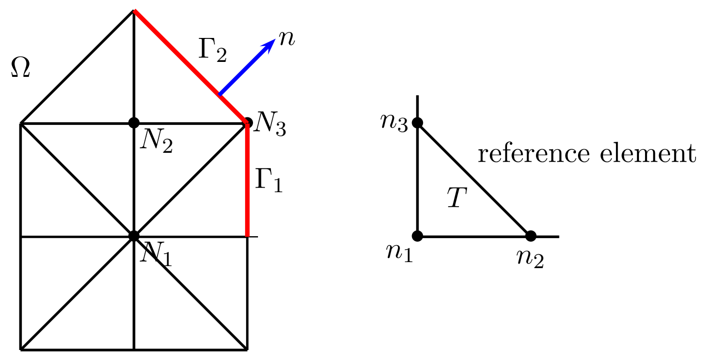
$$\iint_\Omega (\nabla u_h \cdot \nabla v) dxdy = \iint_\Omega f v \, dxdy, \qquad \forall v \in V_h^0(\Omega). \tag{10.2.4}$$

The approximation procedure follows the path generalizing the 1D-case. Below we illustrate the procedure for an example of a mixed boundary value problem in two dimensions and derive the corresponding coefficient matrices.

Example 10.2. *Formulate the cG(1) piecewise continuous Galerkin method for the boundary value problem*

$$\begin{aligned} -\Delta u + u =& f, \qquad x \in \Omega; \\ u =& 0, \qquad x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \\ \nabla u \cdot \mathbf{n} =& 0, \qquad x \in \Gamma_1 \cup \Gamma_2, \end{aligned} \tag{10.2.5}$$

on the domain Ω, with outward unit normal $\mathbf{n}$ at the boundary (as in the Figure below). Compute the matrices for the resulting system of equations using the following mesh with nodes at N_1, N_2 and N_3.
The idea is to use a standard triangle-element T with axi-parallel sides of lenght h.



Solution: Let V be the linear space defined as

$$V := \{v : v \in H^1(\Omega), \;\; v = 0, \;\; \text{on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

We denote the scalar product by $(\cdot, \cdot)$, viz.

$$(f, g) = \int_\Omega f(\mathbf{x}) g(\mathbf{x})\, d\mathbf{x}.$$

Multiplying the differential equation by $v \in V$ and integrating over Ω we get

$$-(\Delta u, v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned} -(\Delta u, v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (\mathbf{n} \cdot \nabla u) v\, ds \\ &= (\nabla u, \nabla v) - \int_{\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)} (\mathbf{n} \cdot \nabla u) v\, ds - \int_{\Gamma_1 \cup \Gamma_2} (\mathbf{n} \cdot \nabla u) v\, ds \\ &= (\nabla u, \nabla v), \qquad \forall v \in V, \end{aligned}$$

where we used the boundary data and the definition of V. Thus the variational formulation reads as: find $u \in V$ such that

$$(\nabla u, \nabla v) + (u, v) = (f, v), \qquad \forall v \in V.$$

Let V_h be the finite element space consisting of continuous piecewise linear functions, on the partition as in the Figure in the example, satisfying the boundary condition $v = 0$ on $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$: Then, the cG(1) method for (10.2.5) *is formulated as: find $u_h \in V_h$ such that*

$$(\nabla u_h, \nabla v) + (u_h, v) = (f, v) \qquad \forall v \in V_h.$$

Making the "Ansatz" $h_h(x) = \sum_{j=1}^{3} \xi_j \varphi_j(x)$, where φ_j are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^{3} \xi_j \Big(\int_\Omega \nabla\varphi_j \cdot \nabla\varphi_i \, dx + \int_\Omega \varphi_j \varphi_i \, dx \Big) = \int_\Omega f \varphi_i \, dx, \quad i = 1, 2, 3,$$

or in matrix form,

$$(S + M)\xi = F,$$

where $S_{ij} = (\nabla\varphi_i, \nabla\varphi_j)$ is the stiffness matrix, $M_{ij} = (\varphi_i, \varphi_j)$ is the mass matrix, and $F_i = (f, \varphi_i)$ is the load vector. We shall first compute the mass and stiffness matrices for the standard element T with the side length h (see Figure). The local basis functions and their gradients are given by

$$\begin{aligned}
\phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla\phi_1(x_1, x_2) &= -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\
\phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla\phi_2(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
\phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla\phi_3(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\end{aligned}$$

Hence, with $|T| = \int_T dz = h^2/2$,

$$\begin{aligned}
m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 \, dx_1 dx_2 = \frac{h^2}{12}, \\
s_{11} &= (\nabla\phi_1, \nabla\phi_1) = \int_T |\nabla\phi_1|^2 \, dx = \frac{2}{h^2} |T| = 1.
\end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 \, dx = \frac{|T|}{3} \sum_{j=1}^{3} \phi_1(\hat{x}_j)^2 = \frac{h^2}{6} \Big(0 + \frac{1}{4} + \frac{1}{4} \Big) = \frac{h^2}{12},$$

where $\hat{x}_j$ are the midpoints of the edges. Similarly we can compute the other elements and thus obtain the local mass and stiffness matrices for the standard element T:

$$m = \frac{h^2}{24}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad s = \frac{1}{2}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices M and S from the local ones m and s, through identifying the contribution from each element sharing the same node to the global matrices. For instance, since the node N_1 is common to 8 triangles and represents vertices corresponding to n_2 or n_3 in reference element T so we have

$$M_{11} = 8m_{22} = \frac{8}{12}h^2, \qquad S_{11} = 8s_{22} = 4,$$

Similarly

$$M_{12} = 2m_{12} = \frac{1}{12}h^2, \qquad S_{12} = 2s_{12} = -1,$$
$$M_{13} = 2m_{23} = \frac{1}{12}h^2, \qquad S_{13} = 2s_{23} = 0,$$
$$M_{22} = 4m_{11} = \frac{4}{12}h^2, \qquad S_{22} = 4s_{11} = 4,$$
$$M_{23} = 2m_{12} = \frac{1}{12}h^2, \qquad S_{23} = 2s_{12} = -1,$$
$$M_{33} = 3m_{22} = \frac{3}{12}h^2, \qquad S_{33} = 3s_{22} = 3/2.$$

The remaining matrix elements are obtained by symmetry $M_{ij} = M_{ji}$, $S_{ij} = S_{ji}$. Hence,

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3/2 \end{bmatrix}.$$

10.3 Constructing finite element spaces

The simplest finite element space consists of continuous piecewise linear basis functions on a partition of a bounded domain $\Omega \in \mathbb{R}^d$, $d = 1, 2, 3$. This however, is not sufficient to capture all important features of a solution and many problems are better approximated considering higher spectral orders which involves functional values and some first, second and higher order derivatives on the sides and at the vertices of the elements.

In the one dimensional case, because of the simplicity of the geometry, we could manage to set up approximation procedures relying on elementary calculus and detailed formalism was unnecessary. A general and rigorous approach to finite elements sought for a somewhat detailed structure. Below we give a general framework constructing a *finite element method*:

Definition 10.3. *By a finite element we mean a triple* (K, P_K, Σ) *where*

K	*is a geometric object, i.e., a triangle, a quadrilateral, a tetrahedron*
P_K	*is a finite-dimensional linear (vector) space of functions defined on* K
Σ	*is a set of degrees of freedom*

and a function $v \in P_K$ *is uniquely determined by the degrees of freedom* Σ.

In this setting $\Sigma = \{\sigma_1, \dots, \sigma_k\}$ is a basis for P_K' (the set of nodal variables), where P_K' stands for the algebraic dual of P_K.

Definition 10.4. *Let* (K, P_K, Σ) *be a finite element, and let* $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$ *be basis for* P_K *dual to* Σ*:*

$$\sigma_i(\varphi_j) = \delta_{ij}, \qquad 1 \le i, j \le k.$$

Then, $\{\varphi_j\}_{j=1}^k$ *is called a nodal basis for* P_K.

For the simplest case of the linear approximation in one-dimension we have that (K, P_K, Σ) is a finite element with

$K = I$	an interval
$P_K = P_1(I)$	is a polynomial of degree 1 on I
Σ	is the values at the endpoints of I.

In this setting we can identify our previous, simple one-dimensional basis as follows.

Example 10.3. *For* $K = (0,1)$, $P_K = P_1(K)$ *and* $\Sigma = \{\sigma_1, \sigma_2\}$ *with* $\sigma_1(v) = v(0)$ *and* $\sigma_2(v) = v(1)$ *for all* $v \in P_1(K)$, $(K, P_1(K), \Sigma)$ *is a finite element, with nodal basis* $\varphi_1(x) = 1 - x$ *and* $\varphi_2(x) = x$.

Now consider a subdivision $\mathcal{T}_h = \{K\}$ of a bounded domain $\Omega \subset \mathbb{R}^d, d = 2, 3$, into elements K (triangles or quadrilaterals for $d = 2$ and tetrahedrons for $d = 3$). The commonly used finite element spaces V_h consists of piecewise polynomials satisfying

$$\begin{aligned} V_h \subset H^1(\Omega) &\Longleftrightarrow V_h \subset C^0(\bar{\Omega}) \\ V_h \subset H^2(\Omega) &\Longleftrightarrow V_h \subset C^1(\bar{\Omega}). \end{aligned} \tag{10.3.1}$$

Remark 10.1. *The first equivalence in* (10.3.1) *is due to the fact that the functions* $v \in V_h$ *are polynomials on each element* $K \in \mathcal{T}_h$ *so that for a continuous* v *across the common boundary* $\partial K_i \cap \partial K_j$ *of any pair of adjusent elements* K_i *and* K_j *we have that* $D^\alpha v$, $|\alpha| = 1$, *exist and are piecewise continuous, i.e.* $v \in H^1(\Omega)$. *On the other hand, if* v *is not continuous across an inter-element boundary, then* $v \notin C^0(\bar{\Omega})$ *and* $D^\alpha v, |\alpha| = 1$, *is a* δ *function across the boundary of discontinuity. Hence* $D^\alpha v \notin L_2(\Omega)$, $|\alpha| = 1$, *consequently* $v \notin H^1(\Omega)$. *The second relation in* (10.3.1) *is justified similarly.*

To define a finite element space V_h on a triangulation $\mathcal{T}_h = \{K\}$ of the domain Ω we may need to specify

$\bullet$	function values
$\circ$	values of the first derivatives
$\bigcirc$	values of the second derivatives
$/$	values of the normal derivatives
$\nearrow$	value of the mixed derivatives $\partial^2 v/(\partial x_1 \partial x_2)$.

In tables below we have collected some of the commonly used finite elements. In the one-dimensional problems we have considered the first two rows in Table 10.1:

Degrees of freedom Σ	Function space P_k	Degree of continuity of V_h
2	$P_1(K)$	C^0
3	$P_2(K)$	C^0
 4	$P_3(K)$	C^1

Table 10.1: some one-dimensional finite elements

As for the two- and three-dimensional cases we have listed function spaces of 5th spectral order on triangular elements (Table 10.2; 2d case), up to third spectral order on quadrilaterals (Table 10.3; 2d case) and finally up to second spectral order on tetrahedrons (Table 10.4; 3d case).

10.4 The interpolant

An approach to approximation error. Given a differential equation with an unknown exact solution u in a certain infinite dimensional space, e.g., a Sobolev space V. We construct an approximate solution $\tilde{u}$ in an adequate finite dimensional subspace $\tilde{V}$, e.g., in a finite element space V_h. In the previous chapters we demonstrated two procedures (*a priori* and *a posteriori*] estimating the error $e := u - u_h$ (assuming $\tilde{u} := u_h$) in some energy norm for one-dimensional problems. A general approach reads as follows: given a simply identifiable approximation $\hat{u} \in \tilde{V}$ for which we can derive an error estimate for $\eta := u - \hat{u}$ in an adequate norm $\|\cdot\|$, we may write the approximation error as

$$e := u - \tilde{u} = u - \hat{u} + \hat{u} - \tilde{u} \equiv \eta + \xi,$$

and hence we have that

$$\|e\| \le \|u - \hat{u}\| + \|\hat{u} - \tilde{u}\| = \|\eta\| + \|\xi\|.$$

The next task is to bound $\|\xi\|$ by $\|\eta\|$. Note that $\xi \in \tilde{V}$, since both $\tilde{u}$ and $\hat{u}$ are in this finite dimensional space.

The crucial step in this procedure is the estimate of the approximation error η, where $u \in V$ and $\hat{u} \in \tilde{V}$. Generally the estimation of the error ξ (error between elements inside the finite dimensional space $\tilde{V}$) is much easier. In constructive approaches $\hat{u}$ is chosen as the *interpolant* of u (a function in $\tilde{V}$ with the same nodal values as u). Then, one needs to further formalize the concept of interpolation error (in particular give a clue how the interpolation constant may look like). This yields an estimation for $\|\eta\|$, and an estimate of the form $\|\xi\| \le C\|\eta\|$ will finish the job. Here, to derive interpolation error in higher dimensions we describe the interpolant in a more general setting. To this end we recall some standard definitions:

Definition 10.5. *Let (K, P_K, Σ) be a finite element and let $\{\varphi_j\}_{j=1}^k$ be a basis for P_K dual to Σ. For a function v for which all $\sigma_i \in \Sigma$, $i = 1, \dots, k$, are defined, we define the local interpolant $\mathcal{I}v$ of v as*

$$\mathcal{I}v = \sum_{i=1}^{k} \sigma_i(v)\varphi_i.$$

Example 10.4. *Consider the standard triangle of example 10.2. We want to find the local interpolant $\mathcal{I}_T v$ of the function $v(x, y) = (1 + x^2 + y^2)^{-1}$.*

By the definition we have that

$$\mathcal{I}_T v = \sigma_1(v)\phi_1 + \sigma_2(v)\phi_2 + \sigma_3(v)\phi_3. \tag{10.4.1}$$

As in the example 10.2 we can easily verify that $\phi_1(x, y) = 1 - x/h - y/h$, $\phi_2(x, y) = x/h$ and $\phi_3(x, y) = y/h$. Further, here we have that $\sigma_1(v) =$

$v(0,0) = 1$, $\sigma_2(v) = v(h,0) = (1+h^2)^{-1}$ and $\sigma_3(v) = v(0,h) = (1+h^2)^{-1}$. Inserting in (10.4.1), *we get*

$$\mathcal{I}_T(v) = 1 - \frac{x+y}{h}\Big(1 - \frac{1}{1+h^2}\Big).$$

Lemma 10.2. *(The properties of the local interpolation)*

i) The mapping $v \mapsto \mathcal{I}_K v$ is linear.

ii) $\sigma_i(\mathcal{I}_K(v)) = \sigma_i(v),\ i = 1, \dots, k$.

iii) $\mathcal{I}_K v = v$ for $v \in P_K$, i.e., $\mathcal{I}_K$ is idempotent: $\mathcal{I}_K^2 = \mathcal{I}_K$.

Proof. i) Follows from the fact that each $\sigma_i : v \mapsto \sigma_i(v)$, $i = 1, \dots, k$ is linear.

ii) For $i = 1, \dots, k$, we have that

$$\sigma_i(\mathcal{I}_K(v)) = \sigma_i\Big(\sum_{j=1}^k \sigma_j(v)\varphi_j\Big) = \sum_{j=1}^k \sigma_j(v)\sigma_i(\varphi_j) = \sum_{j=1}^k \sigma_j(v)\delta_{ij} = \sigma_i(v)$$

iii) That $\mathcal{I}_K v = v$ for $v \in P_K$, follows using ii), via $\sigma_i(v - \mathcal{I}_K(v)) = 0,\ i = 1, \dots, k$. Then, since $\mathcal{I}_K(v) \in P_K$,

$$\mathcal{I}_K^2 v = \mathcal{I}_K(\mathcal{I}_K(v)) = \mathcal{I}_K(v),$$

and the proof is complete.

□

To extend the local interpolant to a global computational domain we need an adequate partitioning strategy which we introduce below

Definition 10.6. *Let Ω be a bounded open set in $\mathbb{R}^n$ with subdivision $\mathcal{T}$. Assume that each subdomain K is associated with a finite element (K, P_K, Σ) with the shape function P_K and nodal variables Σ. If m is the order of highest partial derivatives involved in the nodal variables then, for $v \in C^m(\bar{\Omega})$ we define the global interpolant $\mathcal{I}_h v$ on $\bar{\Omega}$ by*

$$\mathcal{I}_h v|_{K_i} = \mathcal{I}_{K_i} v, \qquad \forall K_i \in \mathcal{T}.$$

Definition 10.7. *An interpolant has continuity of order r, if $\mathcal{I}_h v \in C^r(\bar{\Omega})$ for all $v \in C^m(\bar{\Omega})$. The space*

$$\{\mathcal{I}_h v : v \in C^m(\bar{\Omega})\}$$

is called a C^r finite element space.

Finally, to be able to compare global interpolation operators on different elements, we introduce the following definition of *affine equivalence*.

Definition 10.8. *Let $K \subset \Omega \subset \mathbb{R}^n$, suppose that (K, P_K, Σ) is a finite element and $F(x) = \mathbf{A}\mathbf{x} + \mathbf{b}$ where A is a non-singular $n \times n$ matrix and $\mathbf{x}$ and $\mathbf{b}$ are n-component column vectors. Then, the finite element $(\hat{K}, \hat{P}_K, \hat{\Sigma})$ is affine equivalence to (K, P_K, Σ) if :*

a) $F(K) = \hat{K}$;

b) $F^* \hat{P}_K = P_K$ *and*

c) $F_* \Sigma = \hat{\Sigma}$.

Here F^* is the pull-back of F defined by $F^*(\hat{v}) = \hat{v} \circ F$, and F_* is the push-forward of F defined by $(F_*\Sigma)(\hat{v}) = \Sigma(F^*(\hat{v})) = \Sigma(\hat{v} \circ F)$.

Example 10.5. *Lagrange elements on triangles (see Table 10.2) with appropriate choice of edge and interior nodes are affine equivalent. The same is true for Hermite elements on triangles.*

10.4.1 Error estimates for piecewise linear interpolation

Deriving error estimates, both in *a priori* and *a posteriori* setting, rely on using, in a certain order, combination of some fundamental concepts as: variatioal formulation, finite element formulation, Galerkin orthogonality, Cauchy-Schwarz inequalty and interpolation estimates. These concepts, except interpolation estimates, obey a universal form regardless of the dimension. In this section we make a straightforward generalization of the one-dimensional linear interpolation estimates on an interval to a two-dimensional linear interpolation on a triangle.

As in the 1D case, our estimates indicate that the *linear* interpolation errors depend on the second order, this time, partial derivatives of the function being interpolated, (i.e., the *curvature* of the function), the mesh size and also the shape of the triangle.

We consider a triangulation $\mathcal{T}_h = \{K\}$ of a two-dimensional polygonal domain Ω. Let v_i, $i = 1, 2, 3$ be the vertices of the triangle K and f a continuous function defined on K. Define the linear interpolant $\mathcal{I}_K f \in \mathcal{P}^1(K)$ by

$$\mathcal{I}_K f(v_i) = f(v_i), \qquad i = 1, 2, 3. \tag{10.4.2}$$

Note that $\mathcal{I}_K$ is used for single element K (as in Figure 10.8), whereas $\mathcal{I}_h$ is employed in a partition $\mathcal{T}_h$. In higher dimensional case, for an elliptic problem, satisfying the conditions in the Reisz representation theorem (weak formulation with a symmetric, coercieve, bilinear form and a linear form as in the definition 3.19), a typical error estimate is a generalization of the a priori error estimate for two-point boundary value problems:

$$\|u - u_h\|_V \leq C\|u - v\|_V, \qquad \forall v \in V_h. \tag{10.4.3}$$

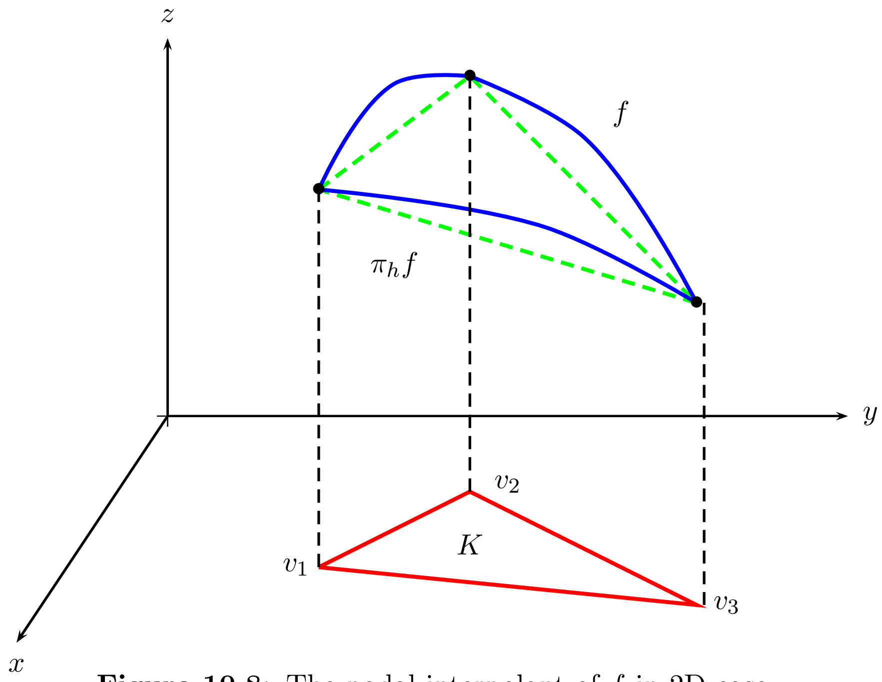


Figure 10.8: The nodal interpolant of f in 2D case

Choosing v as the interpolant of u: $v = \mathcal{I}_h u \in V_h$ the right hand side in (10.4.3) is replace by $\|u - \mathcal{I}_h u\|_V$, where $\mathcal{I}_h u \in V_h$ is chosen with the same degrees of freedom as that of $u_h \in V_h$. This would reduced the global estimate (10.4.3) of the problem to local estimates of the form $\|u - \mathcal{I}_h\|_K$, $\forall K \in \mathcal{T}_h$.

To proceed we need to define the concept of *star-shaped* domain:

Definition 10.9. *A bounded open set in $\Omega \subset \mathbb{R}^n$ is called star-shaped with respect to $B \subset \Omega$, if for all $x \in \Omega$ the closed convex hull of $\{x\} \cup B$ is a subset of Ω.*

We now state some basic interpolation results that will be frequently used in the error estimates. We shall only give some sketcky proofs. Detailed proofs can be found in, e.g., [10], [20] and [53].

A key result in finite element error analysis is the *Bramble-Hilbert Lemma*. It rely of the following proposition:

Proposition 10.1. *(polynomial approximation) For every $v \in W_p^m(\Omega)$ there exists a polynomial Q of degree $\leq m-1$ and a constant Λ_j such that*

$$|v - Q|_{W_p^j(\Omega)} \leq \Lambda_j |v|_{W_p^m(\Omega)}, \quad j = 0, \dots, m-1.$$

Bramble and Hilbert gave a non-constructive proof of this propostion based on Hahn-Banach Theorem. We skip the proof and refer the reader to a constructive version given by Süli in [53].

Lemma 10.3. *(Bramble-Hilbert Lemma) Suppose that Ω is a bounded open set in $\mathbb{R}^n$ and assume that Ω is star-shaped with respect to every point in a set $B \subset \Omega$ of positive measure. Let ℓ be a bounded linear functional on the Sobolev space $W_p^m(\Omega)$, $m \geq 1$, $1 < p < \infty$, such that $\ell(Q) = 0$ for any polynomial Q of degree $\leq m-1$. Then there esists a constant $C > 0$ such that*

$$|\ell(v)| \leq C|v|_{W_p^m(\Omega)}, \qquad \forall v \in W_p^m(\Omega).$$

Proof. By the assuptions there exists a $C_0 > 0$ such that

$$|\ell(v)| \leq C_0\|v\|_{W_p^m(\Omega)}, \qquad \forall v \in W_p^m(\Omega).$$

By the linearity of ℓ and using the fact that $\ell(Q) = 0$ for any polynomial Q of degree $\leq m-1$ we have that

$$\begin{aligned}
|\ell(v)| = |\ell(v-Q)| &\leq C_0\|v-Q\|_{W_p^m(\Omega)} \\
&= C_0\Big(\sum_{j=0}^{m} |v-Q|^p_{W_p^j(\Omega)}\Big)^{1/p} \\
&= C_0\Big(\sum_{j=0}^{m-1} |v-Q|^p_{W_p^j(\Omega)} + |v|^p_{W_p^m(\Omega)}\Big)^{1/p} \\
&\leq C_0\Big(\sum_{j=0}^{m-1} |v-Q|_{W_p^j(\Omega)} + |v|_{W_p^m(\Omega)}\Big).
\end{aligned}$$

Now, by the above proposition we have that

$$|\ell(v)| \leq C_0\Big(1 + \sum_{j=0}^{m-1} \Lambda_j\Big)|v|_{W_p^m(\Omega)},$$

and the proof is complete choosing $C = C_0\Big(1 + \sum_{j=0}^{m-1} \Lambda_j\Big)$. □

A Corollary of the Bramble-Hillber lemma, which is crucial in determining the interpolation constant, reads as follows:

Corollary 10.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set of diameter d which is star-shaped with respect to every point of an open ball $B \subset \Omega$ of diameter μd, $\mu \in (0,1]$. Suppose that $1 < p < \infty$ and $0 \leq j < m$, $m \geq 1$. If $v \in W_p^m(\Omega)$ then*

$$inf_{Q\in\mathcal{P}_{m-1}}|v-Q|_{W_p^j(\Omega)} \leq C(m,n,p,j,\mu)d^{m-j}|v|_{W_p^m(\Omega)},$$

where

$$C(m,n,p,j,\mu) = \mu^{-n/p}(\#\{\alpha : |\alpha| = j\})\frac{p(m-j)}{p-1}\Big(\sum_{|\beta|=m-j}(\beta!)^{-p'}\Big)^{1/p}$$

where $1/p + 1/p' = 1$, and $\#A$ denotes the number of elements in A.

To derive a basic interpolation error estimate we define the *norm of the local interpolation operator* $\mathcal{I}_K : C^l(\bar{K}) \to W_p^m(\Omega)$ by

$$\sigma(K) = \sup_{v \in C^l(\bar{K})} \frac{\|\mathcal{I}_K v\|_{W_p^m(\Omega)}}{\|v\|_{C^l(\bar{K})}}.$$

Now we are ready to state and prove our interpolation theorem,

Theorem 10.1. *Let (K, P_K, Σ) be a finite element satisfying the following conditions:*

i) K is star-shaped with respect to some ball cotained in K;

ii) $\mathcal{P}_{m-1} \subset P_K \subset W_\infty^m(K)$,

iii) $\Sigma \subset (C^l(\bar{K}))'$.

Suppose that $1 < p < \infty$ and $m - l - (n/p) > 0$. Then for $0 \le j \le m$ and $v \in W_p^m(K)$ we have

$$|v - \mathcal{I}_K v|_{W_p^j(K)} \le C(m, n, p, \mu, \sigma(\hat{K})) h_K^{m-j} |v|_{W_p^m(\Omega)},$$

where h_K is the diameter of K, $\hat{K} = \{x/h_K : x \in K\}$ and μ is the largest real number in the interval $(0, 1]$ such that all balls of diameter μh_K is contained in K.

Proof. We may assume, without loss of generality that $K = \hat{K}$ and *diam* $K = 1$, the general case follows by an scaling argument. Further, we note that the local interpolation operator is well defined on $W_p^m(K)$ by the Sobolev embedding theorem and, using Soblev inequality, there exists a constant $C = C_{m,n,p}$ such that for all $v \in W_p^m(K)$,

$$\|v\|_{C^l(\bar{K})} \le C_{m,n,p} \|v\|_{W_p^m(K)}, \qquad \text{for } m - l > n/p,\ 1 \le p < \infty.$$

Now given $x \in B$ we define a *Taylor expansion* as

$$P_m(v)(x, y) = \sum_{|\beta| \le m} D^\beta v(x) \frac{(y - x)^\beta}{\beta!}$$

where $(y - x)^\beta := (y_1 - x_1)^{\beta_1} \dots (y_n - x_n)^{\beta_n}$. Let $Q_m v$ be the averaging operator defined by

$$Q_m(v)(y) = \frac{1}{|B|} \int_B P_m(v)(x, y)\, dx.$$

Then, since $\mathcal{I}_K f = f$ for any $f \in P_K$, and $Q_m v \in \mathcal{P}_{m-1} \subset P_K$, we have

$$\mathcal{I}_K Q_m v = Q_m v.$$

Thus, by the sobolev embedding theorem

$$\begin{aligned}\|v-\mathcal{I}_K v\|_{W_p^m(K)} &\le \|v-Q_m v\|_{W_p^m(K)} + \|Q_m v-\mathcal{I}_K v\|_{W_p^m(K)}\\ &= \|v-Q_m v\|_{W_p^m(K)} + \|\mathcal{I}_K(Q_m v-\mathcal{I}_K v)\|_{W_p^m(K)}\\ &\le \|v-Q_m v\|_{W_p^m(K)} + \sigma(K)\|Q_m v-\mathcal{I}_K v\|_{C^l(\bar K)}\\ &\le (1+C_{m,n,p}\sigma(K))\|v-Q_m v\|_{W_p^m(K)},\end{aligned}$$

Finally, by Proposition 10.1 and Corollary 10.1 we have that

$$\|v-\mathcal{I}_K v\|_{W_p^m(K)} \le C(m,n,p,\mu,\sigma(K))|v|_{W_p^m(K)},$$

and we have the desired result. □

Under certain condition on the subdivision $\mathcal{T}=\{K\}$ of the computational domain Ω one may remove the $\sigma(K)$ dependence from the constant $C(m,n,p,\mu,\sigma(K))$. This however, is beyond our goal in this text and for such investigations we refer the reader to [10] and [53].

Now we return to some two-dimensional analogous of the interpolation estimtes in 1d.

Theorem 10.2. *If f has continuous second order partial derivatives, then*

$$\|f-\mathcal{I}_K f\|_{L_\infty(K)} \le \frac{1}{2}h_K^2||D^2 f||_{L_\infty(K)}, \tag{10.4.4}$$

$$||\nabla(f-\mathcal{I}_K f)||_{L_\infty(K)} \le \frac{3}{\sin(\alpha_K)}h_K||D^2 f||_{L_\infty(K)}, \tag{10.4.5}$$

where h_K is the largest side of K, α_K is the smallest angle of K, and

$$D^2 f = \Big(\sum_{i,j=1}^{2}(\frac{\partial^2 f}{\partial x_i\partial x_j})^2\Big)^{1/2}.$$

Remark 10.2. *Note that the gradient estimate* (10.4.5) *deteriorates for small* $\sin(\alpha_K)$*; i.e. for the thinner triangle K. This phenomenon is avoided assuming a quasi-uniform triangulation, where there is a minimum angle requirement for the triangles, viz.*

$$\sin(\alpha_K) \ge C, \qquad \textit{for some constant } C\in(0,1). \tag{10.4.6}$$

10.4.2 The L_2 and Ritz projections

Definition 10.10. *Let V be an arbitrary function space defined over Ω. The L_2 projection $Pu\in V$ of a function $u\in L_2(\Omega)$ is defined by*

$$(u-Pu,v)=0, \qquad \forall v\in V. \tag{10.4.7}$$

Usually, as a special case, we take $V = V_h$ *the space of all continuous piecewise linear functions on a triangulation* $\mathcal{T}_h = \{K\}$ *of the domain* Ω, *and the equation* (10.4.7) *is then written as*

$$(u - P_h u, v) = 0, \qquad \forall v \in V_h. \tag{10.4.8}$$

This means that, the error $u - P_h u$ *is orthogonal to* V_h. (10.4.8) *yields a linear system of equations for the coefficients of* $P_h u$ *with respect to the nodal basis of* V_h.

Advantages of the L_2 projection to the nodal interpolation

• The L_2 projection $P_h u$ is well defined for $u \in L_2(\Omega)$, whereas the nodal interpolant $\pi_h u$ in general requires u to be continuous. Therefore the L_2 projection is an alternative for the nodal interpolation for, e.g., discontinuous L_2 functions.

• Letting $v \equiv 1$ in (10.4.7) we have that

$$\int_\Omega P_h u \, dx = \int_\Omega u \, dx. \tag{10.4.9}$$

Thus the L_2 projection conserves the *total mass*, whereas, in general, the nodal interpolation operator does not preserve the total mass.

• The L_2 projection does not need to satisfy the (Dirichlet) boundary condition of the function, whereas an interpolant has to.

• Finally we have the following error estimate for the L_2 projection:

Theorem 10.3. *For a* u *with square integrable second derivative the* L_2 *projection* P_h *satisfies*

$$\|u - P_h u\| \le C_i \|h^2 D^2 u\|. \tag{10.4.10}$$

Proof. For a $v \in V_h$, we have using (10.4.8) and the Cauchy's inequality that

$$\begin{aligned} \|u - P_h u\|^2 &= (u - P_h u, u - P_h u) \\ &(u - P_h u, u - v) + (u - P_h u, v - P_h u) = (u - P_h u, u - v) \\ &\le \|u - P_h u\| \|u - v\|. \end{aligned} \tag{10.4.11}$$

This yields

$$\|u - P_h u\| \le \|u - v\|, \qquad \forall v \in V_h. \tag{10.4.12}$$

Thus $P_h u$ is at least as close to u (is an at least as good approximation in the L_2-norm) as $\pi_h u$. Now choosing $v = \pi_h u$ and recalling the interpolation theorem above we get the desired estimate (10.4.10). □

A model problem in 2 dimensions. Let Ω be a bounded polygonal domain in $\mathbb{R}^2$ and consider the model problem

$$\mathcal{A}u := -\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega. \tag{10.4.13}$$

Assume that the coefficient $a = a(x)$ is smooth with $a(x) \geq \alpha > 0$ in $\bar{\Omega}$ and $f \in L_2(\Omega)$. Then the variational formulation for (10.4.13) reads as: find $u \in H_0^1(\Omega)$ such that

$$a(u, v) = (f, v), \qquad \forall v \in H_0^1(\Omega), \tag{10.4.14}$$

where

$$a(u, v) = \int_\Omega a\nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f, v) = \int_\Omega f v \, dx,$$

we can easily verify that this problem has a unique solution in $H_0^1(\Omega)$ which, for convex domain Ω satisfies the stability estimate

$$\|u\|_2 \leq C\|f\|. \tag{10.4.15}$$

As usual the finite element approximation yields a discrete version of the variational formulation (10.4.14) as

$$a(u_h, v) = (f, v), \qquad \forall v \in V_h^0, \tag{10.4.16}$$

and we can prove the following error estimat

Theorem 10.4. *Let Ω be convex and let u and u_h be the solutions for* (10.4.14) *and* (10.4.16)*, respectively. Then*

$$\|u - u_h\| \leq Ch^2\|u\|_2. \tag{10.4.17}$$

Proof. . Let $e = u - u_h$. We shall use the dual problem

$$\mathcal{A}\phi = e, \quad \text{in } \Omega \qquad \phi = 0, \quad \text{on } \quad \partial\Omega. \tag{10.4.18}$$

Using the stability estimate (10.4.15) we have that

$$\|\phi\|_2 \leq C\|e\|. \tag{10.4.19}$$

Further by an standard estimate we have that

$$\|e\| \leq Ch|e|_1 \leq Ch^2\|u\|_2. \tag{10.4.20}$$

□

Definition 10.11 (Ritz or elliptic projection)**.** *Let $R_h : H_0^1 \to V_h^0$ be the orthogonal projection with respect to the energy inner product, so that*

$$a(R_h v - v, \varphi) = 0, \qquad \forall \varphi \in V_h^0, \quad v \in H_0^1. \tag{10.4.21}$$

Then the operator R_h is called the Ritz projection or elliptic projection. One may verify, using a Galerkin orthogonality, that the finite element solution u_h of (10.4.16) *is exactly the Ritz projection of the analytic solution u, i.e., $u_h = R_h u$.*

Using (10.4.17) one can verify that, see, e.g., [33],

$$\|u - R_h u\| + h|u - R_h u|_1 \leq Ch^s \|u\|_s, \quad \forall v \in H^s \cap H_0^1. \tag{10.4.22}$$

10.5 Exercises

Problem 10.1. *Show that the function $u : \mathbb{R}^2 \to \mathbb{R}$ given by $u(x) = \log(|x|^{-1})$, $x \neq 0$ is a solution to the Laplace equation $\Delta u(x) = 0$.*

Problem 10.2. *Show that the Laplacian of a C^2 function $u : \mathbb{R}^2 \to \mathbb{R}$ in the polar coordinates is written by*

$$\Delta u = \frac{1}{r}\frac{\partial}{\partial r}\Big(r\frac{\partial u}{\partial r}\Big) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}. \tag{10.5.1}$$

Problem 10.3. *Show using (10.5.1) that the function $u = a\log(r) + b$, where a and b are arbitrary constants is a solution of the Laplace equation $\Delta u(x) = 0$ for $x \neq 0$. Are there any other solutions of the Laplace equation in $\mathbb{R}^2$ which are invariant under rotation (i.e. it depends only on $r = |x|$)?*

Problem 10.4. *For a given triangle K, determine the relation between the smallest angle τ_K, the triangle diameter h_K and the diameter ρ_K of the largest inscribed circle.*

Problem 10.5. *Prove that a linear function in $\mathbb{R}^2$ is uniquely determined by its values at three points as long as they don't lie on a straight line.*

Problem 10.6. *Let K be a triangle with nodes $\{a^i\}$, $i = 1, 2, 3$ and let the midpoints of the edges be denoted by $\{a^{ij}, 1 \leq i < j \leq 3\}$.*
a) Show that a function $v \in \mathcal{P}^1(K)$ is uniquely determined by the degrees of freedom: $\{v(a^{ij}), 1 \leq i < j \leq 3\}$.
b) Are functions continuous in the corresponding finite element space of piecewise linear functions?

Problem 10.7. *Prove that if K_1 and K_2 are two neighboring triangles and $w_1 \in \mathcal{P}^2(K_1)$ and $w_2 \in \mathcal{P}^2(K_2)$ agree at three nodes on the common boundary (e.g., two endpoints and a midpoint), then $w_1 \equiv w_2$ on the common boundary.*

Problem 10.8. *Assume that the triangle K has nodes at $\{v^1, v^2, v^3\}$, $v^i = (v_1^i, v_2^i)$, the element nodal basis is the set of functions $\lambda_i \in \mathcal{P}^1(K)$, $i = 1, 2, 3$ such that*

$$\lambda_i(v^j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Compute the explicit formulas for λ_i.

Problem 10.9. *Let K be a triangular element. Show the following identities, for j, $k = 1, 2$, and $x \in K$,*

$$\sum_{i=1}^{3} \lambda_i(x) = 1, \qquad \sum_{i=1}^{3} (v_j^i - x_j)\lambda_i(x) = 0, \tag{10.5.2}$$

$$\sum_{i=1}^{3} \frac{\partial}{\partial x_k} \lambda_i(x) = 0, \qquad \sum_{i=1}^{3} (v_j^i - x_j) \frac{\partial \lambda_i}{\partial x_k} = \delta_{jk}, \tag{10.5.3}$$

where $v^i = (v_1^i, v_2^i)$, $i = 1, 2, 3$ are the vertices of K, $x = (x_1, x_2)$ and $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ otherwise.

Problem 10.10. *Using* (10.5.2)*, we obtain a representation for the interpolation error of the form*

$$f(x) - \pi_h f(x) = -\sum_{i=1}^{3} r_i(x)\lambda_i(x). \tag{10.5.4}$$

Prove that the remainder term $r_i(x)$ can be estimated as

$$|r_i(x)| \leq \frac{1}{2} h_K \|D^2 f\|_{L_\infty(K)}, \qquad i = 1, 2, 3. \tag{10.5.5}$$

Hint: (I) Note that $|v^i - x| \leq h_K$. (II) Start applying Cauchy's inequality to show that

$$\sum_{ij} x_i c_{ij} x_j = \sum_i x_i \sum_j c_{ij} x_j.$$

Problem 10.11. *τ_K is the smallest angle of a triangular element K. Show that*

$$\max_{x \in K} |\nabla \lambda_i(x)| \leq \frac{2}{h_K \sin(\tau_K)}.$$

Problem 10.12. *The Euler equation for an incompressible inviscid fluid of density can be written as*

$$u_t + (u \cdot \nabla)u + \nabla p = f, \qquad \nabla \cdot u = 0, \tag{10.5.6}$$

where $u(x, t)$ is the velocity and $p(x, t)$ the pressure of the fluid at the pint x at time t and f is an applied volume force (e.g., a gravitational force). The second equation $\nabla \cdot u = 0$ expresses the incopressibility. Prove that the first equation follows from the Newton's second law.

Hint: Let $u = (u_1, u_2)$ with $u_i = u_i(x(t), t)$, $i = 1, 2$ and use the chain rule to derive $\dot{u}_i = \frac{\partial u_i}{\partial x_1} u_1 + \frac{\partial u_i}{\partial x_2} u_2 + \frac{\partial u_i}{\partial t}$, $\quad i = 1, 2$.

Problem 10.13. *Prove that if $u : \mathbb{R}^2 \to \mathbb{R}^2$ satisfies $rot\, u := \left(\frac{\partial u_2}{\partial x_1}, -\frac{\partial u_1}{\partial x_2}\right) = 0$ in a convex domian $\Omega \subset \mathbb{R}^2$, then there is a scalar function φ defined on Ω such that $u = \nabla\varphi$ in Ω.*

Problem 10.14. *Prove that $\int_\Omega rot\, u\, dx = \int_\Gamma \mathbf{n} \times u\, ds$, where Ω is a subset of $\mathbb{R}^3$ with boundary Γ with outward unit normal $\mathbf{n}$.*

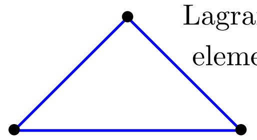
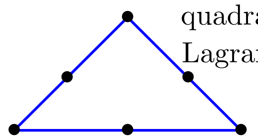
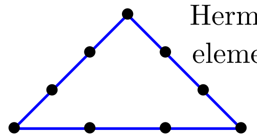
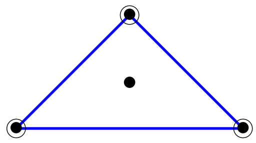
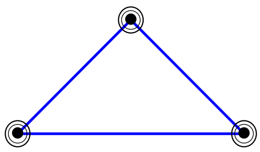
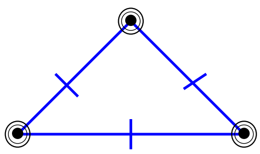
Degrees of freedom Σ	Function space P_k	Degree of continuity of V_h
 Lagrange element 3	$P_1(K)$	C^0
 quadratic Lagrande 6	$P_2(K)$	C^0
 Hermite element 10	$P_3(K)$	C^0
 10	$P_3(K)$	C^0
 18	$P_5'(K)$	C^1
 21	$P_5(K)$	C^1

Table 10.2: Some 2-dimensional finite elements with triangular elements

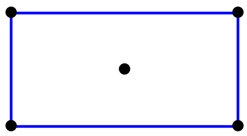
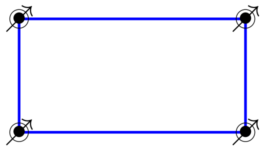
Degrees of freedom Σ	Function space P_k	Degree of continuity of V_h
 4	$Q_1(K)$	C^0
 9	$Q_2(K)$	C^0
 16	$Q_3(K)$	C^1

Table 10.3: Some 2-dimensional finite elements with quadrilateral elements

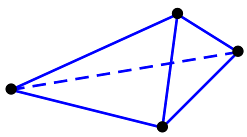
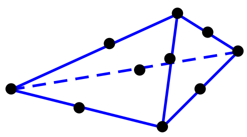
Degrees of freedom Σ	Function space P_k	Degree of continuity of V_h
 4	$P_1(K)$	C^0
 10	$P_2(K)$	C^0

Table 10.4: Some 3-dimensional finite elements with tetrahedron elements

Chapter 11

The Poisson Equation

In this chapter we shall extend the studies for the one-dimensional problem of the stationary heat conduction in previous chapters and solve the Poisson equation in higher dimensions, i.e.,

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d, \quad d = 2, 3 \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11.0.1}$$

where Ω is a bounded domain in $\mathbb{R}^d$, $d = 2$, 3, with polygonal boundary $\Gamma = \partial\Omega$. For a motivation of studying the Poisson equation and its applications, see Chapter 2. Several phenomena in science and engineering are modeled by the Poisson's equation. Below are some of the most common cases:

• *Electrostatics.* To describe the components of the Maxwell's equations, associating the electric- and potential fields $E(x)$ and $\varphi(x)$ to the charge density ρ:

$$\left.\begin{cases} \nabla \cdot E = \rho, \quad \text{in } \Omega & \text{(Coulomb's law)} \\ \nabla \times E = 0, & \text{(Faraday's law)} \implies E = \nabla\varphi \end{cases}\right\} \implies \Delta\varphi = \rho,$$

and with a Dirichlet boundary condition $\varphi = g$ on $\partial\Omega$.

• *Fluid mechanics.* The velocity field u of a rotation-free fluid flow satisfies $\nabla \times u = 0$ and hence u is a, so called, gradient field: $u = \nabla\varphi$, with φ being a scalar potential field. The rotation-free incompressible fluid flow satisfies $\nabla \cdot u = 0$, which yields the Laplace's equation $\Delta\varphi = 0$ for its potential. At a solid boundary, this problem will be associated with homogeneous Neumann boundary condition, due to the fact that in such boundary the normal velocity is zero.

• *Statistical physics.* The random motion of particles inside a container Ω until they hit the boundary is described by the probability $u(x)$ of a particle starting at the point $x \in \Omega$ winding up to stop at some point on $\partial\Omega$, where $u(x) = 1$ means that it is certain and $u(x) = 0$ means that it never happens. It turns out that u solves the Laplace's equation $\Delta u = 0$, with discontinuity at the boundary: $u = 0$ on $\partial\Omega_0$ and $u = 1$ on $\partial\Omega_1$ where $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$.

Below we give a brief introduction to the *Green's function approach* to the solution of (11.0.1), and then construct an approximate solution of (11.0.1) using finite element methods. We shall also prove stability results and derive a priori and a posteriori error estimates of the finite element approximations.

11.1 The Fundamental Solution

A classical approach to solve PDEs is through Fourier techniques, e.g., multidimensional Fourier series, Laplace and Fourier transforms and separation of variables technique. For real problems, this however is often a cumbersome procedure. An alternative approach is through using Green's functions. Below we shall introduce a general approach and compute the Green's functions for the Laplacian in some common geometries using the fundamental solution and *method of images*. Below we derive fundamental solution for the Poisson equation through deriving rotationaly symmetric solution $u(\mathbf{x}) = v(r)$, $r = |\mathbf{x}|$, $\mathbf{x} = (x_1, x_2, \dots, x_n)$ for the Laplace's equation $\Delta u(\mathbf{x}) = 0$, i.e.,

$$\Delta u(\mathbf{x}) = v''(r) + \frac{n-1}{r} v'(r) = 0.$$

This yields the following ordinary differential equation for v:

$$\frac{v''}{v'} = -\frac{n-1}{r},$$

so that

$$\log v' = -(n-1)\log r + C.$$

Hence

$$v(r) = \begin{cases} \frac{p}{r^{n-2}} + q, & \text{for } n \geq 3 \\ p \log r + q, & \text{for } n = 2. \end{cases}$$

The fundamental solution $\Phi(\mathbf{x})$ for the Laplace's equation is defined by setting $q = 0$. However, to use $\Phi(\mathbf{x})$ to determine the fundamental solution for the Poisson's equation we choose p that normalizes $\Phi(\mathbf{x})$:

$$\Phi(\mathbf{x}) = \begin{cases} \frac{1}{n(n-2)|B_1(n)|} \frac{1}{|\mathbf{x}|^{n-2}}, & \text{for } n \geq 3 \\ -\frac{1}{2\pi} \log |\mathbf{x}|, & \text{for } n = 2, \end{cases}$$

where $B_1(n)$ is the volume of the unit ball in $\mathbb{R}^n$.

Remark 11.1. *Note that Φ is integrable on bounded sets (even though it is not defined at $\mathbf{x} = 0$) and has integrable singularity at $\mathbf{x} = 0$. Nevertheless, the singularity of $-\Delta\Phi$ is not integrable, but it is the singular Dirac's δ function.*

Now we formulate the fundamental solution for the Possion's equation in $\mathbb{R}^n$.

Theorem 11.1. *Suppose that $f \in C^2(\mathbb{R}^n)$ has compact support. Then the solution for the Poisson's equation*

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \qquad \mathbf{x} \in \mathbb{R}^n, \tag{11.1.1}$$

is given by the convolution product of f with the fundamental solution Φ of the corresponding Laplace's equation:

$$u(\mathbf{x}) = (\Phi * f)(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y})\, d\mathbf{y}.$$

We skip the proof of this theorem and refer the reader to [47].

11.1.1 Green's Functions

We shall give a swift introduction to a general framework. A more thorough study is via distribution theory which is beyond the scope of this text. As we demonstrated in the solution approach for boundary value problems in the one-space dimension, Green's functions provide us with an integral representation for solution of the boundary value problems. Let $u : \mathbb{R}^n \to \mathbb{R}$ and consider the problem

$$\mathcal{L}u = f \quad \text{ in } \Omega, \qquad Bu = g \text{ on } \partial\Omega. \tag{11.1.2}$$

Then, in *distribution sense*, the fundamental solution $\Phi(\mathbf{x}, \mathbf{y})$ for (11.1.2) satisfies

$$\mathcal{L}\Phi(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \qquad \mathbf{x},\, \mathbf{y} \in \mathbb{R}^n. \tag{11.1.3}$$

Note that here Φ is defined on $\mathbb{R}^n \times \mathbb{R}^n$. To see the advantage of this representation we assume that for each $\mathbf{x} \in \mathbb{R}^n$ the function Φ is integrable in $\mathbf{y}$, then

$$v(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} = (\Phi(\mathbf{x}, \mathbf{y}), f(\mathbf{y})). \tag{11.1.4}$$

Now since

$$\mathcal{L}v(\mathbf{x}) = \mathcal{L}(\Phi(\mathbf{x}, \mathbf{y}), f(\mathbf{y})) = (\delta(\mathbf{x} - \mathbf{y}), f(\mathbf{y})) = f(\mathbf{x}),$$

thus v satisfies the equation $\mathcal{L}v = f$ in $\mathbb{R}^n$.

The remaining issue is that, in general, $v(\mathbf{x})$ does not satisfy the boundary condition $Bv = g$. So, as in elementary calculus, we need to construct a solution by adding the solution $w(\mathbf{x})$ of the corresponding homogeneous equation ($\mathcal{L}w = 0$) to $v(\mathbf{x})$, so that $u(\mathbf{x}) = v(\mathbf{x}) + w(\mathbf{x})$ and $Bu = g$. This imposes the boundary condition on w, viz. $Bw = g - Bv$ (see Chapter 2). Thus, knowing the fundamental solution, the boundary value problem (11.1.2) is reduced to the problem

$$\mathcal{L}w = 0 \quad \text{in } \Omega, \qquad Bw = g - Bv \quad \text{on } \partial\Omega.$$

It is the boundary condition for w that can be smoothly tackled using the Green's function for the operator $\mathcal{L}$ with the boundary operator B. In this setting the Green's function $G(\mathbf{x}, \mathbf{y})$ is defined as the solution of the problem

$$\begin{cases} \mathcal{L}G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), & \mathbf{x} \in \Omega, \\ BG(\mathbf{x}, \mathbf{y}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \tag{11.1.5}$$

for each $\mathbf{y} \in \bar{\Omega}$. Note that unlike (11.1.3), here $\mathbf{x} \in \Omega$. Now, as in the one dimensional case, we construct G using the fundamental solution that defines a $\mathbf{y}$-parametric function $\Phi^{\mathbf{y}}(\mathbf{x})$, viz.

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}) - \Phi^{\mathbf{y}}(\mathbf{x}),$$

then,

$$\begin{cases} \mathcal{L}\Phi^{\mathbf{y}}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ B\Phi^{\mathbf{y}}(\mathbf{x}) = B\Phi(\mathbf{x}, \mathbf{y}), & \mathbf{x} \in \partial\Omega. \end{cases} \tag{11.1.6}$$

Thus we may write the solution of (11.1.2) as $u = v + w$ where

$$\begin{cases} v(\mathbf{x}) = \displaystyle\int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} & \text{and} \\ \mathcal{L}w = 0 \quad \text{in } \Omega & \text{with } Bw = g \quad \text{on } \partial\Omega. \end{cases} \tag{11.1.7}$$

We use the above approach to solve the Dirichlet boundary value problem:

Theorem 11.2 (Green's Function for Laplace's equation)**.** *Let $u \in C^2(\bar{\Omega})$ be the solution for*

$$-\Delta u = f \quad \text{in } \Omega, \qquad u = g \text{ on } \partial\Omega, \tag{11.1.8}$$

then

$$u(\mathbf{x}) = \int_\Omega G(\mathbf{x}, \mathbf{y}) f(\mathbf{y})\, d\mathbf{y} - \int_{\partial\Omega} \frac{\partial G}{\partial \nu_{\mathbf{y}}}(\mathbf{x}, \mathbf{y}) g(\mathbf{y})\, d\sigma_{\mathbf{y}}. \tag{11.1.9}$$

Here G is the Green's function given as

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Delta\varphi^{\mathbf{y}}(\mathbf{x}),$$

where $\Phi(\mathbf{x}-\mathbf{y})$ *is the fundamental solution and* $\varphi^{\mathbf{y}}(\mathbf{x})$ *is the solution for the problem*

$$\begin{cases} \Delta\varphi^{\mathbf{y}}(\mathbf{x}) = 0, & \text{in } \Omega \\ \Delta\varphi^{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x}-\mathbf{y}), & \text{on } \partial\Omega. \end{cases} \tag{11.1.10}$$

For a proof see [47].

11.1.2 Method of Images

Below we give examples of Green's function for Laplacian that is obtained using fundamental solution in symmetric domains. The idea is for each $\mathbf{x} \in \Omega$ to construct image points $\tilde{\mathbf{x}} \notin \Omega$ so that the function $\Phi(C\tilde{\mathbf{x}}-\mathbf{y})$ cancels $\Phi(\mathbf{x}-\mathbf{y})$ at the boundary $\partial\Omega$ for some constant C (or a function $C(\mathbf{x})$). Then, $\Phi(C\tilde{\mathbf{x}}-\mathbf{y})$ is harmonic with respect to $\mathbf{y}$, and we can set $\Delta\varphi^{\mathbf{y}}(\mathbf{x}) = \Phi(C\tilde{\mathbf{x}}-\mathbf{y})$. Below we demonstrate two examples from [47] constructing $\Phi(C\tilde{\mathbf{x}}-\mathbf{y})$.

Example 11.1. *(method of images for half-space) Consider the half-space* $\Omega := \{\mathbf{x} \in \mathbb{R}^n : x_n > 0\}$. *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in \Omega$ *and define its image* $\tilde{\mathbf{x}} = (x_1, \ldots, x_{n-1}, -x_n)(\notin \Omega)$. *Then,*

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(\tilde{\mathbf{x}}-\mathbf{y}), \quad x_n \geq 0,\ y_n \geq 0,$$

will be established as the Green's function if $G(\mathbf{x},\mathbf{y}) = 0$ *for* $y_n = 0$, $x_n > 0$. *Now since for* $y_n = 0$ *we have that*

$$|\mathbf{y}-\mathbf{x}| = \sqrt{\sum_{j=1}^{n-1}(y_j - x_j)^2 + x_n^2} = |\mathbf{y}-\tilde{\mathbf{x}}|,$$

hence, we deduce that $\Phi(\mathbf{x}-\mathbf{y}) = \Phi(\tilde{\mathbf{x}}-\mathbf{y})$.

Example 11.2. *(method of images for unit ball) Let* $\Omega := \{\mathbf{x} : |\mathbf{x}| \leq 1\}$, *be the unit ball in* $\mathbb{R}^n$, *and define the image* $\tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}^2|$ *of a given* $\mathbf{x} \in \Omega$. *Note that* $\tilde{\mathbf{x}}$ *is a scaling followed by a reflection through the boundary. Let now*

$$G(\mathbf{x},\mathbf{y}) = \Phi(\mathbf{x}-\mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}}-\mathbf{y})). \tag{11.1.11}$$

We need to show that $G(\mathbf{x},\mathbf{y})$ *defined as* (11.1.11) *is the Green's function for the unit ball with Dirichlet boundary condition. This is equivalent to showing*

$$G(\mathbf{x},\mathbf{y}) = 0, \quad \mathbf{x} \in \Omega,\ \mathbf{y} \in \partial\Omega.$$

Hence, it suffices to establish

$$|\mathbf{x}||\tilde{\mathbf{x}}-\mathbf{y}| = |\mathbf{x}-\mathbf{y}|, \quad \text{for } 0 < |\mathbf{x}| < 1,\ \tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}^2|, \quad \text{and } |\mathbf{y}| = 1,$$

which follows from the identity

$$|\mathbf{x}|^2|\tilde{\mathbf{x}}-\mathbf{y}|^2 = \left|\frac{\mathbf{x}}{|\mathbf{x}|} - |\mathbf{x}|\mathbf{y}\right|^2 = |\mathbf{x}|^2 - 2\mathbf{y}\cdot\mathbf{x} + 1 = |\mathbf{x}-\mathbf{y}|^2.$$

11.2 Stability

To derive stability estimates for (11.0.1) we shall assume an underlying general vector space V (to be specified below) of functions as the solution space. We multiply the equation by $u \in V$ and integrate over Ω to obtain

$$-\int_{\Omega} (\Delta u) u dx = \int_{\Omega} f\, u\, dx. \tag{11.2.1}$$

Using Green's formula and the boundary condition, $u = 0$ on Γ, we get that

$$\|\nabla u\|^2 = \int_{\Omega} f u \leq \|f\|\ \|u\|, \tag{11.2.2}$$

where $\|\cdot\|$ denotes the usual $L_2(\Omega)$-norm.

To derive a, so called, *weak stability estimate* for (11.0.1) we combine the Poincare inequality with the inequality (11.2.2) and get

$$\|\nabla u\| \leq C_{\Omega} \|f\|. \tag{11.2.3}$$

11.3 Error Estimates for the CG(1) FEM

We start formulating the *weak form* for the problem (11.0.1): We multiply the equation by a test function v, integrate over $\Omega \subset \mathbb{R}^2$ and use Green's formula. Then, the *variational formulation* reads as follows: find $u \in H_0^1(\Omega)$ such that

$$(VF): \quad \int_{\Omega} \nabla u \cdot \nabla v\, dx = \int_{\Omega} f v\, dx, \quad \forall v \in H_0^1(\Omega). \tag{11.3.1}$$

To prepare for a finite element method we approximate the exact solution $u(x)$ by a suitable discrete solution $U(x)$ in a certain finite dimensional subspace of $H_0^1(\Omega)$. To this end, let $\mathcal{T}_h = \{K : \bigcup K = \Omega\}$ be a triangulation of the domain Ω by elements K with the maximal diameter $h = \max_K \operatorname{diam}(K)$. To be specific we shall consider the cG(1) method describing continuous piecewise linear approximations for the solution $u \in H_0^1(\Omega)$ in a finite dimensional subspace. Thus we have the test and trial function space given as

$$V_h^0 = \{v(x) : v \text{ is continuous, piecewise linear on } \mathcal{T}_h, \text{ and } v = 0 \text{ on } \Gamma = \partial\Omega\}.$$

Observe that $V_h^0 \subset H_0^1(\Omega)$ is associated to the chosen partition $\mathcal{T}_h$. We assume n interior partition nodes with $\varphi_j, j = 1, 2, \ldots, n$, as the corresponding basis functions with the property that, each $\varphi_j(x),\ j = 1, 2, \ldots, n$, is continuous in Ω, linear on each K, and

$$\varphi_j(N_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \tag{11.3.2}$$

Here $N_1, N_2, \ldots, N_n$ are the interior nodes in the triangulation, and $\dim V_h^0 = n$. Now we set the approximate solution $u_h(x)$ to be a linear combination of the basis functions $\varphi_j,\ j = 1, \ldots, n$:

$$u_h(x) = u_{h,1}\varphi_1(x) + u_{h,2}\varphi_2(x) + \ldots + u_{h,n}\varphi_n(x), \tag{11.3.3}$$

and seek the coefficients $u_{h,j} = u_h(N_j)$, i.e., the nodal values of $u_h(x)$ (which are not necessarily $= u(N_j)$), at the nodes $N_j,\ 1 \le j \le n$, so that

$$\int_\Omega \nabla u_h \cdot \nabla v\, dx = \int_\Omega f\, v\, dx, \quad \forall v \in V_h^0, \tag{11.3.4}$$

or equivalently

$$(FEM) \qquad \int_\Omega \nabla u_h \cdot \nabla \varphi_i\, dx = \int_\Omega f\, \varphi_i\, dx, \quad \text{for} \quad i = 1, 2, \ldots, n. \tag{11.3.5}$$

Note that every $v \in V_h^0$ can be represented by a linear combination of $\{\varphi_j\}_{j=1}^n$:

$$v(x) = v(N_1)\varphi_1(x) + v(N_2)\varphi_2(x) + \ldots + v(N_n)\varphi_n(x). \tag{11.3.6}$$

The equations (11.3.5) (or (11.3.4)) are the finite element formulation for (11.0.1).

Theorem 11.3 ($cG(1)$ a priori error estimate for the gradient $\nabla u - \nabla u_h$)**.** *Let $e = u - u_h$ represent the error in the above continuous, piecewise linear, finite element approximation of the solution for* (11.0.1). *Then there is constant C, independent of h, such that*

$$\|\nabla e\| = \|\nabla(u - u_h)\| \le C\|h\, D^2 u\|. \tag{11.3.7}$$

Proof. For the error $e = u - u_h$ we have $\nabla e = \nabla(u - u_h) = \nabla u - \nabla u_h$. Subtracting (11.3.4) from the (11.3.1) where we restrict v to V_h^0, we obtain the *Galerkin Orthogonality* relation

$$\int_\Omega (\nabla u - \nabla u_h) \cdot \nabla v\, dx = \int_\Omega \nabla e \cdot \nabla v\, dx = 0, \qquad \forall v \in V_h^0. \tag{11.3.8}$$

Further we may write

$$\|\nabla e\|^2 = \int_\Omega \nabla e \cdot \nabla e\, dx = \int_\Omega \nabla e \cdot \nabla(u - u_h)\, dx = \int_\Omega \nabla e \cdot \nabla u\, dx - \int_\Omega \nabla e \cdot \nabla u_h\, dx.$$

Now using the Galerkin orthogonality (11.3.8), since $u_h \in V_h^0$ we have the last integral above: $\int_\Omega \nabla e \cdot \nabla u_h\, dx = 0$. Hence removing the vanishing ∇u_h-term and inserting $\int_\Omega \nabla e \cdot \nabla v\, dx = 0$ for any arbitrary $v \in V_h^0$ we get

$$\|\nabla e\|^2 = \int_\Omega \nabla e \cdot \nabla u dx - \int_\Omega \nabla e \cdot \nabla v dx = \int_\Omega \nabla e \cdot \nabla(u - v) dx \le \|\nabla e\|\, \|\nabla(u - v)\|.$$

Thus

$$\|\nabla(u-u_h)\| \le \|\nabla(u-v)\|, \quad \forall v \in V_h^0. \tag{11.3.9}$$

That is, in the L_2-norm, the gradient of the finite element solution, ∇u_h is closer to the gradient of the exact solution: ∇u, than the gradient ∇v of any other v in V_h^0. In other words, measuring in H_0^1-norm, the error

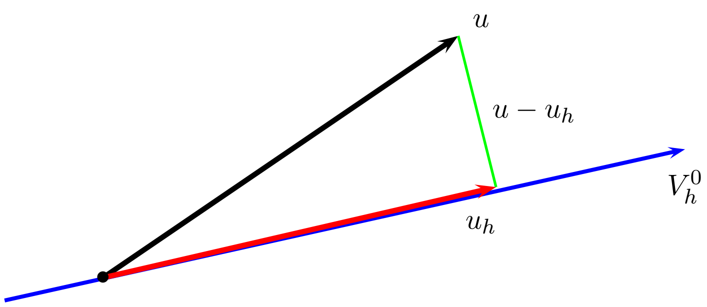


Figure 11.1: The orthogonal (L_2) projection of u on V_h^0.

$u-u_h$ is orthogonal to V_h^0. It is possible to show that there is a $v \in V_h^0$ (an interpolant), such that

$$\|\nabla(u-v)\| \le C\|h\, D^2 u\|, \tag{11.3.10}$$

where $h = h(x) = \text{diam}(K)$ for $x \in K$ and C is a constant independent of h. This is the case, for example, if v interpolates u at the nodes N_i.

Combining (11.3.9) and (11.3.10) we end up with

$$\|\nabla e\| = \|\nabla(u-u_h)\| \le C\|h\, D^2 u\|, \tag{11.3.11}$$

which is indicating that the error is small if $h(x)$ is sufficiently small depending on D^2u.

$\square$

To prove an a priori error estimate *for the solution* we shall use the following result:

Lemma 11.1 (regularity lemma)**.** *Assume that Ω has no reentrent corners (Ω is convex). We have for $u \in H^2(\Omega)$; with $u = 0$ or $(\frac{\partial u}{\partial n} = 0)$ on $\partial\Omega$, that*

$$\|D^2 u\| \le C_\Omega \, \|\Delta u\|, \tag{11.3.12}$$

where

$$D^2 u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}. \tag{11.3.13}$$

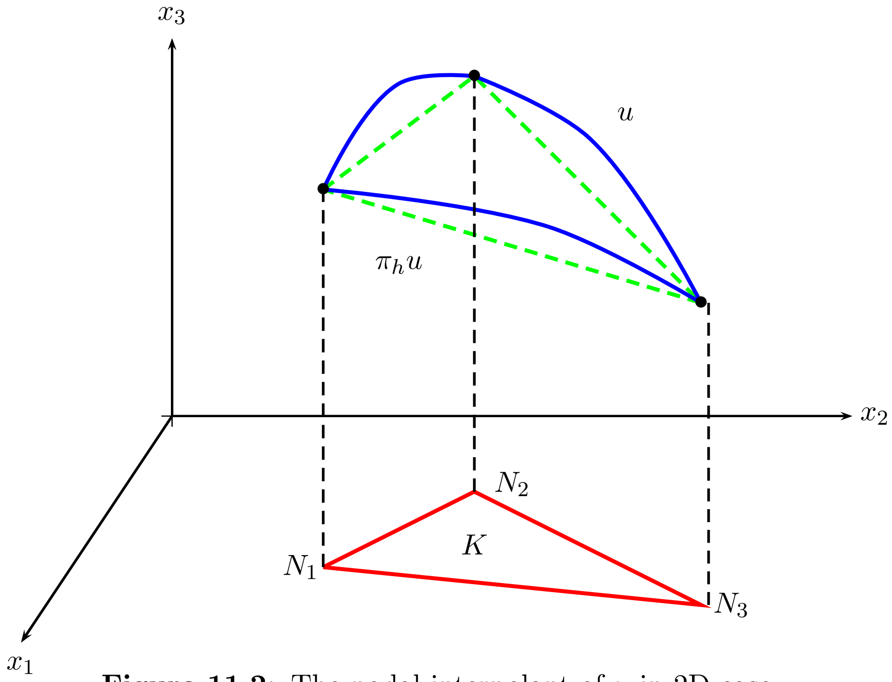


Figure 11.2: The nodal interpolant of u in 2D case

We postpone the proof of this lemma and first derive the error estimate.

Theorem 11.4 ($cG(1)$ a priori error estimate for the solution $e = u - u_h$). *For a general mesh we have the following a priori error estimate for the solution of the Poisson equation* (11.0.1)*,*

$$\|e\| = \|u - u_h\| \le C^2 C_\Omega \, (\max_\Omega h) \, \|h \, D^2 u\|, \tag{11.3.14}$$

where C is a constant (generated twice).

Proof. Let φ be the solution of the *dual problem* for (11.0.1).

$$\begin{cases} -\Delta\varphi = e, & \text{in } \Omega \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \tag{11.3.15}$$

Then, we have using Green's formula that

$$\begin{aligned} \|e\|^2 = \int_\Omega e(-\Delta\varphi) dx = \int_\Omega \nabla e \cdot \nabla\varphi \, dx = \int_\Omega \nabla e \cdot \nabla(\varphi - v) \, dx \\ \le \|\nabla e\| \|\nabla(\varphi - v)\|, \qquad\qquad \forall v \in V_h^0, \end{aligned} \tag{11.3.16}$$

where in the last equality we have used the Galerkin orthogonality. We now choose v (e.g. as an interpolant of φ) such that

$$\|\nabla(\varphi - v)\| \le C\|h \, D^2\varphi\| \le Ch\|D^2\varphi\|. \tag{11.3.17}$$

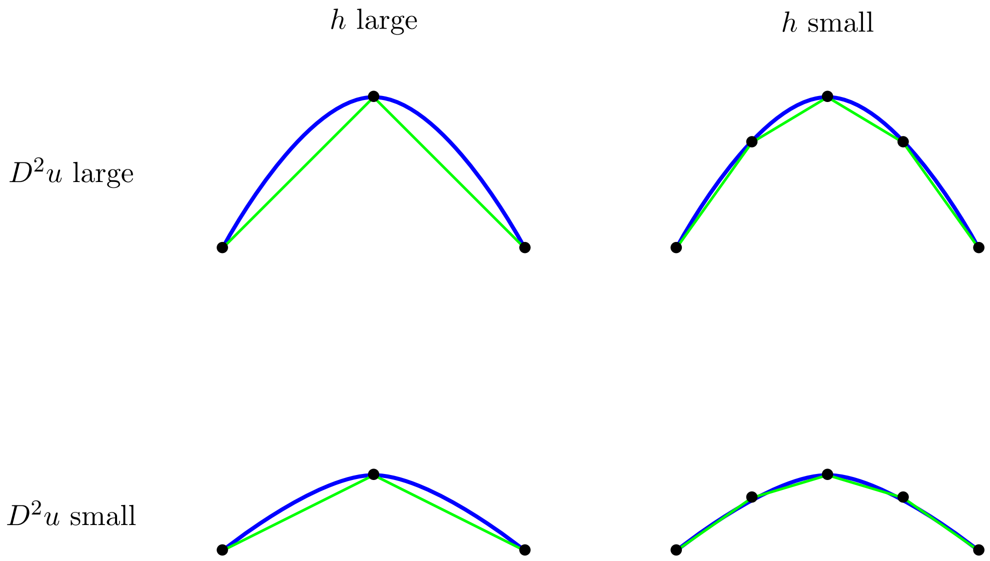


Figure 11.3: The adaptivity principle: to refine mesh for large D^2u

Applying the lemma to φ, we get

$$\|D^2\varphi\| \leq C_\Omega \,\|\Delta\varphi\| = C_\Omega \|e\|. \tag{11.3.18}$$

Finally, combining (11.3.11)-(11.3.18) yields

$$\begin{aligned}\|e\|^2 &\leq \|\nabla e\|\,\|\nabla(\varphi - v)\| \leq \|\nabla e\|\, Ch\,\|D^2\varphi\| \\ &\leq \|\nabla e\|\, ChC_\Omega \|e\| \leq C\|h\,D^2u\|\,C\,hC_\Omega\|e\|.\end{aligned} \tag{11.3.19}$$

Thus we have obtained the desired a priori error estimate

$$\|e\| = \|u - u_h\| \leq C^2\,C_\Omega\,h\,\|h\,D^2u\|. \tag{11.3.20}$$

□

Corollary 11.1 (strong stability estimate)**.** *Using the Lemma 11.1, for a piecewise linear approximation, the a priori error estimate* (11.3.20) *can be written as the following strong stability estimate,*

$$\|u - u_h\| \leq C^2\,C_\Omega\,h^2\,\|f\|. \tag{11.3.21}$$

Theorem 11.5 ($cG(1)$ *a posteriori* error estimate)**.** *Let u be the solution of the Poisson equation* (11.0.1) *and u_h its continuous piecewise linear finite element approximation. Then, there is a constant C, independent of u and h, such that*

$$\|u - u_h\| \leq C\,\|h^2 r\|, \tag{11.3.22}$$

where $r = f + \Delta_h u_h$ is the residual with Δ_h being the discrete Laplacian operator defined by

$$(\Delta_h u_h, v) = \sum_{K\in\mathcal{T}_h} (\nabla u_h, \nabla v)_K. \tag{11.3.23}$$

Proof. We start considering the following dual problem

$$\begin{cases} -\Delta\varphi(x) = e(x), & x \in \Omega, \\ \varphi(x) = 0, \quad x \in \partial\Omega, & e(x) = u(x) - u_h(x). \end{cases} \tag{11.3.24}$$

Then $e(x) = 0,\ \forall x \in \partial\Omega$, and using Green's formula we have that

$$\|e\|^2 = \int_\Omega e\, e\, dx = \int_\Omega e(-\Delta\varphi)dx = \int_\Omega \nabla e \cdot \nabla\varphi\, dx. \tag{11.3.25}$$

Thus, by the Galerkin orthogonality (11.3.8) and using the boundary data: $v(x) = \varphi(x) = 0$, $\forall x \in \partial\Omega$, we get

$$\begin{aligned} \|e\|^2 &= \int_\Omega \nabla e \cdot \nabla\varphi\, dx - \int_\Omega \nabla e \cdot \nabla v dx = \int_\Omega \nabla e \cdot \nabla(\varphi - v)\, dx \\ &= \int_\Omega (-\Delta e)(\varphi - v)\, dx \le \|h^2 r\|\, \|h^{-2}(\varphi - v)\| \\ &\le C\, \|h^2 r\|\, \|\Delta\varphi\| \le C\, \|h^2 r\|\, \|e\|, \end{aligned} \tag{11.3.26}$$

where we use the fact that the $-\boldsymbol{\Delta} e = -\Delta u + \Delta_h u_h + [\frac{\partial u_h}{\partial \mathbf{n}}] = f + \Delta_h u_h = r$ is the residual and choose v as an interpolant of φ, so that

$$\|h^{-2}(\varphi - v)\| \le C\|D^2\varphi\| \le C\, C_\Omega \|\boldsymbol{\Delta}\varphi\|.$$

Thus, the final result reads as

$$\|u - u_h\| \le C_\Omega\, \|h^2 r\|. \tag{11.3.27}$$

□

Corollary 11.2 (strong stability estimate)**.** *Observe that for piecewise linear approximations $\Delta u_h = 0$ but $\Delta_h u_h \neq 0$ on each element K. If $\Delta_h u_h = 0$ on each element K then $r \equiv f$ and our a posteriori error estimate above can be viewed as a strong stability estimate*

$$\|e\| \le C\, \|h^2 f\|. \tag{11.3.28}$$

We now return to the proof of Lemma 10.2. One can show that, in the lemma for a convex Ω the constant $C_\Omega \le 1$. Otherwise $C_\Omega > 1$ and increases with the degree of the singularity of a corner point at the boundary of the domain Ω. With Ω having a reenterant we have that $C_\Omega = \infty$.

11.3.1 Proof of the regularity Lemma

Proof. Let now Ω be a rectangular domain and set $u = 0$ on $\partial\Omega$, then

$$\|\Delta u\|^2 = \int_\Omega (u_{xx} + u_{yy})^2 dxdy = \int_\Omega (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2)\, dxdy. \quad (11.3.29)$$

Applying Green's formula (in x-direction) for the rectangular domain Ω we can write (note that partial derivatives of higher order appears),

$$\int_\Omega u_{xx}u_{yy}dxdy = \int_{\partial\Omega} (u_x \cdot n_x)u_{yy}\, ds - \int_\Omega u_x \underbrace{u_{yyx}}_{=u_{xyy}} dxdy. \quad (11.3.30)$$

Once again by Green's formula ("with $v = u_x$ and $\Delta w = u_{xyy}$"),

$$\int_\Omega u_x u_{xyy} dxdy = \int_{\partial\Omega} u_x(u_{yx} \cdot n_y)ds - \int_\Omega u_{xy}u_{xy}\, dxdy. \quad (11.3.31)$$

Inserting (11.3.31) in (11.3.30) yields

$$\int_\Omega u_{xx}u_{yy}\, dxdy = \int_{\partial\Omega} (u_x u_{yy} n_x - u_x u_{yx} n_y)ds + \int_\Omega u_{xy}u_{xy}\, dxdy. \quad (11.3.32)$$

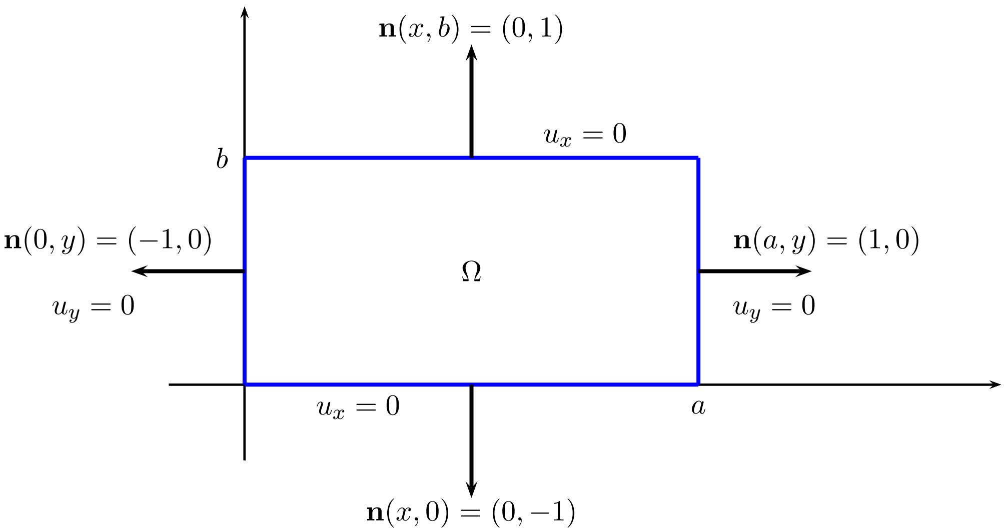


Figure 11.4: Rectangular domain Ω with outward unit normals to its sides

Now, as we can see from the figure, we have that $(u_x u_{yy} n_x - u_x u_{yx} n_y) = 0$, on $\partial\Omega$ and hence

$$\int_\Omega u_{xx}u_{yy}dxdy = \int_\Omega u_{xy}u_{xy}dxdy = \int_\Omega u_{xy}^2\, dxdy. \quad (11.3.33)$$

Thus, in this case,

$$\|\Delta u\|^2 = \int_\Omega (u_{xx} + u_{yy})^2 dxdy = \int_\Omega (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) dxdy = \|D^2 u\|^2,$$

and the proof is complete by a constant $C_\Omega \equiv 1$. The general case of a polygonal domain, easily, follows from this proof. $\square$

11.4 Exercises

Problem 11.1. *Derive stability estimate for the Neumann boundary value problem:*

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{11.4.1}$$

Problem 11.2. *Show that* $\|\nabla(u - u_h)\| \leq C\|hr\|$.

Problem 11.3. *Verify that for* v *being the interpolant of* φ, *we have*

$$\|e\| \leq C\,\|h^2 f\| \times \begin{cases} \|h^{-2}\,(\varphi - v)\| & \leq C\,\|\Delta\varphi\|, \quad \text{and} \\ \|h^{-1}(\varphi - v)\| & \leq C\,\|\nabla\varphi\|. \end{cases} \tag{11.4.2}$$

Problem 11.4. *Derive the estimate corresponding to (11.3.27) for the one-dimensional case.*

Problem 11.5. *Consider the following two dimensional problem:*

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_N \end{cases} \tag{11.4.3}$$

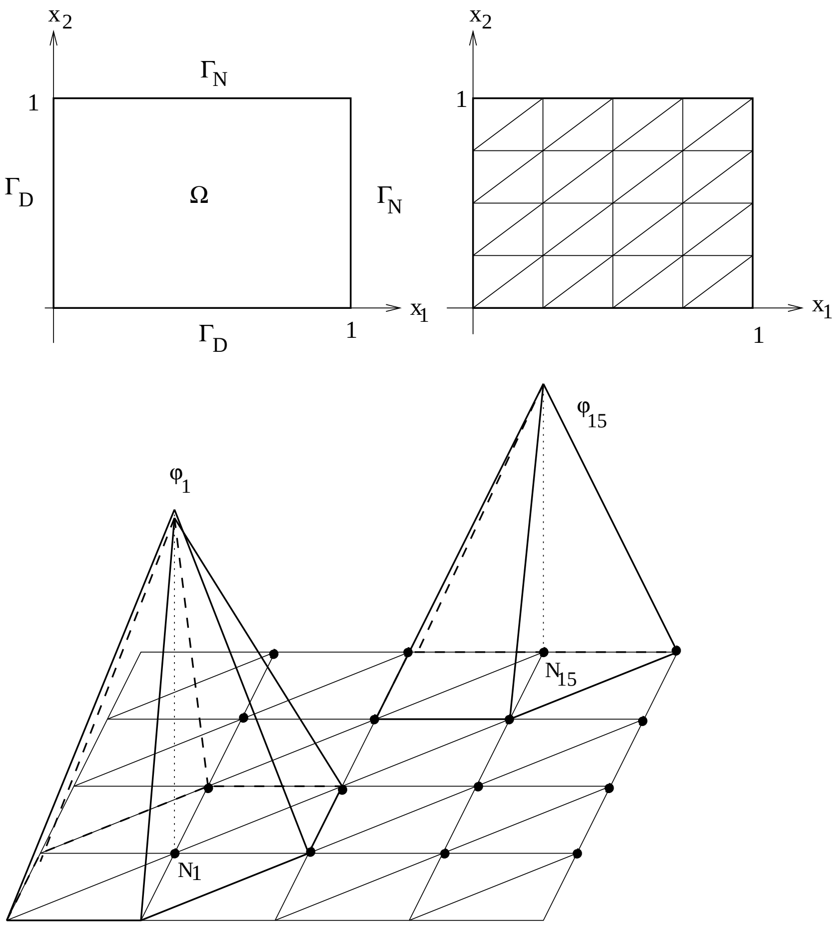
See figure below

Triangulate Ω *as in the figure and let*

$$U(x) = U_1\varphi_1(x) + \ldots + U_{16}\varphi_{16}(x),$$

where $x = (x_1, x_2)$ *and* φ_j, $j = 1, \ldots 16$ *are the basis functions, see Figure below, and determine* $U_1, \ldots U_{16}$ *so that*

$$\int_\Omega \nabla U \cdot \nabla\varphi_j dx = \int_\Omega \varphi_j dx, \quad j = 1, 2, \ldots, 16.$$



Problem 11.6. *Generalize the procedure in the previous problem to the following case with mesh-size=h.*

$$\begin{cases} -\nabla(a\nabla u) = f, & \textit{in } \Omega \\ u = 0, & \textit{on } \Gamma_D \\ a\frac{\partial u}{\partial n} = 7, & \textit{on } \Gamma_N \end{cases}, \quad \textit{where} \quad \begin{cases} a = 1 \quad \textit{for } x_1 < \frac{1}{2} \\ a = 2 \quad \textit{for } x_1 > \frac{1}{2} \\ f = x_2. \end{cases}$$

Problem 11.7. *Consider the Dirichlet problem*

$$-\nabla\cdot(a(x)\nabla u(x)) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \qquad u(x) = 0, \ \textit{for } x \in \partial\Omega.$$

Assume that c_0 *and* c_1 *are constants such that* $0 < c_0 \leq a(x) \leq c_1, \ \forall x \in \Omega$ *and let* U *be a Galerkin approximation of* u *in a subspace* M *of* $H_0^1(\Omega)$*. Prove the a priori error estimate*

$$||u - U||_{H_0^1(\Omega)} \leq \frac{c_1}{c_0} \inf_{\chi\in M} ||u - \chi||_{H_0^1(\Omega)}.$$

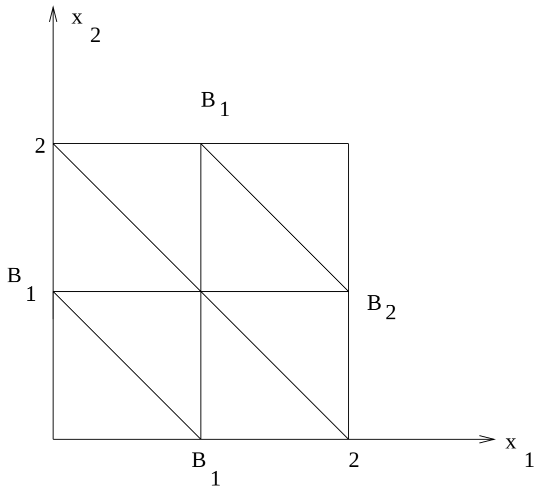
Problem 11.8. *Determine the stiffness matrix and load vector if the cG(1) finite element method applied to the Poisson's equation on a triangulation with triangles of side length 1/2 in both x_1- and x_2-directions:*

$$\begin{cases} -\Delta u = 1, & in \quad \Omega = \{(x_1, x_2): \ 0 < x_1 < 2, \ 0 < x_2 < 1\}, \\ u = 0, & on \quad \Gamma_1 = \{(0, x_2)\} \cup \{(x_1, 0)\} \cup \{(x_1, 1)\}, \\ \frac{\partial u}{\partial n} = 0, & on \quad \Gamma_2 = \{(2, x_2) : 0 \le x_2 \le 1\}. \end{cases}$$

Problem 11.9. *Let $\Omega = (0,2)\times(0,2)$, $B_1 = \partial\Omega \setminus B_2$ and $B_2 = \{2\}\times(0,2)$. Determine the stiffness matrix and load vector in the cG(1) solution for the problem*

$$\begin{cases} -\frac{\partial^2 u}{\partial x_1^2} - 2\frac{\partial^2 u}{\partial x_2^2} = 1, & in\ \Omega = (0,2)\times(0,2), \\ u = 0, \quad on\ B_1, & \frac{\partial u}{\partial x_1} = 0, \quad on\ B_2, \end{cases}$$

with piecewise linear approximation applied on the triangulation below:



Problem 11.10. *Determine the stiffness matrix and load vector if the cG(1) finite element method with piecewise linear approximation is applied to the following Poisson's equation with mixed boundary conditions:*

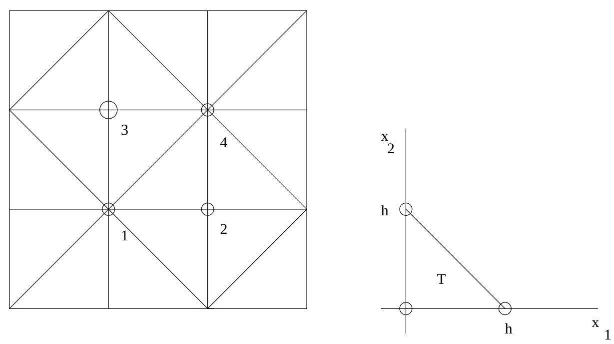
$$\begin{cases} -\Delta u = 1, & on \quad \Omega = (0,1)\times(0,1), \\ \frac{\partial u}{\partial n} = 0, & for \quad x_1 = 1, \\ u = 0, & for \quad x \in \partial\Omega \setminus \{x_1 = 1\}, \end{cases}$$

on a triangulation with triangles of side length 1/4 in the x_1-direction and 1/2 in the x_2-direction.

Problem 11.11. *Formulate the cG(1) method for the boundary value problem*

$$-\Delta u + u = f, \quad x \in \Omega; \qquad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting system of equations using the following uniform mesh:



Chapter 12

The Initial Boundary Value Problems in $\mathbb{R}^N$

This chapter is devoted to the study of time dependent problems formulated in spatial domain in $\mathbb{R}^N$, $N \geq 2$. More specifically we shall consider the finite element methods for two fundamental equations: the heat equation and the wave equation. In this setting we have already studied time discretizations as well as a concise finite element analysis of the Laplace's equation in Chapters 8 and 11, respectively. Here we shall derive certain stability estimates for these equations and construct fully discrete schemes for temporal and spatial discretizations. We also formulate some convergence results that can be viewed as a combination of error estimates in Chapters 8 and 11. Detailed proofs for a priori and a posteriori error estimates are direct generalization of the results of Chapter 9 for one space dimension and can be found in, e.g., Thomeé [55] and Eriksson et al. in [20]

12.1 The heat equation in $\mathbb{R}^N$

In this section we shall study the stability of the heat equation in $\mathbb{R}^N$, $N \geq 2$. Here our concern will be those aspects of the stability estimates for the higher dimensional case that are not a direct consequence of the study of the one-dimensional problem addressed in Chapter 9.

The initial value problem for the heat conduction can be formulated as

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = f(x,t), & \text{in } \mathbb{R}^d, d = 1,2,3 \quad (DE) \\ u(x,0) = u_0(x), & \text{for } x \in \mathbb{R}^d. \quad (IC) \end{cases} \tag{12.1.1}$$

The equation (12.1.1) is of parabolic type with significant *smoothing* and *stability* properties. It can also be used as a model for a variety of physical phenomena involving *diffusion processes*. Some important physical properties of (12.1.1) are addressed in Chapter 2. Here, we shall focus on the stability concept and construction of the fully discrete schemes, which is a combination of the discretization procedure in Chapters 7 and 11. For details in a priori and a posteriori analysis we refer the reader to Chapter 16 in [20].

12.1.1 The fundamental solution

We consider the homogeneous heat conduction in the general d-dimensional Euclidean space:

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = 0, & \text{for } (x,t) \in \mathbb{R}^d \times (0,\infty) \\ u(x,0) = u_0(x), & \text{for } \ x \in \mathbb{R}^d. \end{cases} \tag{12.1.2}$$

First we seek the *fundamental solution* of (12.1.2) in the two-dimensional case and when the initial data is assumed to be the Dirac δ-function: $u_0(x) = \delta_0(x)$. We denote this solution by $E(x,t)$. One may easily verify that

$$E(x,t) = \frac{1}{(4\pi t)} \exp\Big(-\frac{|x|^2}{4t}\Big). \tag{12.1.3}$$

Then, for an arbitrary initial data u_0 the solution for (12.1.2), with $d = 2$, can be expressed in terms of fundamental solution and is obtained through convolution between u_0 and E:

$$u(x,t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} u_0(y) \exp\Big(-\frac{|x-y|^2}{4t}\Big)\, dy. \tag{12.1.4}$$

For the general problem (12.1.2), using the d-dimensional Fourier transform, see [33], we have the d-dimensional versions of (12.1.3) and (12.1.4), viz

$$E(x,t) = \frac{1}{(4\pi t)^{d/2}} \exp\Big(-\frac{|x|^2}{4t}\Big), \quad x \in \mathbb{R}^d, \ \ t > 0, \tag{12.1.5}$$

and

$$u(x,t) =: (E(t)u_0)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} u_0(y) \exp\Big(-\frac{|x-y|^2}{4t}\Big)\, dy. \tag{12.1.6}$$

Here, if we let $\eta = \frac{y-x}{\sqrt{4t}}$, we get

$$\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\Big(-\frac{|x-y|^2}{4t}\Big)\, dy = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|\eta|^2}\, d\eta = 1, \tag{12.1.7}$$

consequently, we have that

$$|u(x,t)| \le \|u_0\|_{L_\infty(\mathbb{R}^d)}, \qquad \text{for} \quad t > 0. \tag{12.1.8}$$

Now, we return to the original, non-homogeneous, heat equation (12.1.1) with $f \neq 0$ and recall that E, defined as in (12.1.6), is interpreted as an operator acting on initial data u_0. This can be extended to f and using the *Duhamel's principle*, see [33], the solution to the problem (12.1.1) can be expressed as

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(x,s)\,ds. \tag{12.1.9}$$

Then as usual, it is easy to verify, (see below), that

$$\|u(t)\| \le \|u_0\| + \int_0^t \|f(s)\|\,ds, \tag{12.1.10}$$

which yields the uniqueness in the standard way.

12.2 Stability

Here we consider the following initial boundary value problem for the homogeneous heat equation in a bounded domain $\Omega \subset \mathbb{R}^2$,

$$\begin{cases} \dot u(x,t) - \Delta u(x,t) = 0, & \text{for } (x,t) \in \Omega \times (0,\infty) \\ u(x,t) = 0, & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \end{cases} \tag{12.2.1}$$

The stability estimates for the equation (12.2.1) are summarized in theorem below. It roughly states a control of certain energy norms of the solution u by the L_2 norm of the initial data u_0, where some of the estimates below up in the vicinity of the initial time $t = 0$.

Theorem 12.1 (Energy estimates)**.** *The solution u of the problem* (12.2.1) *satisfies the following stability estimates*

$$\max\Big(\|u(\cdot,t)\|^2,\ 2\int_0^t \|\nabla u\|^2(s)\,ds\Big) \le \|u_0\|^2 \tag{12.2.2}$$

$$\|\nabla u(\cdot,t)\| \le \frac{1}{\sqrt{2}\,t}\|u_0\| \tag{12.2.3}$$

$$\Big(\int_0^t s\|\Delta u(\cdot,s)\|^2\,ds\Big)^{1/2} \le \frac{1}{2}\|u_0\| \tag{12.2.4}$$

$$\|\Delta u(\cdot,t)\| \le \frac{1}{\sqrt{2}\,t}\|u_0\| \tag{12.2.5}$$

$$\textit{For } \varepsilon > 0 \textit{ small,} \quad \int_\varepsilon^t \|\dot u(\cdot,s)\|\,ds \le \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\,\|u_0\|. \tag{12.2.6}$$

Note that (12.2.6) *indicates strong instabilities for ε (as well as t) close to 0. This yields the splitting strategy used in error analysis below.*

Proof. To derive the first two estimates in (12.2.2) we multiply (12.1.1) by u and integrate over Ω, to get

$$\int_\Omega \dot{u}u\,dx - \int_\Omega (\Delta u)u\,dx = 0. \tag{12.2.7}$$

Note that $\dot{u}u = \frac{1}{2}\frac{d}{dt}u^2$. Further, using Green's formula with the Dirichlet boundary data, $u = 0$ on Γ, we have that

$$-\int_\Omega (\Delta u)u\,dx = -\int_\Gamma (\nabla u \cdot \mathbf{n})\,u\,ds + \int_\Omega \nabla u \cdot \nabla u\ dx = \int_\Omega |\nabla u|^2\,dx. \tag{12.2.8}$$

Thus equation (12.2.7) can be written in the following two equivalent forms:

$$\frac{1}{2}\frac{d}{dt}\int_\Omega u^2 dx + \int_\Omega |\nabla u|^2 dx = 0 \iff \frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\nabla u\|^2 = 0, \tag{12.2.9}$$

where $\|\cdot\|$ denotes the $L_2(\Omega)$ norm. We substitute t by s and integrate the equation (12.2.9) over $s \in (0,t)$ to get

$$\frac{1}{2}\int_0^t \frac{d}{ds}\|u\|^2(s)ds + \int_0^t \|\nabla u\|^2(s)ds = \frac{1}{2}\|u\|^2(t) - \frac{1}{2}\|u\|^2(0) + \int_0^t \|\nabla u\|^2 ds = 0.$$

Hence, inserting the initial data $u(0) = u_0$ we have

$$\|u\|^2(t) + 2\int_0^t \|\nabla u\|^2(s)\,ds = \|u_0\|^2. \tag{12.2.10}$$

In particular, we have our first two stability estimates

$$\|u\|(t) \le \|u_0\|, \qquad \text{and} \qquad \int_0^t \|\nabla u\|^2(s)\,ds \le \frac{1}{2}\|u_0\|. \tag{12.2.11}$$

To derive (12.2.3) and (12.2.4), since $\dot{u} = \Delta u$, and we need to estimate the L_2-norm of $\dot{u}$, so we multiply the equation, $\dot{u} - \Delta u = 0$, by $-t\,\Delta u$ and integrate over Ω to get

$$-t\int_\Omega \dot{u}\,\Delta u\,dx + t\int_\Omega (\Delta u)^2\,dx = 0. \tag{12.2.12}$$

Using Green's formula ($u = 0$ on $\Gamma \implies \dot{u} = 0$ on Γ) yields

$$\int_\Omega \dot{u}\,\Delta u\,dx = -\int_\Omega \nabla\dot{u} \cdot \nabla u\,dx = -\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2, \tag{12.2.13}$$

so that (12.2.12) can be rewritten as

$$t\,\frac{1}{2}\,\frac{d}{dt}\|\nabla u\|^2 + t\,\|\Delta u\|^2 = 0. \tag{12.2.14}$$

Now using the relation $t\,\frac{d}{dt}\|\nabla u\|^2 = \frac{d}{dt}(t\|\nabla u\|^2) - \|\nabla u\|^2$, (12.2.14) can be rewritten as

$$\frac{d}{dt}\Big(t\|\nabla u\|^2\Big) + 2t\|\Delta u\|^2 = \|\nabla u\|^2. \tag{12.2.15}$$

Once again we substitute t by s and integrate over $(0,t)$ to get

$$\int_0^t \frac{d}{ds}\Big(s\|\nabla u\|^2(s)\Big)\,ds + 2\int_0^t s\|\Delta u\|^2(s)ds = \int_0^t \|\nabla u\|^2(s)ds.$$

Using (12.2.11), the last equality is estimated as

$$t\,\|\nabla u\|^2(t) + 2\int_0^t s\,\|\Delta u\|^2(s)\,ds \le \frac{1}{2}\|u_0\|^2. \tag{12.2.16}$$

In particular, we have

$$\|\nabla u\|(t) \le \frac{1}{\sqrt{2t}}\|u_0\| \qquad \text{and} \qquad \Big(\int_0^t s\|\Delta u\|^2(s)\,ds\Big)^{1/2} \le \frac{1}{2}\|u_0\|,$$

which are our third and fourth stability estimates (12.2.3) and (12.2.4). The stability estimate (12.2.5) is proved analogously. Now using (12.1.1): ($\dot u = \Delta u$) and (12.2.5) we may write

$$\int_\varepsilon^t \|\dot u\|(s)ds \le \frac{1}{\sqrt{2}}\|u_0\|\int_\varepsilon^t \frac{1}{s}\,ds = \frac{1}{\sqrt{2}}\ln\frac{t}{\varepsilon}\|u_0\| \tag{12.2.17}$$

or more carefully

$$\begin{aligned}\int_\varepsilon^t \|\dot u\|(s)ds &= \int_\varepsilon^t \|\Delta u\|(s)ds = \int_\varepsilon^t 1\,\|\Delta u\|(s)ds = \int_\varepsilon^t \frac{1}{\sqrt{s}}\,\sqrt{s}\|\Delta u\|(s)ds\\ &\le \Big(\int_\varepsilon^t s^{-1}\,ds\Big)^{1/2}\Big(\int_\varepsilon^t s\|\Delta u\|^2(s)\,ds\Big)^{1/2}\\ &\le \frac{1}{2}\sqrt{\ln\frac{t}{\varepsilon}}\|u_0\|,\end{aligned}$$

where in the last two inequalities we use Cauchy Schwarz inequality and (12.2.4), respectively. This yields (12.2.6) and the proof is complete. □

Problem 12.1. *Show that $\|\nabla u(t)\| \le \|\nabla u_0\|$ (the stability estimate for the gradient). Hint: Assume $f = 0$, multiply* (12.1.1) *by $-\Delta u$ and integrate over Ω.*

Is this inequality valid for $u_0 =$ constant?

Problem 12.2. *Derive the corresponding estimate for Neumann boundary condition:*

$$\frac{\partial u}{\partial n} = 0. \tag{12.2.18}$$

12.3 The finite element for heat equation

12.3.1 The semidiscrete problem

Here we consider only the space discretization, hence the problem is continuous in time, and is referred to as a *spatially semidiscrete* problem. We focus on a 2-dimensional heat conduction formulated for a bounded convex domain $\Omega \subset \mathbb{R}^2$ with smooth boundary Γ. Hence we consider the *non-homogeneous* initial-boundary value problem (12.2.1).

$$\begin{cases} \dot{u}(x,t) - \Delta u(x,t) = f(x,t), & \text{for } \ x \in \Omega \subset \mathbb{R}^2, \quad (DE) \\ u(x,t) = 0, & x \in \Gamma, \quad 0 < t \leq T, \quad (BC) \\ u(x,0) = u_0(x), & \text{for } x \in \Omega. \quad (IC) \end{cases} \tag{12.3.1}$$

Using Green's formula we have the following weak formulation:

$$(u_t, \varphi) + (\nabla u, \nabla \varphi) = (f, \varphi), \qquad \forall \varphi \in H_0^1(\Omega), \quad t > 0. \tag{12.3.2}$$

Let now $\mathcal{T}_h$ be a triangulation of Ω of the types introduced in Chapter 10 (see Tabels 1-3), with the interior nodes $\{x_j\}_{j=1}^{M_h}$. Further, let V_h be the continuous piecewise linear functions on $\mathcal{T}_h$ vanishing on $\partial\Omega$. Finally, let $\{\psi_j\}_{j=1}^{M_h}$ be the standard basis of V_h corresponding to the nodes $\{x_j\}_{j=1}^{M_h}$.

Then, we formulate an approximate problem as: for each t find $u_h(t) = u_h(\cdot, t) \in V_h$ such that

$$\begin{aligned} (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) = (f, \chi), \qquad \forall \chi \in V_h, \quad t > 0, \\ u_h(0) = u_{0,h}, \end{aligned} \tag{12.3.3}$$

where $u_{0,h}$ in an approximation of u_0.

Thus the problem (12.3.3) is equivalent to find the coefficients $\xi_j(t)$ in the representation

$$u_h(x,t) = \sum_{j=1}^{M} \xi_j(t)\psi_j(x),$$

such that

$$\sum_{j=1}^{M} \xi'(t)(\psi_j, \psi_i) + \sum_{j=1}^{M} \xi(t)(\nabla\psi_j, \nabla\psi_i) = (f(t), \psi_i), \quad i = 1, \dots, M_h,$$

and, with η_j denoting the nodal values of the initial approximation $u_{0,h}$:

$$\xi_j(0) = \eta_j, \qquad j = 1, \dots, M_h.$$

In the matrix form this can be written as

$$M\xi'(t) + S\xi(t) = b(t), \quad t > 0, \quad \text{and} \quad \xi(0) = \eta, \tag{12.3.4}$$

where $M = (m_{ij}) = ((\psi_j, \psi_i))$ is the mass matrix, $S = (s_{ij}) = ((\nabla\psi_j, \nabla\psi_i))$ is the stiffness matrix, and $b = (b_i) = ((f, \psi_i))$ is the load vector. Recall that both mass and stiffness matrices are symmetric and positive definite, in particular M is invertible and hence (12.3.4) can be written as a system of ordinary differential equations, viz.

$$\xi'(t) + M^{-1}S\xi(t) = M^{-1}b(t), \quad t > 0, \quad \text{and} \quad \xi(0) = \eta, \tag{12.3.5}$$

which has a unique solution for $t > 0$.

We can easily derive the counterpart of the stability estimates for the semi-discrete problem (12.3.3) with χ replaced by u_h:

$$\begin{aligned} (u_{h,t}, u_h) + (\nabla u_h, \nabla u_h) &= (f, u_h), \qquad \forall\chi \in S_h, \quad t > 0, \\ u_h(0) &= u_{0,h}, \end{aligned} \tag{12.3.6}$$

and end up with the $L_2(\Omega)$ estimate

$$\|u_h(t)\| \le \|u_{0,h}\| + \int_0^t \|f\|\, ds. \tag{12.3.7}$$

To proceed, we introduce the *discrete Laplacian* operator $\Delta_h : V_h^0 \to V_h^0$, as in a discrete analogue of Green's formula:

$$(-\Delta_h\psi, \chi) = (\nabla\psi, \nabla\chi), \quad \forall\psi,\, \chi \in V_h^0. \tag{12.3.8}$$

One can show that the operator Δ_h is self-adjoint and $-\Delta_h$ is positive definite with respect to the L_2-inner product. Let now P_h denote the L_2-projection onto V_h^0, then the equation (12.3.3) can be written as

$$(u_{h,t} - \Delta_h u_h - P_h f, \chi) = 0, \quad \forall\chi \in V_h^0.$$

Now since $u_{h,t} - \Delta_h u_h - P_h f \in V_h^0$ taking $\chi = u_{h,t} - \Delta_h u_h - P_h f$, we thus obtain

$$u_{h,t} - \Delta_h u_h = P_h f, \qquad t > 0, \quad u_h(0) = u_{0,h}. \tag{12.3.9}$$

Now considering $E_h(t)$ as the solution operator of homogeneous (12.3.9), with $f = 0$, we have that $u_h(t) = E_h(t)u_{0,h}$. Recall that the Duhamel's principle reads as

$$u_h(t) = E_h(t)u_{0,h} + \int_0^t E_h(t-s)P_h f(s)\, ds. \tag{12.3.10}$$

Recalling the interpolant $\mathcal{I}_h : \mathcal{C}_0(\bar{\Omega}) \to V_h$ and the $L_2(\Omega)$ version of the error bound stated in Theorem 10.2, we can now prove the following error estimate:

Theorem 12.2. *Let u_h and u be the solutions of* (12.3.3) *and* (12.3.1) *respectively. Then we have the following $L_2(\Omega)$ estimate*

$$\|u_h(t) - u(t)\| \le \|u_{0,h} - u_0\| + Ch^2\Big(\|u_0\|_2 + \int_0^t \|u_t\|_2\, ds\Big).$$

Proof. We recall the common estimate for interpolation $\mathcal{I}_h$, L_2-projection P_h and elliptic (Ritz) projection R_h,

$$\|v - v_h\| \le Ch^2\|v\|, \quad \text{where } v_h = \mathcal{I}_h v,\ v_h = P_h v, \text{ or } v_h = R_h v. \tag{12.3.11}$$

Below we shall use the Ritz projection and rewrite the error as

$$u - u_h = (u - R_h u) + (R_h u - u_h) = \rho + \theta. \tag{12.3.12}$$

Now, recalling (10.4.22), the first term is estimated as

$$\begin{aligned} \|\rho(t)\| \le& Ch^2\|u(t)\| = Ch^2\|u_0 + \int_0^t u_t\, ds\|_2 \\ \le& Ch^2\Big(\|u_0\|_2 + \int_0^t \|u_t\|_2\, ds\Big). \end{aligned} \tag{12.3.13}$$

As for a bound for θ, using the notation $a(w,\xi) := (\nabla w, \nabla \xi)$ and the definition of R_h, we have

$$\begin{aligned} (\theta_t, \varphi) + a(\theta,\varphi) &= -(u_{h,t},\varphi) - a(u_h,\varphi) + (R_h u_{h,t},\varphi) + a(R_h u_h,\varphi) \\ &= -(f,\varphi) + (R_h u_t,\varphi) + a(u,\varphi) = (-u_t + R_h u_t, \varphi), \end{aligned} \tag{12.3.14}$$

which, using the fact that R_h commutes with time differentiation, i.e., $R_h u_t = (R_h u)_t$, yields

$$(\theta_t,\varphi) + a(\theta,\varphi) = -(\rho_t,\varphi), \qquad \forall \varphi \in V_h. \tag{12.3.15}$$

Now using the variational formulation (12.3.2) and the finite element formulation for the discrete problem (12.3.6) together with the estimate (12.3.7) we get

$$\|\theta(t)\| \le \|\theta(0)\| + \int_0^t \|\rho_t\|\, ds \tag{12.3.16}$$

We have that

$$\begin{aligned} \|\theta(0)\| &= \|R_h u_0 - u_{0,h}\| \le \|R_h u_0 - u_0\| + \|u_0 - u_{h,0}\| \\ &\le \|u_0 - u_{h,0}\| + Ch^2\|u_0\|_2. \end{aligned} \tag{12.3.17}$$

Further

$$\|\rho_t\| = \|u_t - R_h u_t\| \le Ch^2\|u\|_2, \tag{12.3.18}$$

Combining the above estimates we get the desired result. $\square$

The gradient estimate counterpart has a similar form, viz:

Theorem 12.3. *Let u_h and u be the solutions of* (12.3.3) *and* (12.3.1) *respectively. Then we have the following $L_2(\Omega)$ estimate for the gradient of the solution*

$$|u_h(t) - u(t)|_1 \leq |u_{0,h} - u_0|_1 + Ch\Big[\|u_0\|_2 + \|u(t)\|_2 + \Big(\int_0^t \|u_t\|_1^2\, ds\Big)^{1/2}\Big].$$

Proof. Let $\rho(t)$ be defined as in (12.3.12). Then using (10.4.22) we have

$$|\rho(t)|_1 = |u - R_h u|_1 \leq Ch\|u(t)\|_2. \tag{12.3.19}$$

To estimate $\nabla\theta$ we use (12.3.15) with $\varphi = \theta_t$. Then

$$\|\theta_t\|^2 + \frac{1}{2}\frac{d}{dt}|\theta|_1^2 = -(\rho_t, \theta_t) \leq \frac{1}{2}(\|\rho_t\|^2 + \|\theta_t\|^2), \tag{12.3.20}$$

and hence

$$\frac{d}{dt}|\theta|_1^2 \leq \|\rho_t\|^2, \tag{12.3.21}$$

which integrating in t, yields

$$|\theta(t)|_1^2 \leq |\theta(0)|_1^2 + \int_0^t \|\rho_t\|^2\; ds \leq (|R_h u_0 - u_0|_1^2 + |u_0 - u_{0,h}|_1^2) + \int_0^t \|\rho_t\|^2\; ds.$$

Now recalling the elliptic projection estimate:

$$\|R_h w - w\| + h|R_h w - w|_1 \leq Ch^s\|w\|_a, \qquad s = 1, 2, \tag{12.3.22}$$

we get

$$|\theta(t)|_1 \leq |u_0 - u_{0,h}|_1 + Ch\Big[\|u_0\|_2 + \Big(\int_0^t \|u_t\|^2\, ds\Big)^{1/2}\Big], \tag{12.3.23}$$

which completes the proof. $\square$

12.3.2 A fully discrete algorithm

For the time discretization of (12.1.1) we shall consider a partition of the time interval $I = [0, T] : 0 = t_0 < t_1 < \ldots < t_N = T$ into sub-intervals $I_n = (t_{n-1}, t_n)$ of length $k_n = t_n - t_{n-1}$. Then, we divide the *space-time slab* $S_n = \Omega \times I_n$ into prisms $K \times I_n$, where $K \in \mathcal{T}_n$ and $\mathcal{T}_n$ is a triangulation of Ω with the mesh function h_n. We construct a finite element mesh based on approximations by continuous piecewise linear functions in space and discontinuous polynomials of degree r in time. We shall refer to this method as $cG(1)dG(r)$ method. We define the trial space as,

$$W_k^{(r)} = \{v(x,t) : v|_{S_n} \in W_{kn}^{(r)}\}, \quad n = 1, 2, \ldots, N,$$

where

$$W_{kn}^{(r)} = \{v(x,t) : v(x,t) = \sum_{j=0}^{r} t^j \psi_j(x), \ \ \psi_j \in V_n, \ (x,t) \in S_n\},$$

with $V_n = V_{h_n}$ is the space of continuous piecewise linear functions vanishing on Γ associated to $\mathcal{T}_n$. Generally, the functions in $W_k^{(r)}$ are discontinuous across the discrete time levels t_n with jump discontinuities $[w_n] = w_n^+ - w_n^-$ and $w_m^{\pm} = \lim_{s\to 0^{\pm}} w(t_n + s)$.

The $cG(1)dG(r)$ method is based on a variational formulation for (12.2.1), viz. find $U \in W_k^{(r)}$ such that for $n = 1, 2, \dots, N$,

$$\int_{I_n} \Big((\dot{U}, v) + (\nabla U, \nabla v) \Big)\, dt + \Big([U_{n-1}], v_{n-1}^+ \Big) = \int_{I_n} (f, v)\, dt, \quad \forall v \in W_{kn}^{(r)}, \tag{12.3.24}$$

where $U_0^- = u_0$ and$(\cdot,\cdot)$ is the $L_2(\Omega)$ inner product.

Using the discrete Laplacian Δ_{h_n}, in the case of $r = 0$, we may write (12.3.24) as follows: find $U_n \in V_n$ such that

$$(I - k_n \Delta_n) U_n = P_n U_{n-1} + \int_{I_n} P_n f\, dt, \tag{12.3.25}$$

where $U_n = U_n^- = U|_{I_n} \in V_n$, and P_n is the $L_2(\Omega)$ projection onto V_n. Note that the *initial data* $U_{n-1} \in V_{n-1}$ from the previous time interval I_{n-1} is projected into the space V_n. If $V_{n-1} \subset V_n$, then $P_n U_{n-1} = U_{n-1}$. For the case $r = 1$, writing $U(t) = \Phi_n + (t - t_{n-1})\Psi_n$ on I_n with Φ_n, $\Psi_n \in V_n$, the variational formulation (12.3.24) takes the following form

$$\begin{cases} (I - k_n \Delta_n)\Phi_n + (I - \dfrac{k_n}{2}\Delta_n)\Psi_n = P_n U_{n-1} + \displaystyle\int_{I_n} P_n f\, dt, \\ (\dfrac{1}{2} I - \dfrac{k_n}{3}\Delta_n)\Psi_n - \dfrac{k_n}{2}\Delta_n \Phi_n = \displaystyle\int_{I_n} \dfrac{t - t_{n-1}}{k_n} P_n f\, dt, \end{cases} \tag{12.3.26}$$

and gives a system of equations for Φ_n and Ψ_n.

12.3.3 Constructing the discrete equations

To construct the matrix equation that determines U_n in the case $r = 0$ according (12.3.25), we let $\varphi_{n,j}$ denote the nodal basis for V_n associated with J_n interior nodes of $\mathcal{T}_n$, then we can write U_n as

$$U_n = \sum_{j=1}^{J_n} \xi_{n,j} \varphi_{n,j}.$$

Here $\xi_{n,j}$ are the nodal values of U_n. Let us denote the vector coefficients by $\xi_n = (\xi_{n,j})$. We denote the $J_n \times J_n$ mass matrix by M_n, stiffness matrix by S_n and the $J_n \times 1$ data vector by b_n, with the associated elements

$$(M_n)_{i,j} = (\varphi_{n,j}, \varphi_{n,i}), \quad (S_n)_{i,j} = (\nabla\varphi_{n,j}, \nabla\varphi_{n,i}), \quad (b_n)_i = \int_{t_{n-1}}^{t_n} (f, \varphi_{n,i}),$$

for $1 \le i,\, j \le J_n$. Finally, we denote the $J_n \times J_{n-1}$ matrix $M_{n-1,n}$ with entries

$$(M_{n-1,n})_{ij} = (\varphi_{n,j}, \varphi_{n-1,i}), \quad 1 \le i \le J_n,\ 1 \le j \le J_{n-1}. \qquad (12.3.27)$$

Now, the discrete equation for $cG(1)dG(0)$ approximation on I_n reads as follows: fins ξ_n such that

$$(M_n + k_n S_n)\xi_n = M_{n-1,n}\xi_{n-1} + b_n. \qquad (12.3.28)$$

One can easily verify that the coefficient matrix of the system (12.3.28) is sparse, symmetric, and positive definite, which can be solved by a direct or an iterative algorithm.

12.3.4 An a priori error estimate: Fully discrete problem

To derive a priori (or a posteriori) error estimates for the heat equation in higher dimensions is a rather involved procedure. Below we outline the prerequisites and formulate an a priori error estimate theorem without a proof. Detailed studies of the error analysis, for both heat and wave equations, can be found in [20].

We assume that there are positive constants α_i, $i = 1, 2$, with α_2 sufficiently small, such that for $n = 1, \dots, N$,

$$\alpha_1 k_n \le k_{n+1} \le \alpha_1^{-1} k_n, \qquad (12.3.29)$$

$$\alpha_1 h_n(x) \le h_{n+1}(x) \le \alpha_1^{-1} h_n, \quad x \in \Omega \qquad (12.3.30)$$

$$(\max_{x\in\bar{\Omega}} h(x))^2 \le \alpha_2 k_n, \qquad (12.3.31)$$

The relation (12.3.31) enters if $V_{n-1} \not\subseteq V_n$. An a priori error estimate is then given as:

Theorem 12.4. *If Ω is convex and α_2 sufficiently small, then there is a constant C_i depending only on τ (which is the minimum angle in the triangulation of Ω) and α_i, $i = 1, 2$, such that for $N \ge 1$, and under the assumptions* (12.3.29)-(12.3.31)*, we have*

$$\|u(t_N) - U_N\| \le C_i L_N \max_{1\le n\le N} (k_n \|\dot{u}\|_{I_n} + \|h_n^2 D^2 u\|_{I_n}), \qquad (12.3.32)$$

where

$$L_N = 2 + \max_{1\le n\le N} \max\Big\{ \Big(\log(\frac{t_n}{k_n})\Big)^{1/2},\ \log(\frac{t_n}{k_n}) \Big\}.$$

12.4 Exercises

Problem 12.3. *Consider the following general form of the heat equation*

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), & \text{for } x \in \Omega,\ \ 0 < t \le T, \\ u(x,t) = 0, & \text{for } x \in \Gamma,\ \ 0 < t \le T, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \tag{12.4.1}$$

where $\Omega \in \mathbb{R}^2$ *with boundary* Γ*. Let* $\tilde{u}$ *be the solution of* (12.4.1) *with a modified initial data* $\tilde{u}_0(x) = u_0(x)\varepsilon(x)$*.*

a) *Show that* $w := \tilde{u} - u$ *solves* (12.4.1) *with initial data* $w_0(x) = \varepsilon(x)$*.*

b) *Give estimates for the difference between* u *and* $\tilde{u}$*.*

c) *Prove that the solution of* (12.4.1) *is unique.*

Problem 12.4. *Formulate the equation for* $cG(1)dG(1)$ *for the two-dimensional heat equation using the discrete Laplacian.*

Problem 12.5. *In two dimensions the heat equation, in the case of radial symmetry, can be formulated as* $r\dot{u} - (ru'_r)'_r = rf$*, where* $r = |x|$ *and* $w'_r = \frac{\partial w}{\partial r}$*.*

a) *Verify that* $u = \frac{1}{4\pi t}\exp(-\frac{r^2}{4t})$ *is a solution for the homogeneous equation* ($f = 0$) *with the initial data being the Dirac* δ *function* $u(r,0) = \delta(r)$*.*

b) *Sketching* $u(r,t)$ *for* $t = 1$ *and* $t = 0.01$*, deduce that* $u(r,t) \to 0$ *as* $t \to 0$ *for* $r > 0$*.*

c) *Show that* $\int_{\mathbb{R}^2} u(x,t)\,dx = 2\pi \int_0^\infty u(r,t)\,r\,dr = 1$ *for all* t*.*

d) *Determine a stationary solution to the heat equation with data*

$$f = \begin{cases} 1/(\pi\varepsilon)^2, & \text{for} \quad r < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

e) *Determine the fundamental solution corresponding to* $f = \delta$*, letting* $\varepsilon \to 0$*.*

Problem 12.6. *Consider the Schrödinger equation*

$$iu_t - \Delta u = 0, \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

where $i = \sqrt{-1}$ *and* $u = u_1 + iu_2$*.*

a) Show that the total probability $\int_\Omega |u|^2$ *is independent of the time.*

Hint: Multiplying by $\bar{u} = u_1 - iu_2$*, and consider the imaginary part*

b) Consider the corresponding eigenvalue problem, i.e, find the eigenvalue λ *and the corresponding eigenfunction* $u \neq 0$ *such that*

$$-\Delta u = \lambda u \quad \text{in } \Omega, \qquad u = 0, \quad \text{on } \partial\Omega.$$

Show that $\lambda > 0$ *and give the relationship between the norms* $\|u\|$ *and* $\|\nabla u\|$ *for the corresponding eigenfunction* u*.*

c) Determine (in terms of the smallest eigenvalue λ_1*), the smallest possible value for the constant* C *in the Poincare estimate*

$$\|u\| \leq C\|\nabla u\|,$$

derived for all solutions u *vanishing at the boundary (*$u = 0$, *on* $\partial\Omega$*).*

Problem 12.7. *Consider the initial-boundary value problem*

$$\begin{cases} u_t(x,t) - \Delta u(x,t) = f(x,t), & \text{for } x \in \Omega,\ t > 0, \\ u(x,t) = 0, & \text{for } x \in \Gamma,\ t > 0, \\ u(x,0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \tag{12.4.2}$$

a) Prove (with $\|u\| = (\int_\Omega u^2\,dx)^{1/2}$*) that*

$$\|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2\,ds \leq \|u_0\|^2 + \int_0^t \|f(s)\|^2\,ds$$
$$\|\nabla u(t)\|^2 + \int_0^t \|\Delta u(s)\|^2\,ds \leq \|\nabla u_0\|^2 + \int_0^t \|f(s)\|^2\,ds$$

b) Formulate $dG(0) - cG(1)$ *method for this problem.*

Problem 12.8. *Formulate and prove* $dG(0) - cG(1)$ *a priori and a posteriori error estimates for the two dimensional heat equation (cf. the previous problem) that uses lumped mass and midpoint quadrature rule.*

Problem 12.9. *Consider the convection problem*

$$\beta \cdot \nabla u + \alpha u = f, \qquad x \in \Omega, \quad u = g, \quad x \in \Gamma_-, \tag{12.4.3}$$

Define the outflow Γ_+ *and inflow* Γ_- *boundaries. Assume that* $\alpha - \frac{1}{2}\nabla \cdot \beta \geq c > 0$*. Show the following stability estimate*

$$c\|u\|^2 \int_{\Gamma_+} n \cdot \beta u^2\,ds\,dt \leq \|u_0\|^2 + \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |n \cdot \beta| g^2\,ds. \tag{12.4.4}$$

Hint: Show first that

$$2(\beta \cdot \nabla u, u) = \int_{\Gamma_+} n \cdot \beta\, u^2\, ds - \int_{\Gamma_-} \|n \cdot \beta\|\, u^2\, ds - ((\nabla \cdot \beta)u, u).$$

Formulate the streamline diffusion for this problem.

Problem 12.10. *Consider the convection problem*

$$\begin{aligned} &\dot{u} + \beta \cdot \nabla u + \alpha u = f, && x \in \Omega, \quad t > 0,\\ &u = g, && x \in \Gamma_-, \quad t > 0, \\ &u(x, 0) = u_0(x), && x \in \Omega, \end{aligned} \tag{12.4.5}$$

where Γ_+ *and* Γ_- *are defined as above. Assume that* $\alpha - \frac{1}{2}\nabla \cdot \beta \geq c > 0$. *Show the following stability estimate*

$$\begin{aligned} &\|u(\cdot, T)\|^2 + c\int_0^T \|u(\cdot, t)\|^2\, dt + \int_0^T \int_{\Gamma_+} n \cdot \beta u^2\, ds\, dt \\ &\qquad \leq \|u_0\|^2 + \frac{1}{c}\int_0^T \|f(\cdot, t)\|^2\, dt + \int_0^T \int_{\Gamma_-} |n \cdot \beta| g^2\, ds\, dt, \end{aligned} \tag{12.4.6}$$

where $\|u(\cdot, T)\|^2 = \int_\Omega u(x, T)^2\, dx$.

12.5 The wave equation in $\mathbb{R}^N$

The study of the wave equation in $\mathbb{R}^N,\ N \geq 2$ is an extension of the results in the one-dimensional case introduced in Chapter 8. Some additional properties of the wave equation are introduced in Chapter 2. For a full finite element study of wave propagation in higher dimensions we refer to [31] and [20]. In this section we shall prove the *law of conservation of energy*, derive spatial and temporal discretizations and give a stability estimate for the wave equation in $\mathbb{R}^N,\ N \geq 2$.

Theorem 12.5 (Conservation of energy)**.** *For the wave equation given by*

$$\begin{cases} \ddot{u} - \Delta u = 0, & \textit{in } \Omega \subset \mathbb{R}^N & (DE) \\ u = 0, & \textit{on } \partial\Omega = \Gamma & (BC) \\ (u = u_0) \ \textit{ and } \ (\dot{u} = v_0) & \textit{in } \Omega, \ \textit{for } t = 0, & (IC) \end{cases} \tag{12.5.1}$$

where $\ddot{u} = \partial^2 u/\partial t^2$, with $t \in [0, T]$, or $t \in (0, \infty)$, we have that

$$\frac{1}{2}\|\dot{u}(\cdot,t)\|^2 + \frac{1}{2}\|\nabla u(\cdot,t)\|^2 = \textit{constant, independent of}\, t, \tag{12.5.2}$$

i.e., the total energy is conserved, where $\frac{1}{2}\|\dot{u}\|^2$ is the kinetic energy, and $\frac{1}{2}\|\nabla u\|^2$ is the potential (elastic) energy.

Proof. We multiply the equation by $\dot{u}$ and integrate over Ω to obtain

$$\int_\Omega \ddot{u}\,\dot{u}\,dx - \int_\Omega \Delta u \cdot \dot{u}\,dx = 0. \tag{12.5.3}$$

Using Green's formula:

$$-\int_\Omega (\Delta u)\dot{u}\,dx = -\int_\Gamma (\nabla u \cdot n)\dot{u}\,ds + \int_\Omega \nabla u \cdot \nabla \dot{u}\,dx, \tag{12.5.4}$$

and the boundary condition $u = 0$ on Γ, (which implies $\dot{u} = 0$ on Γ), we get

$$\int_\Omega \ddot{u} \cdot \dot{u}\,dx + \int_\Omega \nabla u \cdot \nabla \dot{u}\,dx = 0. \tag{12.5.5}$$

Consequently we have that

$$\int_\Omega \frac{1}{2}\frac{d}{dt}(\dot{u}^2)\,dx + \int_\Omega \frac{1}{2}\frac{d}{dt}(|\nabla u|^2)\,dx = 0 \Longleftrightarrow \frac{1}{2}\frac{d}{dt}(\|\dot{u}\|^2 + \|\nabla u\|^2) = 0,$$

and hence

$$\frac{1}{2}\|\dot{u}\|^2 + \frac{1}{2}\|\nabla u\|^2 = \frac{1}{2}\|v_0\|^2 + \frac{1}{2}\|\nabla u_0\|^2,$$

which is the desired result and the proof is complete. □

12.5.1 The weak formulation

We multiply the equation $\ddot{u} - \Delta u = f$ by a test function $v \in H_0^1(\Omega)$ and integrate over Ω to obtain

$$\begin{aligned}\int_\Omega fv\,dx &= \int_\Omega \ddot{u}v\,dx - \int_\Omega \Delta u\,v\,dx\\ &= \int_\Omega \ddot{u}v\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx - \int_{\partial\Omega}(n\cdot\nabla u)v\,ds \qquad (12.5.6)\\ &= \int_\Omega \ddot{u}v\,dv + \int_\Omega \nabla u\cdot\nabla v\,dx.\end{aligned}$$

where we used Green's formula and boundary data on v. Thus we have the following *variational formulation* for the wave equation: Find $u \in H_0^1(\Omega)$ such that for every fixed $t \in [0,T]$,

$$\int_\Omega \ddot{u}v\,dv + \int_\Omega \nabla u\cdot\nabla v\,dx = \int_\Omega fv\,dx, \qquad \forall v \in H_0^1(\Omega). \qquad (12.5.7)$$

12.5.2 The semi-discrete problem

Let V_h^0 be a subspace of $H_0^1(\Omega)$ consisting of continuous piecewise linear functions on a partition $\mathcal{T}_h$ of Ω. The semi-discrete (spatial discretization) counterpart of (12.5.7) reads as follows: find $u_h \in V_h^0$, such that for every fixed $t \in (0,T]$,

$$\int_\Omega \ddot{u}_h v\,dx + \int_\Omega \nabla u_h\cdot\nabla v\,dx = \int_\Omega fv\,dx, \qquad \forall v \in V_h^0. \qquad (12.5.8)$$

Now let $\{\varphi_j\}_{j=1}^N$ be a set of basis functions for V_h^0, thus in (12.5.8) we may choose $v = \varphi_i$, $i = 1,\dots,N$. Next, we make the ansatz

$$u_h(x,t) = \sum_{j=1}^N \xi_j(t)\varphi_j(x), \qquad (12.5.9)$$

where ξ_j, $j = 1,\dots,N$ are N time dependent unknown coefficients. Hence, with $i = 1,\dots,N$ and $t \in I = (0,T]$, (12.5.8) can be written as

$$\sum_{j=1}^N \ddot{\xi}_j(t)\int_\Omega \varphi_j\varphi_i\,dx + \sum_{j=1}^N \xi_j(t)\int_\Omega \nabla\varphi_j\cdot\nabla\varphi_i\,dx = \int_\Omega f\varphi_i\,dx. \qquad (12.5.10)$$

In the matrix form (12.5.10) is written as the $N\times N$ linear system of ODEs,

$$M\ddot{\xi}(t) + S\xi(t) = b(t), \quad t \in I = (0,T], \qquad (12.5.11)$$

where M and S are the usual mass and stiffness matrices, respectively, and b is the load vector.

12.5.3 A priori error estimates for the semi-discrete problem

Theorem 12.6. *The spatial discrete solution u_h given by* (12.5.9) *satisfies the a priori error estimate*

$$\|u(t) - u_h(t)\| \le Ch^2\Big(\|\Delta u(\cdot, t)\| + \int_0^t \|\ddot{u}(\cdot, s)\|\, ds\Big). \tag{12.5.12}$$

Proof. The proof follows the general path of a priori error estimates combined with conservation of energy Theorem 12.2. □

Below we state and prove a general semi-discrete error estimate that contains the proof of the above theorem as well.

Theorem 12.7. *Let u and u_h be the solution of* (12.5.1) *and* (12.5.8), *respectively. Then there is nondecreasing time dependent constant $C(t)$ such that for $t > 0$,*

$$\begin{aligned} &\|u(t) - u_h(t)\| + |u(t) - u_h(t)|_1 + \|u_t(t) - u_{h,t}(t)\| \\ &\quad \le C\Big(|u_{0,h} - R_h u_0|_1 + \|v_h - R_h v\|\Big) \\ &\qquad + C(t)h^2\Big[\|u(t)\|_2 + \|u_t(t)\|_2 + \Big(\int_0^t \|u_{tt}\|_2^2\, ds\Big)^{1/2}\Big]. \end{aligned} \tag{12.5.13}$$

Proof. We start with the, Ritz projection, split

$$u - u_h = (u - R_h u) + (R_h u - u_h) = \rho + \theta.$$

We may bound ρ and ρ_t as in the case of heat equation:

$$\|\rho(t)\| + h|\rho(t)|_1 \le Ch^2\|u(t)\|_2, \qquad \|\rho_t(t)\| \le Ch^2\|u_t(t)\|_2.$$

As for $\theta(t)$, we note that

$$(\theta_{tt}, \psi) + a(\theta, \psi) = -(\rho_{tt}, \psi), \quad \forall \psi \in V_h^0, \quad t > 0. \tag{12.5.14}$$

To separate the effects from the initial values and the data, we let $\theta = \eta + \zeta$ where

$$\begin{cases} (\eta_{tt}, \psi) + a(\eta, \psi) = 0, & \forall \psi \in V_h^0, \quad t > 0 \\ \eta(0) = \theta(0), \qquad \eta_t(0) = \theta_t(0). \end{cases}$$

Then η and η_t are bounded (as in conservation of energy).

As for the contribution from the ζ, we note that by the initial data for η, ζ satisfies (12.5.14) with the data $\zeta(0) = \zeta_t(0) = 0$, where with $\psi = \zeta_t$, we end up with

$$\frac{1}{2}\frac{d}{dt}(\|\zeta_t\|^2 + |\zeta|_1^2) = -(\rho_{tt}, \zeta_t) \le \frac{1}{2}\|\rho_{tt}\|^2 + \le \frac{1}{2}\|\zeta_t\|^2.$$

Then, integrating with respect to t, we have that

$$\|\zeta_t(t)\|^2 + |\zeta(t)|_1^2 \le \int_0^t \|\rho_{tt}\|^2\, ds + \int_0^t \|\zeta_t\|^2\, ds.$$

Finally, using Gronwall's lemma (with $C(t) = e^t$), we get

$$\|\zeta_t(t)\|^2 + |\zeta(t)|_1^2 \le C(t)\int_0^t \|\rho_{tt}\|^2\, ds \le C(t)h^4 \int_0^t \|u_{tt}\|_2^2\, ds,$$

where we used the bound for ρ_{tt}. This bounds the first two terms on the left hand side in theorem. The third term is estimated in a similar way. $\square$

12.5.4 The fully-discrete problem

Here we discretize the ODE system (12.5.11) in the time variable. To this end we let $\eta = \dot{\xi}$ and rewrite (12.5.11) as two first order systems:

$$\begin{aligned} M\dot{\xi}(t) =&\ \ M\eta(t) \\ M\dot{\eta}(t) + S\xi(t) =&\ \ b(t). \end{aligned} \tag{12.5.15}$$

We apply the Crank-Nicolson method to each system, and as in Chapter 7, and end up with

$$M\frac{\xi_n - \xi_{n-1}}{k_n} = M\frac{\eta_n + \eta_{n-1}}{2}, \tag{12.5.16}$$

$$M\frac{\eta_n - \eta_{n-1}}{k_n} + S\frac{\xi_n + \xi_{n-1}}{2} = \frac{b_n + b_{n-1}}{2}. \tag{12.5.17}$$

In block matrix form this can be written as

$$\begin{bmatrix} M & -\frac{k_n}{2}M \\ \frac{k_n}{2}S & M \end{bmatrix}\begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix} = \begin{bmatrix} M & \frac{k_n}{2}M \\ -\frac{k_n}{2}S & M \end{bmatrix}\begin{bmatrix} \xi_{n-1} \\ \eta_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_n}{2}(b_n + b_{n-1}) \end{bmatrix}.$$

Here we may, e.g., choose the starting iterates ξ_0 and η_0 as the nodal interpolations of u_0 and v_0, respectively.

Note that, as we mentioned earlier in Chapter 7, the Crank-Nicolson method is more accurate than the Euler methods and it has the *energy norm preserving property*: it conserves energy. Therefore it is a more suitable method to the numerical study of the wave equation.

12.6 Exercises

Problem 12.11. *Derive a total conservation of energy relation using the Robin type boundary condition:* $\dfrac{\partial u}{\partial n} + u = 0.$

Problem 12.12. *Determine a solution for the following equation*

$$\ddot{u} - \Delta u = e^{it}\delta(x),$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$, $i = \sqrt{-1}$, $x = (x_1, x_2, x_3)$ *and* δ *is the Dirac-delta function.*

Hint: Let $u = e^{it}v(x)$, $v(x) = w(r)/r$ *where* $r = |x|$. *Further* $rv = w \to \frac{1}{4\pi}$ *as* $r \to 0$. *Use spherical coordinates.*

Problem 12.13. *Consider the initial boundary value problem*

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad \dot{u}(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{12.6.1}$$

Rewrite the problem as a system of two equations with a time derivative of order at most 1. Why this modification?

Problem 12.14. *Consider the initial boundary value problem*

$$\begin{cases} \ddot{u} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x,0) = u_0(x), \qquad \dot{u}(x,0) = u_1(x), & x \in \Omega. \end{cases} \tag{12.6.2}$$

Formulate the $cG(1)cG(1)$ *method for this problem. Show that the energy is conserved.*

??

Answers to Exercises

Chapter 11. Poisson equation

11.8

$$A = \begin{pmatrix} 4 & -1 & 0 & 0 \\ -1 & 4 & -1 & 0 \\ 0 & -1 & 4 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix} \qquad b = \frac{1}{8}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

10.9

$$A = \begin{bmatrix} 6 & -1 \\ -1 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

11.10

$$A = \begin{pmatrix} 5 & -2 & 0 & 0 \\ -2 & 5 & -2 & 0 \\ 0 & -2 & 5 & -2 \\ 0 & 0 & -2 & 5/2 \end{pmatrix} \qquad b = \frac{1}{16}\begin{pmatrix} 2 \\ 2 \\ 2 \\ 1 \end{pmatrix}.$$

11.11

$$M = \frac{h^2}{12}\begin{bmatrix} 8 & 1 & 1 & 1 \\ 1 & 4 & 0 & 1 \\ 1 & 0 & 4 & 1 \\ 1 & 1 & 1 & 8 \end{bmatrix}, \qquad S = \begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}.$$

Chapter 12. Initial boundary value problems

12.3 b)

$$\|w(T)\|^2 + 2\int_0^T \|\nabla w\|^2\, dt \le \|\varepsilon\|^2.$$

12.6 c) $1/\sqrt{\lambda_1}$.

12.12 $v = \frac{1}{4\pi}\frac{\cos(r)}{r}$ and the corresponding solution $u = e^{it}\frac{1}{4\pi}\frac{\cos(r)}{r}$.