

**An Introduction to the  
Finite Element Method (FEM)  
for Differential Equations  
Part II: Problems in  $\mathbb{R}^d$ , ( $d > 1$ ).**

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February 5, 2018



# Contents

<b>10</b>	<b>Approximation in several dimensions</b>	<b>5</b>
10.1	Introduction . . . . .	5
10.2	Piecewise linear approximation in 2 D . . . . .	7
10.2.1	Basis functions for the piecewise linears in 2 D . . . . .	8
10.3	Constructing finite element spaces . . . . .	14
10.4	The interpolant . . . . .	17
10.4.1	Error estimates for piecewise linear interpolation . . . . .	19
10.4.2	The $L_2$ and Ritz projections . . . . .	23
10.5	Exercises . . . . .	26
<b>11</b>	<b>The Poisson Equation</b>	<b>31</b>
11.1	The Fundamental Solution . . . . .	32
11.1.1	Green's Functions . . . . .	33
11.1.2	Method of Images . . . . .	35
11.2	Stability . . . . .	36
11.3	Error Estimates for the CG(1) FEM . . . . .	36
11.3.1	Proof of the regularity Lemma . . . . .	42
11.4	Exercises . . . . .	43
<b>12</b>	<b>The Initial Boundary Value Problems in <math>\mathbb{R}^N</math></b>	<b>47</b>
12.1	The heat equation in $\mathbb{R}^N$ . . . . .	47
12.1.1	The fundamental solution . . . . .	48
12.2	Stability . . . . .	49
12.3	The finite element for heat equation . . . . .	52
12.3.1	The semidiscrete problem . . . . .	52
12.3.2	A fully discrete algorithm . . . . .	55
12.3.3	Constructing the discrete equations . . . . .	56
12.3.4	An a priori error estimate: Fully discrete problem . . . . .	57
12.4	Exercises . . . . .	58
12.5	The wave equation in $\mathbb{R}^N$ . . . . .	61
12.5.1	The weak formulation . . . . .	62
12.5.2	The semi-discrete problem . . . . .	62
12.5.3	A priori error estimates for the semi-discrete problem . . . . .	63

12.5.4	The fully-discrete problem . . . . .	64
12.6	Exercises . . . . .	64

# Chapter 10

## Approximation in several dimensions

### 10.1 Introduction

This chapter is, mainly, devoted to piecewise linear finite element approximation in  $2D$ -case. To this end, we shall prove a simple version of the Green's formula useful in weak formulations. We shall demonstrate the finite element structure through a canonical example, introduce some common general finite element spaces and finally extend the interpolation concept to higher dimensions. The study of one-dimensional problems in the previous chapters can be extended to higher dimensional domains in  $\mathbb{R}^n$ ,  $n \geq 2$ . Then, the *mathematical calculus*, being in several variables, becomes somewhat involved. On the other hand, the two and three dimensional cases concern the most relevant models both from physical point of views as well as application aspects. In the sequel we assume that the reader is familiar with the calculus of several variables. A typical problem to study is, e.g. an equation of the form

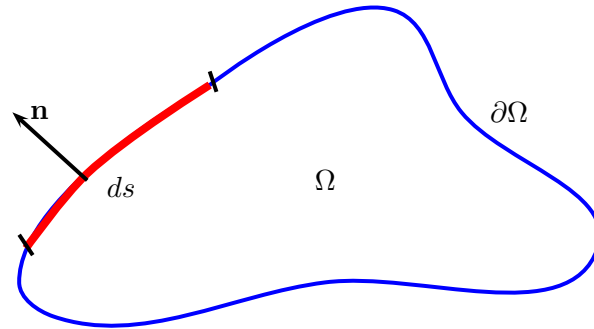
$$\begin{cases} -\Delta u(\mathbf{x}) + a(\mathbf{x})u(\mathbf{x}) = f(\mathbf{x}), & \mathbf{x} \in \Omega \subset \mathbb{R}^n, \quad n \geq 2, \\ u(\mathbf{x}) = 0, & \mathbf{x} \in \partial\Omega. \end{cases} \quad (10.1.1)$$

Here, the discretization procedure, e.g., approximating with piecewise linears with an insight to the error, would require extension of the interpolation estimates from the intervals in  $\mathbb{R}$  (the  $1d$  case) to higher dimensional domains in  $\mathbb{R}^n$ ,  $n \geq 2$ . Our focus in this chapter is on the problems in two space dimensions. Then, To compute the double (multiple) integrals which appear, e.g., in the weak formulations, the frequently used partial integration in the one-dimensional case is now replaced by the *Green's formula* below.

**Lemma 10.1** (Green's formula). *Let  $\Omega$  be an open bounded subset of  $\mathbb{R}^2$ , with a piecewise smooth boundary and  $u \in C^2(\Omega)$  and  $v \in C^1(\Omega)$ , then*

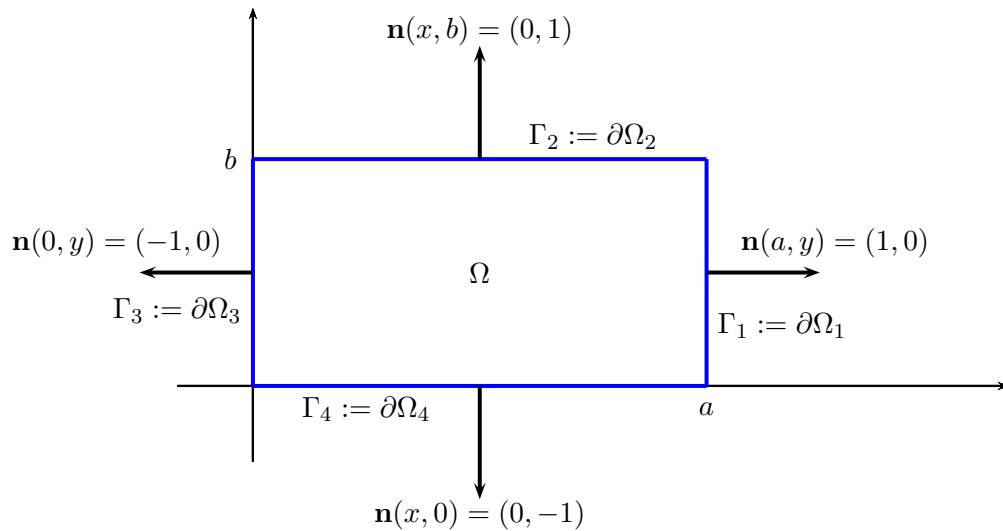
$$\int_{\Omega} (\Delta u)v \, dx \, dy = \int_{\partial\Omega} (\nabla u \cdot \mathbf{n})v \, ds - \int_{\Omega} \nabla u \cdot \nabla v \, dx \, dy, \quad (10.1.2)$$

where  $\mathbf{n} := \mathbf{n}(x, y)$  is the outward unit normal at the boundary point  $\mathbf{x} = (x, y) \in \partial\Omega$  and  $ds$  is a curve element on the boundary  $\partial\Omega$ .



**Figure 10.1:** A smooth domain  $\Omega$  with an outward unit normal  $\mathbf{n}$

*Proof.* Since we shall work with polygonal domains, we give a proof when  $\Omega$  is a rectangular domain, see the figure below. (The classical proof for a general domain  $\Omega$  can be found in any textbook in the calculus of several variables). Then, using integration by parts we have



**Figure 10.2:** A rectangular domain  $\Omega$  with its outward unit normals

$$\begin{aligned}
\iint_{\Omega} \frac{\partial^2 u}{\partial x^2} v \, dx dy &= \int_0^b \int_0^a \frac{\partial^2 u}{\partial x^2}(x, y) v(x, y) \, dx dy \\
&= \int_0^b \left( \left[ \frac{\partial u}{\partial x}(x, y) v(x, y) \right]_{x=0}^a - \int_0^a \frac{\partial u}{\partial x}(x, y) \frac{\partial v}{\partial x}(x, y) \, dx \right) dy \\
&= \int_0^b \left( \frac{\partial u}{\partial x}(a, y) v(a, y) - \frac{\partial u}{\partial x}(0, y) v(0, y) \right) dy \\
&\quad - \iint_{\Omega} \frac{\partial u}{\partial x}(x, y) \frac{\partial v}{\partial x}(x, y) \, dx dy.
\end{aligned}$$

Now identifying  $\mathbf{n}$  on  $\partial\Omega$  viz on  $\Gamma_1 : \mathbf{n}(a, y) = (1, 0)$ ,

$$\text{on } \Gamma_2 : \mathbf{n}(x, b) = (0, 1),$$

$$\text{on } \Gamma_3 : \mathbf{n}(0, y) = (-1, 0),$$

$$\text{on } \Gamma_4 : \mathbf{n}(x, 0) = (0, -1),$$

the first integral on the right hand side above can be written as

$$\left( \int_{\Gamma_1} + \int_{\Gamma_3} \right) \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right)(x, y) \cdot \mathbf{n}(x, y) v(x, y) \, ds.$$

Hence

$$\iint_{\Omega} \frac{\partial^2 u}{\partial x^2} v \, dx dy = \int_{\Gamma_1 \cup \Gamma_3} \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \mathbf{n}(x, y) v(x, y) \, ds - \iint_{\Omega} \frac{\partial u}{\partial x} \frac{\partial v}{\partial x} \, dx dy.$$

Similarly, in the  $y$ -direction we get

$$\iint_{\Omega} \frac{\partial^2 u}{\partial y^2} v \, dx dy = \int_{\Gamma_2 \cup \Gamma_4} \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \cdot \mathbf{n}(x, y) v(x, y) \, ds - \iint_{\Omega} \frac{\partial u}{\partial y} \frac{\partial v}{\partial y} \, dx dy.$$

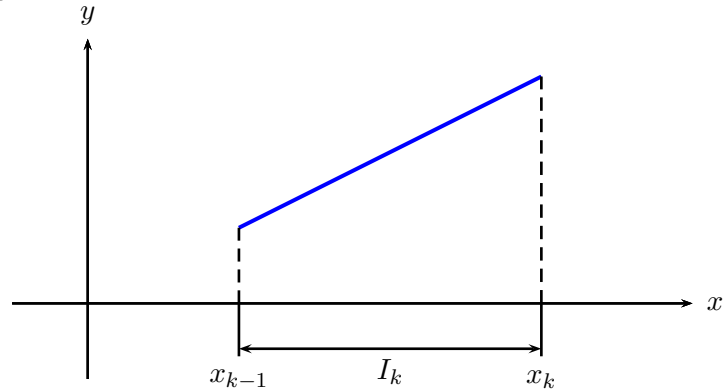
Adding up these two relations gives the desired result.  $\square$

## 10.2 Piecewise linear approximation in 2 D

In this section the objective is to study the piecewise polynomial approximations of the solutions for differential equations in two dimensional spatial domains. For simplicity we introduce the principle ideas in polygonal domains and focus on piecewise linear approximations. Thus we can deal with partitions (*meshes*) without any concern about curved boundaries. At the end of this chapter, in connection with a formal definition to the finite element, we briefly discuss some common extensions of the linear approximations to polynomials both in triangular and quadrilateral elements as well as tetrahedrons in the three dimensional case. See Tables 9.1-9.4

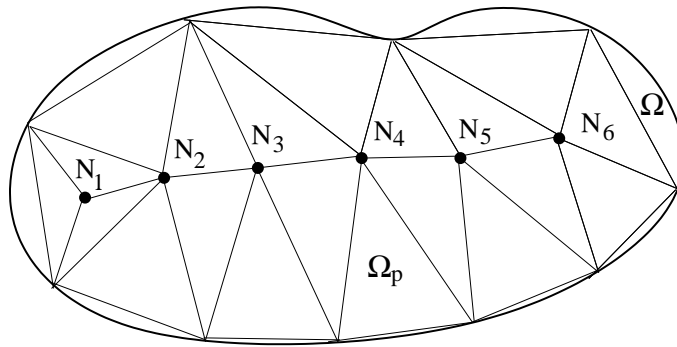
### 10.2.1 Basis functions for the piecewise linears in 2 D

We recall that in the one-dimensional case a function which is linear on a subinterval is uniquely determined by its values at two points (e.g. the endpoints of the subinterval), since there is only one straight line connecting two points.



**Figure 10.3:** A linear function on a subinterval  $I_k = (x_{k-1}, x_k)$ .

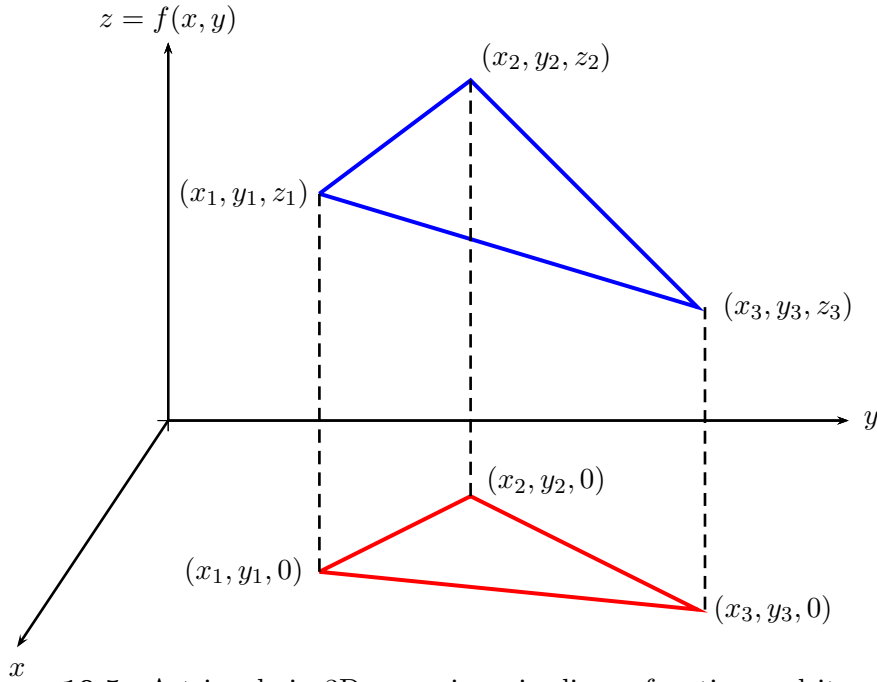
Similarly a plane in  $\mathbb{R}^3$  is uniquely determined by three points. Therefore, for piecewise linear approximation of a function  $u$  defined on a two dimensional polygonal domain  $\Omega_p \subset \mathbb{R}^2$ , i.e.  $u : \Omega_p \rightarrow \mathbb{R}$ , it is natural to make partitions of  $\Omega_p$  into triangular elements and let the sides of the triangles correspond to the endpoints of the intervals in the one-dimensional case.



**Figure 10.4:** Example of triangulation of a 2D domain

The figure 10.4. illustrates a “partitioning”, i.e. *triangulation* of a domain  $\Omega$  with curved boundary where the partitioning concerns only the polygonal domain  $\Omega_p$  generated by  $\Omega$ . Here we have 6 internal nodes  $N_i$ ,  $1 \leq i \leq 6$  and  $\Omega_p$  is the *polygonal* domain inside  $\Omega$ , which is triangulated. The Figure





**Figure 10.5:** A triangle in 3D as a piecewise linear function and its projection in 2D.

10.5 is a piecewise linear function on a single triangle (element) which is determined by its values at the vertices of the triangle. Below we shall give specific definitions of the concepts *subdivision* (in a general setting and any spatial dimension) and *triangulation* in the 2D case.

**Definition 10.1.** A subdivision of a computational domain  $\Omega$  is a finite collection of open sets  $\{K_i\}$  such that

$$a) K_i \cap K_j = \emptyset$$

$$b) \bigcup_i \bar{K}_i = \bar{\Omega}.$$

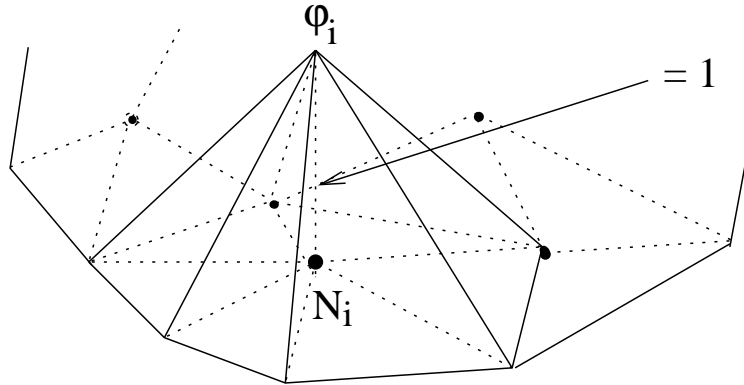
**Definition 10.2.** A triangulation of a polygonal domain  $\Omega$  is a subdivision of  $\Omega$  consisting of triangles with the property that no vertex of any triangle lies in the interior of an edge of another triangle.

Returning to Figure 10.4, for every linear function  $U$  on  $\Omega_p$ ,

$$U(\mathbf{x}) = U_1\varphi_1(\mathbf{x}) + U_2\varphi_2(\mathbf{x}) + \dots + U_6\varphi_6(\mathbf{x}), \quad (10.2.1)$$

where  $U_i = U(N_i)$ ,  $i = 1, 2, \dots, 6$ , are numerical values (nodal values) of  $U$  at the nodes  $N_i$ , and  $\varphi_i(N_i) = 1$ . Further  $\varphi_i(N_j) = 0$  for  $j \neq i$ , and  $\varphi_i(\mathbf{x})$  is linear in  $\mathbf{x}$  in every triangle/element.

Note that with the Dirichlet boundary condition, as in the one-dimensional case, the test functions use  $\varphi_i(\mathbf{x}) = 0$  for  $\mathbf{x} \in \partial\Omega_p$ . Hence, for a Dirichlet boundary value problem, to determine the approximate solution  $U$ , in  $\Omega_p$ , is reduced to find the *nodal values*  $U_1, U_2, \dots, U_6$ , obtained from the corresponding discrete variational formulation.



**Figure 10.6:** A linear basis function in a triangulation in 2D

**Example 10.1.** Let  $\Omega = \{(x, y) : 0 < x < 4, 0 < y < 3\}$  and make a piecewise linear FEM discretization of the following boundary value problem:

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \quad (10.2.2)$$

**Solution.** Using Green’s formula, the variational formulation for the problem (10.2.2) reads as follows: Find the function  $u \in H_0^1(\Omega)$  such that

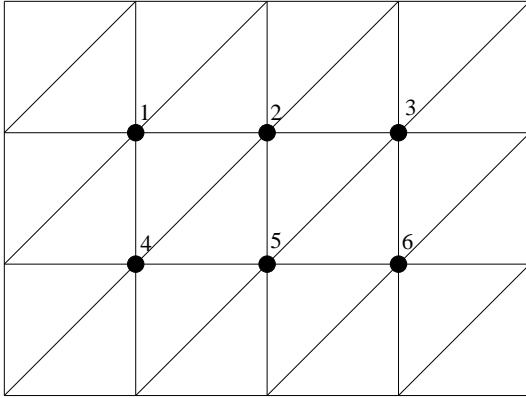
$$\iint_{\Omega} (\nabla u \cdot \nabla v) dx dy = \iint_{\Omega} f v dx dy, \quad \forall v \in H_0^1(\Omega). \quad (10.2.3)$$

Recall that  $H_0^1(\Omega)$  is the usual Hilbert space

$$H_0^1(\Omega) := \{v : v \in L_2(\Omega), |\nabla v| \in L_2(\Omega), \text{ and } v = 0, \text{ on } \Gamma := \partial\Omega\}.$$

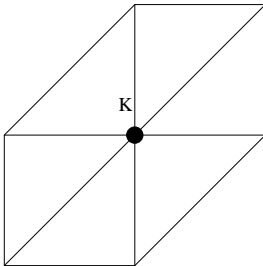
Now we make a test function space of piecewise linears. To this approach we triangulate  $\Omega$  as in the Figure 10.7 and let  $V_h^0 \subset V^0 := H_0^1$  be the space of continuous piecewise linears on the triangulation,  $\mathcal{T}_h$  of  $\Omega$  in Figure 10.7.

$$V_h^0 = \{v \in \mathcal{C}(\Omega) : v \text{ is continuous, linear on each triangle and } v = 0 \text{ on } \partial\Omega\}.$$



**Figure 10.7:** Uniform triangulation of  $\Omega$  with  $h = 1$

Such a discrete solution is uniquely determined by its values at the vertices of the triangles and 0 at the boundary of the domain. In our example we have only 6 inner vertices of interest. Now as in the one-dimensional case we construct continuous piecewise linear basis functions (6 of them in this particular case), with values 1 at one of the (interior) nodes and zero at all other neighboring nodes (both interior and boundary nodes). Then we get the two-dimensional correspondence to the hat-functions  $\varphi_j$ , called *tent functions* as shown in the Figure 10.6.



Then, the finite element method (FEM) for (10.2.2) reads as follows: find  $u_h \in V_h^0$  such that

$$\iint_{\Omega} (\nabla u_h \cdot \nabla v) dx dy = \iint_{\Omega} f v dx dy, \quad \forall v \in V_h^0(\Omega). \quad (10.2.4)$$

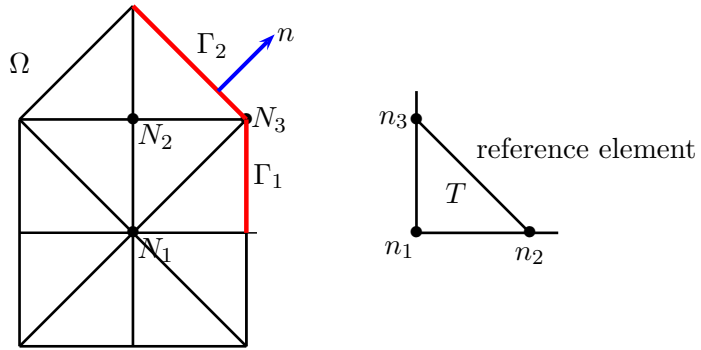
The approximation procedure follows the path generalizing the 1D-case. Below we illustrate the procedure for an example of a mixed boundary value problem in two dimensions and derive the corresponding coefficient matrices.

**Example 10.2.** Formulate the  $cG(1)$  piecewise continuous Galerkin method

for the boundary value problem

$$\begin{aligned} -\Delta u + u &= f, & x \in \Omega; \\ u &= 0, & x \in \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2), \\ \nabla u \cdot \mathbf{n} &= 0, & x \in \Gamma_1 \cup \Gamma_2, \end{aligned} \tag{10.2.5}$$

on the domain  $\Omega$ , with outward unit normal  $\mathbf{n}$  at the boundary (as in the Figure below). Compute the matrices for the resulting system of equations using the following mesh with nodes at  $N_1$ ,  $N_2$  and  $N_3$ . The idea is to use a standard triangle-element  $T$  with *axi-parallel* sides of length  $h$ .



*Solution:* Let  $V$  be the linear space defined as

$$V := \{v : v \in H^1(\Omega), v = 0, \text{ on } \partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)\}.$$

We denote the scalar product by  $(\cdot, \cdot)$ , viz.

$$(f, g) = \int_{\Omega} f(\mathbf{x})g(\mathbf{x}) \, d\mathbf{x}.$$

Multiplying the differential equation by  $v \in V$  and integrating over  $\Omega$  we get

$$-(\Delta u, v) + (u, v) = (f, v), \quad \forall v \in V.$$

Now using Green's formula we have that

$$\begin{aligned} -(\Delta u, v) &= (\nabla u, \nabla v) - \int_{\partial\Omega} (\mathbf{n} \cdot \nabla u)v \, ds \\ &= (\nabla u, \nabla v) - \int_{\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)} (\mathbf{n} \cdot \nabla u)v \, ds - \int_{\Gamma_1 \cup \Gamma_2} (\mathbf{n} \cdot \nabla u)v \, ds \\ &= (\nabla u, \nabla v), \quad \forall v \in V, \end{aligned}$$

where we used the boundary data and the definition of  $V$ . Thus the variational formulation reads as: find  $u \in V$  such that

$$(\nabla u, \nabla v) + (u, v) = (f, v), \quad \forall v \in V.$$

Let  $V_h$  be the finite element space consisting of continuous piecewise linear functions, on the partition as in the figure in the example, satisfying the boundary condition  $v = 0$  on  $\partial\Omega \setminus (\Gamma_1 \cup \Gamma_2)$ : Then, the  $cG(1)$  method for (10.2.5) is formulated as: find  $u_h \in V_h$  such that

$$(\nabla u_h, \nabla v) + (u_h, v) = (f, v) \quad \forall v \in V_h.$$

Making the ‘‘Ansatz’’  $h_h(x) = \sum_{j=1}^3 \xi_j \varphi_j(x)$ , where  $\varphi_j$  are the standard basis functions, we obtain the system of equations

$$\sum_{j=1}^3 \xi_j \left( \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i dx + \int_{\Omega} \varphi_j \varphi_i dx \right) = \int_{\Omega} f \varphi_i dx, \quad i = 1, 2, 3,$$

or in matrix form,

$$(S + M)\xi = F,$$

where  $S_{ij} = (\nabla \varphi_i, \nabla \varphi_j)$  is the stiffness matrix,  $M_{ij} = (\varphi_i, \varphi_j)$  is the mass matrix, and  $F_i = (f, \varphi_i)$  is the load vector. We shall first compute the mass and stiffness matrices for the standard element  $T$  with the side length  $h$  (see Figure). The local basis functions and their gradients are given by

$$\begin{aligned} \phi_1(x_1, x_2) &= 1 - \frac{x_1}{h} - \frac{x_2}{h}, & \nabla \phi_1(x_1, x_2) &= -\frac{1}{h} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \\ \phi_2(x_1, x_2) &= \frac{x_1}{h}, & \nabla \phi_2(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\ \phi_3(x_1, x_2) &= \frac{x_2}{h}, & \nabla \phi_3(x_1, x_2) &= \frac{1}{h} \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Hence, with  $|T| = \int_T dz = h^2/2$ ,

$$\begin{aligned} m_{11} &= (\phi_1, \phi_1) = \int_T \phi_1^2 dx = h^2 \int_0^1 \int_0^{1-x_2} (1 - x_1 - x_2)^2 dx_1 dx_2 = \frac{h^2}{12}, \\ s_{11} &= (\nabla \phi_1, \nabla \phi_1) = \int_T |\nabla \phi_1|^2 dx = \frac{2}{h^2} |T| = 1. \end{aligned}$$

Alternatively, we can use the midpoint rule, which is exact for polynomials of degree 2 (precision 3):

$$m_{11} = (\phi_1, \phi_1) = \int_T \phi_1^2 dx = \frac{|T|}{3} \sum_{j=1}^3 \phi_1(\hat{x}_j)^2 = \frac{h^2}{6} \left( 0 + \frac{1}{4} + \frac{1}{4} \right) = \frac{h^2}{12},$$

where  $\hat{x}_j$  are the midpoints of the edges. Similarly we can compute the other elements and thus obtain the local mass and stiffness matrices as

$$m = \frac{h^2}{24} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad s = \frac{1}{2} \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}.$$

We can now assemble the global matrices  $M$  and  $S$  from the local ones  $m$  and  $s$ , through identifying the contribution from each element sharing the same node to the global matrices. For instance, since the node  $N_1$  is common to 8 triangles and represents vertices corresponding to  $n_2$  or  $n_3$  in reference element  $T$  so we have

$$\begin{aligned} M_{11} &= 8m_{22} = \frac{8}{12}h^2, & S_{11} &= 8s_{22} = 4, \\ \text{Similarly} & & & \\ M_{12} &= 2m_{12} = \frac{1}{12}h^2, & S_{12} &= 2s_{12} = -1, \\ M_{13} &= 2m_{23} = \frac{1}{12}h^2, & S_{13} &= 2s_{23} = 0, \\ M_{22} &= 4m_{11} = \frac{4}{12}h^2, & S_{22} &= 4s_{11} = 4, \\ M_{23} &= 2m_{12} = \frac{1}{12}h^2, & S_{23} &= 2s_{12} = -1, \\ M_{33} &= 3m_{22} = \frac{3}{12}h^2, & S_{33} &= 3s_{22} = 3/2. \end{aligned}$$

The remaining matrix elements are obtained by symmetry  $M_{ij} = M_{ji}$ ,  $S_{ij} = S_{ji}$ . Hence,

$$M = \frac{h^2}{12} \begin{bmatrix} 8 & 1 & 1 \\ 1 & 4 & 1 \\ 1 & 1 & 3 \end{bmatrix}, \quad S = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 4 & -1 \\ 0 & -1 & 3/2 \end{bmatrix}.$$

### 10.3 Constructing finite element spaces

The simplest finite element space consists of continuous piecewise linear basis functions on a partition of a bounded domain  $\Omega \in \mathbb{R}^d$ ,  $d = 1, 2, 3$ . This however, is not sufficient to capture all important properties of a solution and many problems are better approximated considering higher spectral orders which involves functional values and some first, second and/or higher order derivatives on the sides and at the vertices of the elements.

In the one dimensional case, because of the simplicity of the geometry, we could manage to set up approximation procedures relying on elementary calculus and detailed formalism was unnecessary.

Below we give a general framework constructing a *finite element method*:

**Definition 10.3.** *By a finite element we mean a triple  $(K, P_K, \Sigma)$  where*

$K$  is a geometric object, i.e. a triangle, a quadrilateral, a tetrahedron

$P_K$  is a finite-dimensional linear space of functions defined on  $K$

$\Sigma$  is a set of degrees of freedom

and a function  $v \in P_K$  is uniquely determined by the degrees of freedom  $\Sigma$ .

In this setting  $\Sigma = \{\sigma_1, \dots, \sigma_k\}$  is a basis for  $P'_k$  (the set of nodal variables), where  $P'_k$  stands for the algebraic dual of  $P_K$ .

**Definition 10.4.** *Let  $(K, P_K, \Sigma)$  be a finite element, and let  $\{\varphi_1, \varphi_2, \dots, \varphi_k\}$  be basis for  $P_K$  dual to  $\Sigma$ :*

$$\sigma_i(\varphi_j) = \delta_{ij}, \quad 1 \leq i, j \leq k.$$

Then,  $\{\varphi_j\}_{j=1}^k$  is called a nodal basis for  $P_K$ .

For the simplest case of the linear approximation in one-dimension we have that  $(K, P_K, \Sigma)$  is a finite element with

$K = I$  an interval

$P_K = P_1(I)$  is a polynomial of degree 1 on  $I$

$\Sigma$  is the values at the endpoints of  $I$ .

In this setting we can identify our previous, simple one-dimensional basis as follows.

**Example 10.3.** *For  $K = (0, 1)$ ,  $P_K = P_1(K)$  and  $\Sigma = \{\sigma_1, \sigma_2\}$  with  $\sigma_1(v) = v(0)$  and  $\sigma_2(v) = v(1)$  for all  $v \in P_1(K)$ ,  $(K, P_1(K), \Sigma)$  is a finite element, with nodal basis  $\varphi_1(x) = 1 - x$  and  $\varphi_2(x) = x$ .*

Now consider a subdivision  $\mathcal{T}_h = \{K\}$  of a bounded domain  $\Omega \subset \mathbb{R}^d$ ,  $d = 2, 3$ , into elements  $K$  (triangles or quadrilaterals for  $d = 2$  and tetrahedrons for  $d = 3$ ). The commonly used finite element spaces  $V_h$  consists of piecewise polynomials satisfying




$$\begin{aligned} V_h \subset H^1(\Omega) &\iff V_h \subset C^0(\bar{\Omega}) \\ V_h \subset H^2(\Omega) &\iff V_h \subset C^1(\bar{\Omega}). \end{aligned} \tag{10.3.1}$$

**Remark 10.1.** *The first equivalence in (10.3.1) is due to the fact that the functions  $v \in V_h$  are polynomials on each element  $K \in \mathcal{T}_h$  so that for a continuous  $v$  across the common boundary  $\partial K_i \cap \partial K_j$  of any pair of adjacent elements  $K_i$  and  $K_j$  we have that  $D^\alpha v$ ,  $|\alpha| = 1$ , exist and are piecewise continuous, i.e.  $v \in H^1(\Omega)$ . On the other hand, if  $v$  is not continuous across an inter-element boundary, then  $v \notin C^0(\bar{\Omega})$  and hence  $D^\alpha v \notin L_2(\Omega)$ ,  $|\alpha| = 1$ , consequently  $v \notin H^1(\Omega)$ . The second relation in (10.3.1) justifies similarly.*

To define a finite element space  $V_h$  on a triangulation  $\mathcal{T}_h = \{K\}$  of the domain  $\Omega$  we may need to specify

- function values
- values of the first derivatives
- values of the second derivatives
- / values of the normal derivatives
- ↗ value of the mixed derivatives  $\partial^2 v / (\partial x_1 \partial x_2)$ .

In tables below we have collected some of the commonly used finite elements. In one-dimensional problems we considered the first two rows in Table 10.1:

Degrees of freedom $\Sigma$	Function space $P_k$	Degree of continuity of $V_h$
 2	$P_1(K)$	$C^0$
 3	$P_2(K)$	$C^0$
 4	$P_3(K)$	$C^1$

**Table 10.1:** some one-dimensional finite elements

As for the two- and three-dimensional cases we have listed function spaces of 5th spectral order on triangular elements (Table 10.2; 2d case), up to third spectral order on quadrilaterals (table 10.3; 2d case) and finally up to second spectral order on tetrahedrons (table 10.4; 3d case).



## 10.4 The interpolant

**An approach to approximation error:** Given a continuous problem with an unknown exact solution  $u$  in a certain infinite dimensional space, e.g. a Sobolev space  $V$ , we construct an approximate solution  $\tilde{u}$  in an adequate finite dimensional subspace  $\tilde{V}$ , e.g. in a finite element space  $V_h$ . In the previous chapters we have demonstrated two procedures estimating the error  $e := u - u_h$  (assuming  $\tilde{u} := u_h$ ) in some energy norm for simple one dimensional problems.

A somewhat general approach reads as follows: given a simply identifiable approximation  $\hat{u} \in \tilde{V}$  for which we can derive an error estimate for  $\eta := u - \hat{u}$  in an adequate norm  $\|\cdot\|$ . Then we may write the approximation error via this element as

$$e := u - \tilde{u} = u - \hat{u} + \hat{u} - \tilde{u} := \eta + \xi,$$

and hence we have that

$$\|e\| \leq \|u - \hat{u}\| + \|\hat{u} - \tilde{u}\| = \|\eta\| + \|\xi\|.$$

The next task is to bound  $\|\xi\|$  by  $\|\eta\|$ . Note that both  $\tilde{u}$  and  $\hat{u}$  are in the finite dimensional space  $\tilde{V}$ .

The crucial step in this procedure is the estimate of the approximation error  $\eta$ , where  $u \in V$  and  $\hat{u} \in \tilde{V}$ . Generally the estimation of the error  $\xi$  (error between elements inside the finite dimensional space  $\tilde{V}$ ) is much easier. In constructive approaches  $\hat{u}$  is chosen as the *interpolant* of  $u$ . Then, one needs to further formalize the concept of interpolation error (in particular give a clue how the interpolation constant may look like). This yields an estimation for  $\|\eta\|$ , and an estimate of the form  $\|\xi\| \leq C\|\eta\|$  will finish the job. Here, to derive interpolation error in higher dimensions we describe the interpolant in a more general setting. To this end we recall some definitions:

**Definition 10.5.** Let  $(K, P_K, \Sigma)$  be a finite element and let  $\{\varphi_j\}_{j=1}^k$  be a basis for  $P_K$  dual to  $\Sigma$ . For a function  $v$  for which all  $\sigma_i \in \Sigma$ ,  $i = 1, \dots, k$ , are defined, we define the local interpolant  $\mathcal{I}v$  of  $v$  as

$$\mathcal{I}v = \sum_{i=1}^k \sigma_i(v) \varphi_i.$$

**Example 10.4.** Consider the standard triangle of example 10.2. We want to find the local interpolant  $\mathcal{I}_T v$  of the function  $v(x, y) = (1 + x^2 + y^2)^{-1}$ .

By the definition we have that

$$\mathcal{I}_T v = \sigma_1(v) \phi_1 + \sigma_2(v) \phi_2 + \sigma_3(v) \phi_3. \quad (10.4.1)$$

As in the example 10.2 we can easily verify that  $\phi_1(x, y) = 1 - x/h - y/h$ ,  $\phi_2(x, y) = x/h$  and  $\phi_3(x, y) = y/h$ . Further, here we have that  $\sigma_1(v) =$

$v(0,0) = 1$ ,  $\sigma_2(v) = v(h,0) = (1+h^2)^{-1}$  and  $\sigma_3(v) = v(0,h) = (1+h^2)^{-1}$ . Inserting in (10.4.1), we get

$$\mathcal{I}I_T(v) = 1 - \frac{x+y}{h} \left(1 - \frac{1}{1+h^2}\right).$$

**Lemma 10.2.** (The properties of the local interpolation)

- i) The mapping  $v \mapsto \mathcal{I}_K v$  is linear.
- ii)  $\sigma_i(\mathcal{I}_K(v)) = \sigma_i(v)$ ,  $i = 1, \dots, k$ .
- iii)  $\mathcal{I}_K v = v$  for  $v \in P_K$ , i.e.,  $\mathcal{I}_K$  is idempotent:  $\mathcal{I}_K^2 = \mathcal{I}_K$ .

*Proof.* i) Follows from the fact that each  $\sigma_i : v \mapsto \sigma_i(v)$ ,  $i = 1, \dots, k$  is linear.

ii) For  $i = 1, \dots, k$ , we have that

$$\sigma_i(\mathcal{I}_K(v)) = \sigma_i\left(\sum_{j=1}^k \sigma_j(v)\varphi_j\right) = \sum_{j=1}^k \sigma_j(v)\sigma_i(\varphi_j) = \sum_{j=1}^k \sigma_j(v)\delta_{ij} = \sigma_i(v)$$

iii) That  $\mathcal{I}_K v = v$  for  $v \in P_K$ , follows using ii), via  $\sigma_i(v - \mathcal{I}_K(v)) = 0$ ,  $i = 1, \dots, k$ . Then, since  $\mathcal{I}_K(v) \in P_K$ ,

$$\mathcal{I}_K^2 v = \mathcal{I}_K(\mathcal{I}_K(v)) = \mathcal{I}_K(v),$$

and the proof is complete. □

To extend the local interpolant to a global computational domain we need an adequate partitioning strategy which we introduce below

**Definition 10.6.** Let  $\Omega$  be a bounded open set in  $\mathbb{R}^n$  with subdivision  $\mathcal{T}$ . Assume that each subdomain  $K$  is associated with a finite element  $(K, P_K, \Sigma)$  with the shape function  $P_K$  and nodal variables  $\Sigma$ . If  $m$  is the order of highest partial derivatives involved in the nodal variables then, for  $v \in C^m(\bar{\Omega})$  we define the global interpolant  $\mathcal{I}_h v$  on  $\bar{\Omega}$  by

$$\mathcal{I}_h v|_{K_i} = \mathcal{I}_{K_i} v, \quad \forall K_i \in \mathcal{T}.$$

**Definition 10.7.** An interpolant has continuity of order  $r$ , if  $\mathcal{I}_h v \in C^r(\bar{\Omega})$  for all  $v \in C^m(\bar{\Omega})$ . The space

$$\{\mathcal{I}_h v : v \in C^m(\bar{\Omega})\}$$

is called a  $C^r$  finite element space.

Finally, to be able to compare global interpolation operators on different elements, we introduce the following definition of *affine equivalence*.

**Definition 10.8.** Let  $K \subset \Omega \subset \mathbb{R}^n$ , suppose that  $(K, P_K, \Sigma)$  is a finite element and  $F(x) = \mathbf{A}\mathbf{x} + \mathbf{b}$  where  $A$  is a non-singular  $n \times n$  matrix and  $\mathbf{x}$  and  $\mathbf{b}$  are  $n$ -component column vectors. Then, the finite element  $(\hat{K}, \hat{P}_K, \hat{\Sigma})$  is affine equivalence to  $(K, P_K, \Sigma)$  if :

- a)  $F(K) = \hat{K}$ ;
- b)  $F^* \hat{P}_K = P_K$  and
- c)  $F_* \Sigma = \hat{\Sigma}$ .

Here  $F^*$  is the pull-back of  $F$  defined by  $F^*(\hat{v}) = \hat{v} \circ F$ , and  $F_*$  is the push-forward of  $F$  defined by  $(F_* \Sigma)(\hat{v}) = \Sigma(F^*(\hat{v})) = \Sigma(\hat{v} \circ F)$ .

**Example 10.5.** Lagrange elements on triangles (see Table 10.2) with appropriate choice of edge and interior nodes are affine equivalent. The same is true for Hermite elements on triangles.

#### 10.4.1 Error estimates for piecewise linear interpolation

Deriving error estimates, both in a *priori* and a *posteriori* cases, rely on using a certain combination of some fundamental concepts as: variational formulation, finite element formulation, Galerkin orthogonality, Cauchy-Schwarz inequality and interpolation estimates. These concepts, except interpolation estimates, obey a universal form regardless of the dimension. In this section we make a straightforward generalization of the one dimensional linear interpolation estimates on an interval to a two dimensional linear interpolation on a triangle.

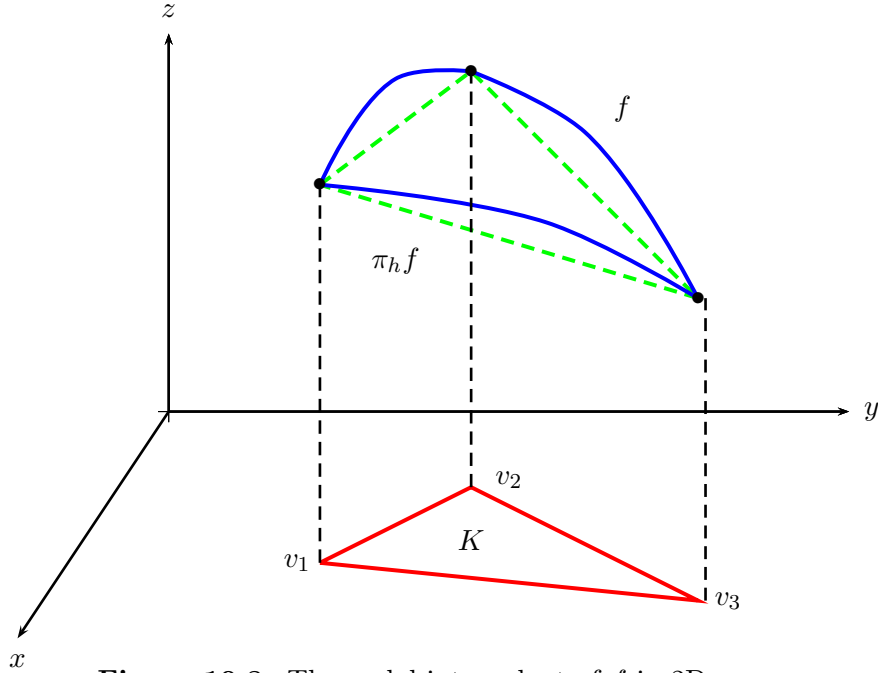
As in the 1D case, our estimates indicate that the interpolation errors depend on the second order, this time, partial derivatives of the function being interpolated, (i.e., the *curvature* of the function), the mesh size and also the shape of the triangle.

Here we assume a triangulation  $\mathcal{T}_h = \{K\}$  of a two dimensional polygonal domain  $\Omega$ . We let  $v_i, i = 1, 2, 3$  be the vertices of the triangle  $K$ . Now we consider a continuous function  $f$  defined on  $K$  and define the linear interpolant  $\mathcal{I}_K f \in \mathcal{P}^1(K)$  by

$$\mathcal{I}_K f(v_i) = f(v_i), \quad i = 1, 2, 3. \quad (10.4.2)$$

Note that  $\mathcal{I}_K$  is used for single element  $K$ , whereas  $\mathcal{I}_h$  is employed in a partition  $\mathcal{T}_h$ . This is illustrated in the Figure 10.5. In higher dimensional case, for an elliptic problem, satisfying the conditions in the Reisz representation theorem (weak formulation with a symmetric, coercive, bilinear form and a linear form as in the definition 2.13), a typical error estimate is a generalization of our a priori error estimates for two-point boundary value problems:

$$\|u - u_h\|_V \leq C \|u - v\|_V, \quad \forall v \in V_h. \quad (10.4.3)$$



**Figure 10.8:** The nodal interpolant of  $f$  in 2D case

Choosing  $v$  as the interpolant of  $u$ :  $v = \mathcal{I}_h u \in V_h$  the right hand side in (10.4.3) is replaced by  $\|u - \mathcal{I}_h u\|_V$ , where  $\mathcal{I}_h u \in V_h$  is chosen with the same degrees of freedom as that of  $u_h \in V_h$ . This would reduce the global estimate (10.4.3) of the problem to local estimates of the form  $\|u - \mathcal{I}_h\|_K$ ,  $\forall K \in \mathcal{T}_h$ .

To proceed we define the concept of *star-shaped* domain:

**Definition 10.9.** A bounded open set in  $\Omega \subset \mathbb{R}^n$  is called *star-shaped* with respect to  $B \subset \Omega$ , if for all  $x \in \Omega$  the closed convex hull of  $\{x\} \cup B$  is a subset of  $\Omega$ .

We shall now state some basic interpolation results that we frequently use in the error estimates. We also give some sketchy proofs. Detailed proofs can be found in, e.g., [10], [20] and [53].

A key result in finite element error analysis is the *Bramble-Hilbert Lemma*. It relies on the following proposition:

**Proposition 10.1.** (*polynomial approximation*) For every  $v \in W_p^m(\Omega)$  there exists a polynomial  $Q$  of degree  $\leq m - 1$  and a constant  $K_j$  such that

$$|v - Q|_{W_p^j(\Omega)} \leq K_j |v|_{W_p^m(\Omega)}, \quad j = 0, \dots, m - 1.$$

Bramble and Hilbert gave a non-constructive proof of this proposition based on Hahn-Banach Theorem. We skip the proof and refer the reader to constructive version given by Suli in [53].

**Lemma 10.3.** (*Bramble-Hilbert Lemma*) *Suppose that  $\Omega$  is a bounded open set in  $\mathbb{R}^n$  and assume that  $\Omega$  is star-shaped with respect to every point in a set  $B \subset \Omega$  of positive measure. Let  $\ell$  be a bounded linear functional on the Sobolev space  $W_p^m(\Omega)$ ,  $m \geq 1$ ,  $1 < p < \infty$ , such that  $\ell(Q) = 0$  for any polynomial  $Q$  of degree  $\leq m - 1$ . Then there exists a constant  $C > 0$  such that*

$$|\ell(v)| \leq C|v|_{W_p^m(\Omega)}, \quad \forall v \in W_p^m(\Omega).$$

*Proof.* By the assumptions there exists a  $C_0 > 0$  such that

$$|\ell(v)| \leq C_0\|v\|_{W_p^m(\Omega)}, \quad \forall v \in W_p^m(\Omega).$$

By the linearity of  $\ell$  and using the fact that  $\ell(Q) = 0$  for any polynomial  $Q$  of degree  $\leq m - 1$  we have that

$$\begin{aligned} |\ell(v)| &= |\ell(v - Q)| \leq C_0\|v - Q\|_{W_p^m(\Omega)} \\ &= C_0 \left( \sum_{j=0}^m |v - Q|_{W_p^j(\Omega)}^p \right)^{1/p} \\ &= C_0 \left( \sum_{j=0}^{m-1} |v - Q|_{W_p^j(\Omega)}^p + |v|_{W_p^m(\Omega)}^p \right)^{1/p} \\ &\leq C_0 \left( \sum_{j=0}^{m-1} |v - Q|_{W_p^j(\Omega)} + |v|_{W_p^m(\Omega)} \right) \end{aligned}$$

Now, by the above proposition we have that

$$|\ell(v)| \leq C_0 \left( 1 + \sum_{j=0}^{m-1} K_j \right) |v|_{W_p^m(\Omega)},$$

and the proof is complete choosing  $C = C_0 \left( 1 + \sum_{j=0}^{m-1} K_j \right)$ .  $\square$

A Corollary of the Bramble-Hilber lemma, which is crucial in determining the interpolation constant, reads as follows:

**Corollary 10.1.** *Let  $\Omega \subset \mathbb{R}^n$  be a bounded open set of diameter  $d$  which is star-shaped with respect to every point of an open ball  $B \subset \Omega$  of diameter  $\mu d$ ,  $\mu \in (0, 1]$ . Suppose that  $1 < p < \infty$  and  $0 \leq j < m$ ,  $m \geq 1$ . If  $v \in W_p^m(\Omega)$  then*

$$\inf_{Q \in \mathcal{P}_{m-1}} |v - Q|_{W_p^j(\Omega)} \leq C(m, n, p, j, \mu) d^{m-j} |v|_{W_p^m(\Omega)},$$

where

$$C(m, n, p, j, \mu) = \mu^{-n/p} (\#\{\alpha : |\alpha| = j\}) \frac{p(m-j)}{p-1} \left( \sum_{|\beta|=m-j} (\beta!)^{-p'} \right)^{1/p}$$

where  $1/p + 1/p' = 1$ , and  $\#A$  denotes the number of elements in  $A$ .

To derive a basic interpolation error estimate we define the *norm of the local interpolation operator*  $\mathcal{I}_K : C^l(\bar{K}) \rightarrow W_p^m(\Omega)$  by

$$\sigma(K) = \sup_{v \in C^l(\bar{K})} \frac{\|\mathcal{I}_K v\|_{W_p^m(\Omega)}}{\|v\|_{C^l(\bar{K})}}.$$

Now we are ready to state and prove our interpolation theorem,

**Theorem 10.1.** *Let  $(K, P_K, \Sigma)$  be a finite element satisfying the following conditions:*

- i)  $K$  is star-shaped with respect to some ball contained in  $K$ ;
- ii)  $\mathcal{P}_{m-1} \subset P_K \subset W_\infty^m(K)$ ,
- iii)  $\Sigma \subset (C^l(\bar{K}))'$ .

Suppose that  $1 < p < \infty$  and  $m - l - (n/p) > 0$ . Then for  $0 \leq j \leq m$  and  $v \in W_p^m(K)$  we have

$$|v - \mathcal{I}_K v|_{W_p^j(K)} \leq C(m, n, p, \mu, \sigma(\hat{K})) h_K^{m-j} |v|_{W_p^m(\Omega)},$$

where  $h_K$  is the diameter of  $K$ ,  $\hat{K} = \{x/h_K : x \in K\}$  and  $\mu$  is the largest real number in the interval  $(0, 1]$  such that all balls of diameter  $\mu h_K$  is contained in  $K$ .

*Proof.* We may assume, without loss of generality that  $K = \hat{K}$  and  $\text{diam } K = 1$ , the general case follows by an scaling argument. Further, we note that the local interpolation operator is well defined on  $W_p^m(K)$  by the Sobolev embedding theorem and, using Sobolev inequality, there exists a constant  $C = C_{m,n,p}$  such that for all  $v \in W_p^m(K)$ ,

$$\|v\|_{C^l(\bar{K})} \leq C_{m,n,p} \|v\|_{W_p^m(K)}, \quad \text{for } m - l > n/p, 1 \leq p < \infty.$$

Now given  $x \in B$  we define a Taylor expansion as

$$P_m(v)(x, y) = \sum_{|\beta| \leq m} D^\beta v(x) \frac{(y-x)^\beta}{\beta!}$$

where  $(y-x)^\beta := (y_1 - x_1)^{\beta_1} \dots (y_n - x_n)^{\beta_n}$ . Let  $Q_m v$  be the averaging operator defined by

$$Q_m(v)(y) = \frac{1}{|B|} \int_B P_m(v)(x, y) dx.$$

Then, since  $\mathcal{I}_K f = f$  for any  $f \in P_K$ , and  $Q_m v \in \mathcal{P}_{m-1} \subset P_K$ , we have

$$\mathcal{I}_K Q_m v = Q_m v.$$

Thus, by the sobolev embedding theorem

$$\begin{aligned}
\|v - \mathcal{I}_K v\|_{W_p^m(K)} &\leq \|v - Q_m v\|_{W_p^m(K)} + \|Q_m v - \mathcal{I}_K v\|_{W_p^m(K)} \\
&= \|v - Q_m v\|_{W_p^m(K)} + \|\mathcal{I}_K(Q_m v - \mathcal{I}_K v)\|_{W_p^m(K)} \\
&\leq \|v - Q_m v\|_{W_p^m(K)} + \sigma(K) \|Q_m v - \mathcal{I}_K v\|_{C^l(\bar{K})} \\
&\leq (1 + C_{m,n,p} \sigma(K)) \|v - Q_m v\|_{W_p^m(K)},
\end{aligned}$$

Finally, by Proposition 10.1 and Theorem 10.1, we have that

$$\|v - \mathcal{I}_K v\|_{W_p^m(K)} \leq C(m, n, p, \mu, \sigma(K)) \|v\|_{W_p^m(K)},$$

and we have the desired result.  $\square$

One may show that under certain condition on the subdivision  $\mathcal{T} = \{K\}$  of the computational domain  $\Omega$  one may remove the  $\sigma(K)$  dependence from the constant  $C(m, n, p, \mu, \sigma(K))$ . This however, is beyond our goal in this text and for such investigations we refer the reader to [10] and [53].

Below we give some two-dimensional analogous of the interpolation estimates in 1d.

**Theorem 10.2.** *If  $f$  has continuous second order partial derivatives, then*

$$\|f - \mathcal{I}_K f\|_{L_\infty(K)} \leq \frac{1}{2} h_K^2 \|D^2 f\|_{L_\infty(K)}, \quad (10.4.4)$$

$$\|\nabla(f - \mathcal{I}_K f)\|_{L_\infty(K)} \leq \frac{3}{\sin(\alpha_K)} h_K \|D^2 f\|_{L_\infty(K)}, \quad (10.4.5)$$

where  $h_K$  is the largest side of  $K$ ,  $\alpha_K$  is the smallest angle of  $K$ , and

$$D^2 f = \left( \sum_{i,j=1}^2 \left( \frac{\partial^2 f}{\partial x_i \partial x_j} \right)^2 \right)^{1/2}.$$

**Remark 10.2.** *Note that the gradient estimate (10.4.5) deteriorates for small  $\sin(\alpha_K)$ ; i.e. for the thinner triangle  $K$ . This phenomenon is avoided assuming a quasi-uniform triangulation, where there is a minimum angle requirement for the triangles, viz.*

$$\sin(\alpha_K) \geq C, \quad \text{for some constant } C \in (0, 1). \quad (10.4.6)$$

#### 10.4.2 The $L_2$ and Ritz projections

**Definition 10.10.** *Let  $V$  be an arbitrary function space defined over  $\Omega$ . The  $L_2$  projection  $Pu \in V$  of a function  $u \in L_2(\Omega)$  is defined by*

$$(u - Pu, v) = 0, \quad \forall v \in V. \quad (10.4.7)$$

Usually, as a special case, we take  $V = V_h$  the space of all continuous piecewise linear functions on a triangulation  $\mathcal{T}_h = \{K\}$  of the domain  $\Omega$ , and the equation (10.4.7) is then written as

$$(u - P_h u, v) = 0, \quad \forall v \in V_h. \quad (10.4.8)$$

This means that, the error  $u - P_h u$  is orthogonal to  $V_h$ . (10.4.8) yields a linear system of equations for the coefficients of  $P_h u$  with respect to the nodal basis of  $V_h$ .

### Advantages of the $L_2$ projection to the nodal interpolation

- The  $L_2$  projection  $P_h u$  is well defined for  $u \in L_2(\Omega)$ , whereas the nodal interpolant  $\pi_h u$  in general requires  $u$  to be continuous. Therefore the  $L_2$  projection is an alternative for the nodal interpolation for, e.g., discontinuous  $L_2$  functions.
- Letting  $v \equiv 1$  in (10.4.7) we have that

$$\int_{\Omega} P_h u \, dx = \int_{\Omega} u \, dx. \quad (10.4.9)$$

Thus the  $L_2$  projection conserves the *total mass*, whereas, in general, the nodal interpolation operator does not preserve the total mass.

- The  $L_2$  projection does not need to satisfy the (Dirichlet) boundary condition, whereas an interpolant has to.
- Finally we have the following error estimate for the  $L_2$  projection:

**Theorem 10.3.** *For a  $u$  with square integrable second derivative the  $L_2$  projection  $P_h$  satisfies*

$$\|u - P_h u\| \leq C_i \|h^2 D^2 u\|. \quad (10.4.10)$$

*Proof.* We have using (10.4.8) and the Cauchy's inequality that

$$\begin{aligned} \|u - P_h u\|^2 &= (u - P_h u, u - P_h u) \\ &= (u - P_h u, u - v) + (u - P_h u, v - P_h u) = (u - P_h u, u - v) \\ &\leq \|u - P_h u\| \|u - v\|. \end{aligned} \quad (10.4.11)$$

This yields

$$\|u - P_h u\| \leq \|u - v\|, \quad \forall v \in V_h. \quad (10.4.12)$$

Thus  $P_h u$  is at least as close to  $u$  (is an at least as good approximation in the  $L_2$ -norm) as  $\pi_h u$ . Now choosing  $v = \pi_h u$  and recalling the interpolation theorem above we get the desired estimate (10.4.10).  $\square$



**A model problem in 2 dimensions.** Let  $\Omega$  be a bounded polygonal domain in  $\mathbb{R}^2$  and consider the model problem

$$\mathcal{A}u := -\nabla \cdot (a\nabla u) = f \quad \text{in } \Omega \quad u = 0 \quad \text{on } \partial\Omega. \quad (10.4.13)$$

Assume that the coefficient  $a = a(x)$  is smooth with  $a(x) \geq \alpha > 0$  in  $\bar{\Omega}$  and  $f \in L_2(\Omega)$ . Then the variational formulation for (10.4.13) reads as: find  $u \in H_0^1(\Omega)$  such that

$$a(u, v) = (f, v), \quad \forall v \in H_0^1(\Omega), \quad (10.4.14)$$

where

$$a(u, v) = \int_{\Omega} a\nabla u \cdot \nabla v \, dx \quad \text{and} \quad (f, v) = \int_{\Omega} f v \, dx,$$

we can easily verify that this problem has a unique solution in  $H_0^1(\Omega)$  which, for convex domain  $\Omega$  satisfies the stability estimate

$$\|u\|_2 \leq C\|f\|. \quad (10.4.15)$$

As usual the finite element approximation yields a discrete version of the variational formulation (10.4.14) as

$$a(u_h, v) = (f, v), \quad \forall v \in V_h^0, \quad (10.4.16)$$

and we can prove the following error estimate

**Theorem 10.4.** *Let  $\Omega$  be convex and let  $u$  and  $u_h$  be the solutions for (10.4.14) and (10.4.16). Then*

$$\|u - u_h\| \leq Ch^2\|u\|_2. \quad (10.4.17)$$

*Proof.* . Let  $e = u - u_h$ . We shall use the dual problem

$$\mathcal{A}\phi = e, \quad \text{in } \Omega \quad \phi = 0, \quad \text{on } \partial\Omega. \quad (10.4.18)$$

Using the stability estimate (10.4.15) we have that

$$\|\phi\|_2 \leq C\|e\|. \quad (10.4.19)$$

Further by an standard estimate we have that

$$\|e\| \leq Ch|e|_1 \leq Ch^2\|u\|_2. \quad (10.4.20)$$

□

**Definition 10.11** (Ritz or elliptic projection). *Let  $R_h : H_0^1 \rightarrow V_h^0$  be the orthogonal projection with respect to the energy inner product, so that*

$$a(R_h v - v, \varphi) = 0, \quad \forall \varphi \in V_h^0, \quad v \in H_0^1. \quad (10.4.21)$$

*Then the operator  $R_h$  is called the Ritz projection or elliptic projection. Then, one may verify, using a Galerkin orthogonality, that the finite element solution  $u_h$  of (10.4.16) is exactly the Ritz projection of the analytic solution  $u$ , i.e.,  $u_h = R_h u$ .*

Using (10.4.17) one can verify that, see, e.g., [33],

$$\|u - R_h u\| + h|u - R_h u|_1 \leq Ch^s \|u\|_s, \quad \forall v \in H^s \cap H_0^1. \quad (10.4.22)$$

## 10.5 Exercises

**Problem 10.1.** Show that the function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $u(x) = \log(|x|^{-1})$ ,  $x \neq 0$  is a solution to the Laplace equation  $\Delta u(x) = 0$ .

**Problem 10.2.** Show that the Laplacian of a  $C^2$  function  $u : \mathbb{R}^2 \rightarrow \mathbb{R}$  in the polar coordinates is written by

$$\Delta u = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2}. \quad (10.5.1)$$

**Problem 10.3.** Show using (10.5.1) that the function  $u = a \log(r) + b$ , where  $a$  and  $b$  are arbitrary constants is a solution of the Laplace equation  $\Delta u(x) = 0$  for  $x \neq 0$ . Are there any other solutions of the Laplace equation in  $\mathbb{R}^2$  which are invariant under rotation (i.e. it depends only on  $r = |x|$ )?

**Problem 10.4.** For a given triangle  $K$ , determine the relation between the smallest angle  $\tau_K$ , the triangle diameter  $h_K$  and the diameter  $\rho_K$  of the largest inscribed circle.

**Problem 10.5.** Prove that a linear function in  $\mathbb{R}^2$  is uniquely determined by its values at three points as long as they don't lie on a straight line.

**Problem 10.6.** Let  $K$  be a triangle with nodes  $\{a^i\}$ ,  $i = 1, 2, 3$  and let the midpoints of the edges be denoted by  $\{a^{ij}, 1 \leq i < j \leq 3\}$ .

a) Show that a function  $v \in \mathcal{P}^1(K)$  is uniquely determined by the degrees of freedom:  $\{v(a^{ij}), 1 \leq i < j \leq 3\}$ .

b) Are functions continuous in the corresponding finite element space of piecewise linear functions?

**Problem 10.7.** Prove that if  $K_1$  and  $K_2$  are two neighboring triangles and  $w_1 \in \mathcal{P}^2(K_1)$  and  $w_2 \in \mathcal{P}^2(K_2)$  agree at three nodes on the common boundary (e.g., two endpoints and a midpoint), then  $w_1 \equiv w_2$  on the common boundary.

**Problem 10.8.** Assume that the triangle  $K$  has nodes at  $\{v^1, v^2, v^3\}$ ,  $v^i = (v_1^i, v_2^i)$ , the element nodal basis is the set of functions  $\lambda_i \in \mathcal{P}^1(K)$ ,  $i = 1, 2, 3$  such that

$$\lambda_i(v^j) = \begin{cases} 1, & i = j \\ 0, & i \neq j. \end{cases}$$

Compute the explicit formulas for  $\lambda_i$ .

**Problem 10.9.** Let  $K$  be a triangular element. Show the following identities, for  $j, k = 1, 2$ , and  $x \in K$ ,

$$\sum_{i=1}^3 \lambda_i(x) = 1, \quad \sum_{i=1}^3 (v_j^i - x_j) \lambda_i(x) = 0, \quad (10.5.2)$$

$$\sum_{i=1}^3 \frac{\partial}{\partial x_k} \lambda_i(x) = 0, \quad \sum_{i=1}^3 (v_j^i - x_j) \frac{\partial \lambda_i}{\partial x_k} = \delta_{jk}, \quad (10.5.3)$$

where  $v^i = (v_1^i, v_2^i)$ ,  $i = 1, 2, 3$  are the vertices of  $K$ ,  $x = (x_1, x_2)$  and  $\delta_{jk} = 1$  if  $j = k$  and  $\delta_{jk} = 0$  otherwise.

**Problem 10.10.** Using (10.5.2), we obtain a representation for the interpolation error of the form

$$f(x) - \pi_h f(x) = - \sum_{i=1}^3 r_i(x) \lambda_i(x). \quad (10.5.4)$$

Prove that the remainder term  $r_i(x)$  can be estimated as

$$|r_i(x)| \leq \frac{1}{2} h_K \|D^2 f\|_{L^\infty(K)}, \quad i = 1, 2, 3. \quad (10.5.5)$$

*Hint: (I) Note that  $|v^i - x| \leq h_K$ . (II) Start applying Cauchy's inequality to show that*

$$\sum_{ij} x_i c_{ij} x_j = \sum_i x_i \sum_j c_{ij} x_j.$$

**Problem 10.11.**  $\tau_K$  is the smallest angle of a triangular element  $K$ . Show that

$$\max_{x \in K} |\nabla \lambda_i(x)| \leq \frac{2}{h_K \sin(\tau_K)}.$$

**Problem 10.12.** The Euler equation for an incompressible inviscid fluid of density can be written as

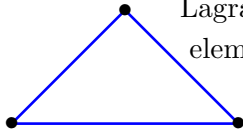
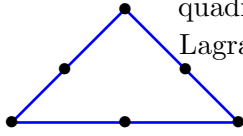
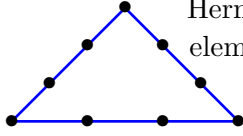
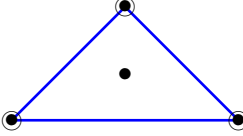
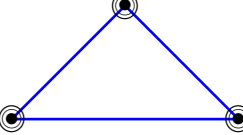
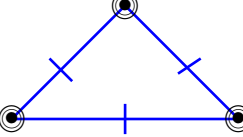
$$u_t + (u \cdot \nabla)u + \nabla p = f, \quad \nabla \cdot u = 0, \quad (10.5.6)$$

where  $u(x, t)$  is the velocity and  $p(x, t)$  the pressure of the fluid at the point  $x$  at time  $t$  and  $f$  is an applied volume force (e.g., a gravitational force). The second equation  $\nabla \cdot u = 0$  expresses the incompressibility. Prove that the first equation follows from the Newton's second law.

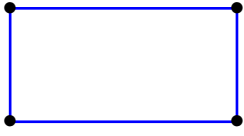
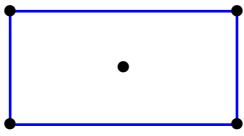
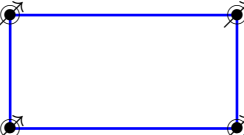
*Hint: Let  $u = (u_1, u_2)$  with  $u_i = u_i(x(t), t)$ ,  $i = 1, 2$  and use the chain rule to derive  $\dot{u}_i = \frac{\partial u_i}{\partial x_1} u_1 + \frac{\partial u_i}{\partial x_2} u_2 + \frac{\partial u_i}{\partial t}$ ,  $i = 1, 2$ .*

**Problem 10.13.** Prove that if  $u : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  satisfies  $\text{rot } u := \left( \frac{\partial u_2}{\partial x_1}, -\frac{\partial u_1}{\partial x_2} \right) = 0$  in a convex domain  $\Omega \subset \mathbb{R}^2$ , then there is a scalar function  $\varphi$  defined on  $\Omega$  such that  $u = \nabla \varphi$  in  $\Omega$ .

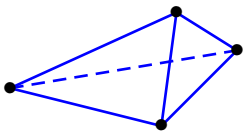
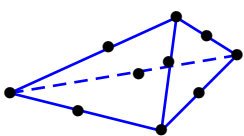
**Problem 10.14.** Prove that  $\int_\Omega \text{rot } u \, dx = \int_\Gamma \mathbf{n} \times u \, ds$ , where  $\Omega$  is a subset of  $\mathbb{R}^3$  with boundary  $\Gamma$  with outward unit normal  $\mathbf{n}$ .

Degrees of freedom $\Sigma$	Function space $P_k$	Degree of continuity of $V_h$
 <p>Lagrange element 3</p>	$P_1(K)$	$C^0$
 <p>quadratic Lagrange 6</p>	$P_2(K)$	$C^0$
 <p>Hermite element 10</p>	$P_3(K)$	$C^0$
 <p>10</p>	$P_3(K)$	$C^0$
 <p>18</p>	$P'_5(K)$	$C^1$
 <p>21</p>	$P_5(K)$	$C^1$

**Table 10.2:** Some 2-dimensional finite elements with triangular elements

Degrees of freedom $\Sigma$	Function space $P_k$	Degree of continuity of $V_h$
	$Q_1(K)$	$C^0$
	$Q_2(K)$	$C^0$
	$Q_3(K)$	$C^1$

**Table 10.3:** Some 2-dimensional finite elements with quadrilateral elements

Degrees of freedom $\Sigma$	Function space $P_k$	Degree of continuity of $V_h$
	$P_1(K)$	$C^0$
	$P_2(K)$	$C^0$

**Table 10.4:** Some 3-dimensional finite elements with tetrahedron elements



# Chapter 11

## The Poisson Equation

In this chapter we shall extend the studies for the one-dimensional problem of the stationary heat conduction in Chapters 4 and 7 to solve the Poisson equation in higher dimensions, i.e.

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^d, \quad d = 2, 3 \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (11.0.1)$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^d$ ,  $d = 2, 3$ , with polygonal boundary  $\Gamma = \partial\Omega$ . For a motivation of studying the Poisson equation and its applications, see Chapter 2. Several phenomena in science and engineering are modeled by the Poisson's equation. Below are some of the most common cases:

- *Electrostatics.* To describe the components of the Maxwell's equations, associating the electric and potential fields: i.e.  $E(x)$  and  $\varphi(x)$  to the charge density  $\rho$ :

$$\left\{ \begin{array}{ll} \nabla \cdot E = \rho, & \text{in } \Omega \quad (\text{Coulomb's law}) \\ \nabla \times E = 0, & (\text{Faraday's law}) \implies E = \nabla\varphi \end{array} \right\} \implies \Delta\varphi = \rho,$$

and with a Dirichlet boundary condition  $\varphi = g$  on  $\partial\Omega$ .

- *Fluid mechanics.* The velocity field  $u$  of a rotation-free fluid flow satisfies  $\nabla \times u = 0$  and hence  $u$  is a, so called, gradient field:  $u = \nabla\varphi$ , with  $\varphi$  being a scalar potential field. The rotation-free incompressible fluid flow satisfies  $\nabla \cdot u = 0$ , which yields to the Laplace's equation  $\Delta\varphi = 0$  for its potential. At a solid boundary, this problem will be associated with homogeneous Neumann boundary condition, due to the fact that in such boundary the normal velocity is zero.

- *Statistical physics.* The random motion of particles inside a container  $\Omega$  until they hit the boundary is described by the probability  $u(x)$  of a particle starting at the point  $x \in \Omega$  winding up to stop at some point on  $\partial\Omega$ , where  $u(x) = 1$  means that it is certain and  $u(x) = 0$  means that it

never happens. It turns out that  $u$  solves the Laplace's equation  $\Delta u = 0$ , with discontinuity at the boundary:  $u = 0$  on  $\partial\Omega_0$  and  $u = 1$  on  $\partial\Omega_1$  where  $\partial\Omega = \partial\Omega_0 \cup \partial\Omega_1$ .

Below we give a brief introduction to the *Green's function approach* to the solution of (11.0.1), and then construct an approximate solution of (11.0.1) using finite element methods. We shall also prove stability results and derive a priori and a posteriori error estimates of the finite element approximations.

## 11.1 The Fundamental Solution

A classical approach to solve PDEs is through Fourier techniques, e.g., multidimensional Fourier series, Laplace and Fourier transforms and separation of variables technique. For real problems, this however is often a cumbersome procedure. An alternative approach is through using Green's functions. Below we shall introduce a general approach and compute the Green's functions for the Laplacian in some common geometries using the fundamental solution and *method of images*. Below we derive fundamental solution for the Poisson equation through deriving rotationally symmetric solution  $u(\mathbf{x}) = v(r)$ ,  $r = |\mathbf{x}|$  for the Laplace's equation  $\Delta u(\mathbf{x}) = 0$ , i.e.

$$\Delta u(\mathbf{x}) = v''(r) + \frac{n-1}{r}v'(r) = 0.$$

This yields the following ode for  $v$ :

$$\frac{v''}{v'} = -\frac{n-1}{r},$$

so that

$$\log v' = -(n-1) \log r + C.$$

Hence

$$v(r) = \begin{cases} \frac{p}{r^{n-2}} + q, & \text{for } n \geq 3 \\ p \log r + q, & \text{for } n = 2. \end{cases}$$

The fundamental solution  $\Phi(\mathbf{x})$  for the Laplace's equation is defined by setting  $q = 0$ . However, to use  $\Phi(\mathbf{x})$  to determine the fundamental solution for the Poisson's equation we choose  $p$  that normalizes  $\Phi(\mathbf{x})$ :

$$\Phi(\mathbf{x}) = \begin{cases} \frac{1}{n(n-2)|B_1(n)|} \frac{1}{|\mathbf{x}|^{n-2}}, & \text{for } n \geq 3 \\ -\frac{1}{2\pi} \log |\mathbf{x}|, & \text{for } n = 2, \end{cases}$$

where  $B_1(n)$  is the volume of the unit ball in  $\mathbb{R}^n$ .



**Remark 11.1.** Note that  $\Phi$  is integrable on bounded sets (even though it is not defined at  $\mathbf{x} = 0$ ) and has integrable singularity at  $\mathbf{x} = 0$ . Nevertheless, the singularity of  $-\Delta\Phi$  is not integrable, but it is the singular Dirac's  $\delta$  function.

Now we formulate the fundamental solution for the Poisson's equation in  $\mathbb{R}^n$ .

**Theorem 11.1.** Suppose that  $f \in C^2(\mathbb{R}^n)$  has compact support. Then the solution for the Poisson's equation

$$-\Delta u(\mathbf{x}) = f(\mathbf{x}) \quad \mathbf{x} \in \mathbb{R}^n, \quad (11.1.1)$$

is given by the convolution product of  $f$  with the fundamental solution  $\Phi$  of the corresponding Laplace's equation:

$$u(\mathbf{x}) = (\Phi * f)(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x} - \mathbf{y})f(\mathbf{y}) d\mathbf{y}.$$

We skip the proof of this theorem and refer the reader to [47].

### 11.1.1 Green's Functions

We shall give a swift introduction to a general framework. A more thorough study is via distribution theory which is beyond the scope of this text. As we demonstrated in the solution approach for boundary value problems in the one-space dimension, Green's functions provide us with an integral representation for solution of the boundary value problems. Let  $u : \mathbb{R}^n \rightarrow \mathbb{R}$  and consider the problem

$$\mathcal{L}u = f \quad \text{in } \Omega \quad Bu = g \quad \text{on } \partial\Omega. \quad (11.1.2)$$

Then, in the distribution sense, the fundamental solution  $\Phi(\mathbf{x}, \mathbf{y})$  for (11.1.2) satisfies

$$\mathcal{L}\Phi(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), \quad \mathbf{x}, \mathbf{y} \in \mathbb{R}^n. \quad (11.1.3)$$

Note that here  $\Phi$  is defined on  $\mathbb{R}^n \times \mathbb{R}^n$ . To see the advantage of this representation, we assume that for each  $\mathbf{x} \in \mathbb{R}^n$ , the function  $\Phi$  is integrable in  $\mathbf{y}$ , then

$$v(\mathbf{x}) = \int_{\mathbb{R}^n} \Phi(\mathbf{x}, \mathbf{y})f(\mathbf{y}) d\mathbf{y} = (\Phi(\mathbf{x}, \mathbf{y}), f(\mathbf{y})). \quad (11.1.4)$$

Now since

$$\mathcal{L}v(\mathbf{x}) = \mathcal{L}(\Phi(\mathbf{x}, \mathbf{y}), f(\mathbf{y})) = (\delta(\mathbf{x} - \mathbf{y}), f(\mathbf{y})) = f(\mathbf{x}),$$

thus  $v$  satisfies the equation  $\mathcal{L}v = f$  in  $\mathbb{R}^n$ .

The remaining issue is that, in general,  $v(\mathbf{x})$  does not satisfy the boundary condition  $Bv = g$ . So, as in elementary calculus, we need to construct

a solution by adding the solution  $w(\mathbf{x})$  of the corresponding homogeneous equation ( $\mathcal{L}w = 0$ ) to  $v(\mathbf{x})$ , so that  $u(\mathbf{x}) = v(\mathbf{x}) + w(\mathbf{x})$  and  $Bu = g$ . This impose the boundary condition on  $w$  viz  $Bw = g - Bv$  (see Chapter 2). Thus, knowing the fundamental solution, the boundary value problem (11.1.2) is reduced to the problem

$$\mathcal{L}w = 0, \quad \text{in } \Omega \quad Bw = g - Bv \quad \text{on } \partial\Omega.$$

It is the boundary condition for  $w$  that can be smoothly tackled using the Green's function for the operator  $\mathcal{L}$  with the boundary operator  $B$ . In this setting the Green's function  $G(\mathbf{x}, \mathbf{y})$  is defined as the solution of the problem

$$\begin{cases} \mathcal{L}G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}), & \mathbf{x} \in \Omega, \\ BG(\mathbf{x}, \mathbf{y}) = 0, & \mathbf{x} \in \partial\Omega, \end{cases} \quad (11.1.5)$$

for each  $\mathbf{y} \in \bar{\Omega}$ . Note that unlike (11.1.3), here  $\mathbf{x} \in \Omega$ . Now, as in the one dimensional case, we construct  $G$  using the fundamental solution that defines a  $\mathbf{y}$ -parametric function  $\Phi^{\mathbf{y}}(\mathbf{x})$ , viz:

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x}, \mathbf{y}) - \Phi^{\mathbf{y}}(\mathbf{x}),$$

then,

$$\begin{cases} \mathcal{L}\Phi^{\mathbf{y}}(\mathbf{x}) = 0, & \mathbf{x} \in \Omega, \\ B\Phi^{\mathbf{y}}(\mathbf{x}) = B\Phi(\mathbf{x}, \mathbf{y}), & \mathbf{x} \in \partial\Omega. \end{cases} \quad (11.1.6)$$

Thus we may write the solution of (11.1.2) as  $u = v + w$  where

$$\begin{cases} v(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y})f(\mathbf{y}) d\mathbf{y} & \text{and} \\ \mathcal{L}w = 0 & \text{in } \Omega \quad \text{with } Bw = g \quad \text{on } \partial\Omega. \end{cases} \quad (11.1.7)$$

We use the above approach to solve the Dirichlet boundary value problem:

**Theorem 11.2** (Green's Function for Laplace's equation). *Let  $u \in C^2(\bar{\Omega})$  be the solution for*

$$-\Delta u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega, \quad (11.1.8)$$

then

$$u(\mathbf{x}) = \int_{\Omega} G(\mathbf{x}, \mathbf{y})f(\mathbf{y}) d\mathbf{y} - \int_{\partial\Omega} \frac{\partial G}{\partial \nu_{\mathbf{y}}}(\mathbf{x}, \mathbf{y})g(\mathbf{y}) d\sigma_{\mathbf{y}}. \quad (11.1.9)$$

Here  $G$  is the Green's function given as

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Delta\phi^{\mathbf{y}}(\mathbf{x}),$$

where  $\Phi(\mathbf{x} - \mathbf{y})$  is the fundamental solution and  $\phi^{\mathbf{y}}(\mathbf{x})$  is the solution for the problem

$$\begin{cases} \Delta \phi^{\mathbf{y}}(\mathbf{x}) = 0, & \text{in } \Omega \\ \Delta \phi^{\mathbf{y}}(\mathbf{x}) = \Phi(\mathbf{x} - \mathbf{y}), & \text{on } \partial\Omega. \end{cases} \quad (11.1.10)$$

For a proof see [47].

### 11.1.2 Method of Images

Below we give examples of Green's function for Laplacian that is obtained using fundamental solution in symmetric domains. The idea is for each  $\mathbf{x} \in \Omega$  to construct image points  $\tilde{\mathbf{x}} \notin \Omega$  so that the function  $\Phi(C\tilde{\mathbf{x}} - \mathbf{y})$  cancels  $\Phi(\mathbf{x} - \mathbf{y})$  at the boundary  $\partial\Omega$  for some constant  $C$  (or a function  $C(\mathbf{x})$ ). Then,  $\Phi(C\tilde{\mathbf{x}} - \mathbf{y})$  is harmonic with respect to  $\mathbf{y}$ , and we can set  $\Delta \phi^{\mathbf{y}}(\mathbf{x}) = \Phi(C\tilde{\mathbf{x}} - \mathbf{y})$ . Below we demonstrate two examples from [47] constructing  $\Phi(C\tilde{\mathbf{x}} - \mathbf{y})$ .

**Example 11.1.** (*method of images for half-space*) Consider the half-space  $\Omega := \{\mathbf{x} \in \mathbb{R}^n : x_n > 0\}$ . Let  $\mathbf{x} = (x_1, \dots, x_n) \in \Omega$  and define its image  $\tilde{\mathbf{x}} = (x_1, \dots, x_{n-1}, -x_n) (\notin \Omega)$ . Then,

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(\tilde{\mathbf{x}} - \mathbf{y}), \quad x_n \geq 0, y_n \geq 0,$$

will be established as the Green's function if  $G(\mathbf{x}, \mathbf{y}) = 0$  for  $y_n = 0, x_n > 0$ . Now since for  $y_n = 0$  we have that

$$|\mathbf{y} - \mathbf{x}| = \sqrt{\sum_{j=1}^{n-1} (y_j - x_j)^2 + x_n^2} = |\mathbf{y} - \tilde{\mathbf{x}}|,$$

and hence  $\Phi(\mathbf{x} - \mathbf{y}) = \Phi(\tilde{\mathbf{x}} - \mathbf{y})$ .

**Example 11.2.** (*method of images for unit ball*) Let  $\Omega := \{\mathbf{x} : |\mathbf{x}| \leq 1\}$ , be the unit ball in  $\mathbb{R}^n$ , and define the image  $\tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|^2$  of a given  $\mathbf{x} \in \Omega$ . Note that  $\tilde{\mathbf{x}}$  is a scaling followed by a reflection through the boundary. Let now

$$G(\mathbf{x}, \mathbf{y}) = \Phi(\mathbf{x} - \mathbf{y}) - \Phi(|\mathbf{x}|(\tilde{\mathbf{x}} - \mathbf{y})). \quad (11.1.11)$$

We need to show that  $G(\mathbf{x}, \mathbf{y})$  defined as (11.1.11) is the Green's function for the unit ball with Dirichlet boundary condition. This is equivalent to showing

$$G(\mathbf{x}, \mathbf{y}) = 0, \quad \mathbf{x} \in \Omega, \mathbf{y} \in \partial\Omega.$$

Hence, it suffices to establish

$$|\mathbf{x}||\tilde{\mathbf{x}} - \mathbf{y}| = |\mathbf{x} - \mathbf{y}|, \quad \text{for } 0 < |\mathbf{x}| < 1, \tilde{\mathbf{x}} = \mathbf{x}/|\mathbf{x}|^2, \quad \text{and } |\mathbf{y}| = 1,$$

which follows from the identity

$$|\mathbf{x}|^2 |\tilde{\mathbf{x}} - \mathbf{y}|^2 = \left| \frac{\mathbf{x}}{|\mathbf{x}|} - |\mathbf{x}|\mathbf{y} \right|^2 = |\mathbf{x}|^2 - 2\mathbf{y} \cdot \mathbf{x} + 1 = |\mathbf{x} - \mathbf{y}|^2.$$

## 11.2 Stability

To derive stability estimates for (11.0.1) we shall assume an underlying general vector space  $V$  (to be specified below) of functions as the solution space. We multiply the equation by  $u \in V$  and integrate over  $\Omega$  to obtain

$$-\int_{\Omega} (\Delta u) u dx = \int_{\Omega} f u dx. \quad (11.2.1)$$

Using Green's formula and the boundary condition,  $u = 0$  on  $\Gamma$ , we get that

$$\|\nabla u\|^2 = \int_{\Omega} f u \leq \|f\| \|u\|, \quad (11.2.2)$$

where  $\|\cdot\|$  denotes the usual  $L_2(\Omega)$ -norm.

To derive a, so called, *weak stability estimate* for (11.0.1) we combine the Poincare inequality with the inequality (11.2.2) and get

$$\|\nabla u\| \leq C_{\Omega} \|f\|. \quad (11.2.3)$$

**Problem 11.1.** *Derive corresponding stability estimate for the following Neumann boundary value problem:*

$$\begin{cases} -\Delta u + u = f, & \text{in } \Omega \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma = \partial\Omega. \end{cases} \quad (11.2.4)$$

## 11.3 Error Estimates for the CG(1) FEM

We start formulating the *weak form* for the problem (11.0.1). We multiply the equation by a test function  $v$ , integrate over  $\Omega \in \mathbb{R}^2$  and use Green's formula. Then, the *variational formulation* reads as follows: find  $u \in H_0^1(\Omega)$  such that

$$(VF) : \int_{\Omega} \nabla u \cdot \nabla v dx = \int_{\Omega} f v dx, \quad \forall v \in H_0^1(\Omega). \quad (11.3.1)$$

We prepare for a finite element method where we shall approximate the exact solution  $u(x)$  by a suitable discrete solution  $U(x)$  in a certain finite dimensional subspace of  $H_0^1(\Omega)$ . To this approach let  $\mathcal{T}_h = \{K : \bigcup K = \Omega\}$  be a triangulation of the domain  $\Omega$  by elements  $K$  with the maximal diameter  $h = \max_K \text{diam}(K)$ . To be specific we shall consider the cG(1) method describing continuous piecewise linear approximations for the solution  $u \in H_0^1(\Omega)$  in a finite dimensional subspace. Thus we have the test and trial function space given as

$$V_h^0 = \{v(x) : v \text{ is continuous, piecewise linear on } \mathcal{T}_h, \text{ and } v = 0 \text{ on } \Gamma = \partial\Omega\}.$$

Observe that  $V_h^0 \subset H_0^1(\Omega)$  is associated to the partition  $\mathcal{T}_h$ . We assume  $n$  interior partition nodes with  $\varphi_j, j = 1, 2, \dots, n$  being the corresponding basis functions, such that  $\varphi_j(x)$  is continuous, linear on each  $K$ , and

$$\varphi_j(N_i) = \begin{cases} 1 & \text{for } i = j \\ 0 & \text{for } i \neq j. \end{cases} \quad (11.3.2)$$

Here  $N_1, N_2, \dots, N_n$  are the interior nodes in the triangulation, and  $\dim V_h^0 = n$ . Now we set the approximate solution  $u_h(x)$  to be a linear combination of the basis functions  $\varphi_j, j = 1, \dots, n$ :

$$u_h(x) = u_{h,1}\varphi_1(x) + u_{h,2}\varphi_2(x) + \dots + u_{h,n}\varphi_n(x), \quad (11.3.3)$$

and seek the coefficients  $u_{h,j} = u_h(N_j)$ , i.e., the nodal values of  $u_h(x)$  (are not necessarily  $= u(N_j)$ ), at the nodes  $N_j, 1 \leq j \leq n$ , so that

$$\int_{\Omega} \nabla u_h \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in V_h^0, \quad (11.3.4)$$

or equivalently

$$(FEM) \quad \int_{\Omega} \nabla u_h \cdot \nabla \varphi_i \, dx = \int_{\Omega} f \varphi_i \, dx, \quad \text{for } i = 1, 2, \dots, n. \quad (11.3.5)$$

Note that every  $v \in V_h^0$  can be represented by a linear combination of  $\{\varphi_j\}_{j=1}^n$ :

$$v(x) = v(N_1)\varphi_1(x) + v(N_2)\varphi_2(x) + \dots + v(N_n)\varphi_n(x). \quad (11.3.6)$$

**Theorem 11.3** (The a priori error estimate for the gradient  $\nabla u - \nabla u_h$ ). *Let  $e = u - u_h$  represent the error in the above continuous, piecewise linear, finite element approximation of the solution for (11.0.1). Then we have*

$$\|\nabla e\| = \|\nabla(u - u_h)\| \leq C\|h D^2 u\|. \quad (11.3.7)$$

*Proof.* For the error  $e = u - u_h$  we have  $\nabla e = \nabla(u - u_h) = \nabla u - \nabla u_h$ . Subtracting (11.3.4) from the (11.3.1) where we restrict  $v$  to  $V_h^0$ , we obtain the *Galerkin Orthogonality*:

$$\int_{\Omega} (\nabla u - \nabla u_h) \cdot \nabla v \, dx = \int_{\Omega} \nabla e \cdot \nabla v \, dx = 0, \quad \forall v \in V_h^0. \quad (11.3.8)$$

Further we may write

$$\|\nabla e\|^2 = \int_{\Omega} \nabla e \cdot \nabla e \, dx = \int_{\Omega} \nabla e \cdot \nabla(u - u_h) \, dx = \int_{\Omega} \nabla e \cdot \nabla u \, dx - \int_{\Omega} \nabla e \cdot \nabla u_h \, dx.$$

Now using the Galerkin orthogonality (11.3.8), since  $u_h \in V_h^0$  we have the last integral above:  $\int_{\Omega} \nabla e \cdot \nabla u_h \, dx = 0$ . Hence removing the vanishing

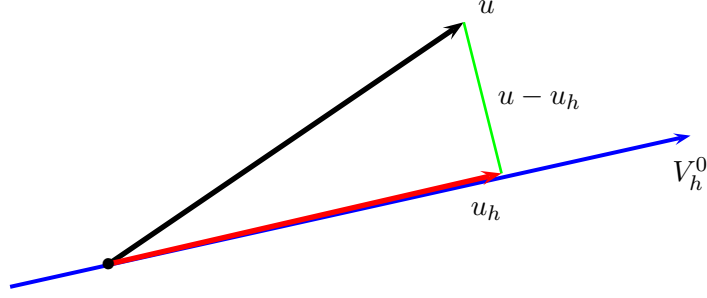
$\nabla u_h$ -term and inserting  $\int_{\Omega} \nabla e \cdot \nabla v \, dx = 0$  for any arbitrary  $v \in V_h^0$  we have that

$$\|\nabla e\|^2 = \int_{\Omega} \nabla e \cdot \nabla u \, dx - \int_{\Omega} \nabla e \cdot \nabla v \, dx = \int_{\Omega} \nabla e \cdot \nabla (u-v) \, dx \leq \|\nabla e\| \|\nabla (u-v)\|.$$

Thus

$$\|\nabla (u - u_h)\| \leq \|\nabla (u - v)\|, \quad \forall v \in V_h^0. \quad (11.3.9)$$

That is, in the  $L_2$ -norm, the gradient of the finite element solution,  $\nabla u_h$  is closer to the gradient of the exact solution:  $\nabla u$ , than the gradient  $\nabla v$  of any other  $v$  in  $V_h^0$ . In other words, measuring in  $H_0^1$ -norm, the error



**Figure 11.1:** The orthogonal ( $L_2$ ) projection of  $u$  on  $V_h^0$ .

$u - u_h$  is orthogonal to  $V_h^0$ . It is possible to show that there is a  $v \in V_h^0$  (an interpolant), such that

$$\|\nabla (u - v)\| \leq C \|h D^2 u\|, \quad (11.3.10)$$

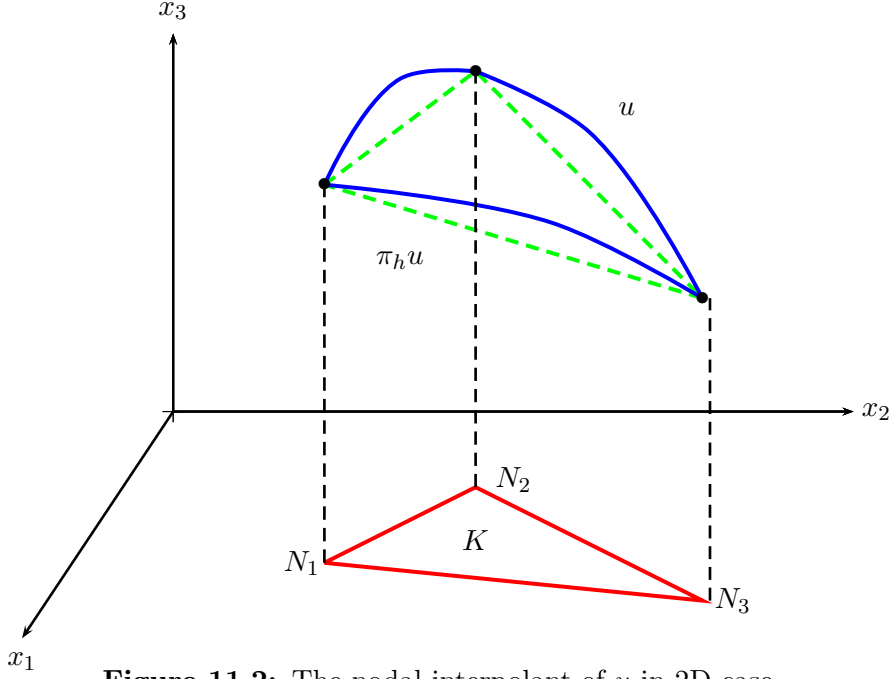
where  $h = h(x) = \text{diam}(K)$  for  $x \in K$  and  $C$  is a constant independent of  $h$ . This is the case, for example, if  $v$  interpolates  $u$  at the nodes  $N_i$ .

Combining (11.3.9) and (11.3.10) we get

$$\|\nabla e\| = \|\nabla (u - u_h)\| \leq C \|h D^2 u\|, \quad (11.3.11)$$

which is indicating that the error is small if  $h(x)$  is sufficiently small depending on  $D^2 u$ . □

To prove an a priori error estimate for the solution we shall use the following result:



**Figure 11.2:** The nodal interpolant of  $u$  in 2D case

**Lemma 11.1** (regularity lemma). *Assume that  $\Omega$  has no reentrant corners ( $\Omega$  is convex). We have for  $u \in H^2(\Omega)$ ; with  $u = 0$  or  $(\frac{\partial u}{\partial n} = 0)$  on  $\partial\Omega$ , that*

$$\|D^2u\| \leq C_\Omega \|\Delta u\|, \quad (11.3.12)$$

where

$$D^2u = (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2)^{1/2}. \quad (11.3.13)$$

We postpone the proof of this lemma and first derive the error estimate.

**Theorem 11.4** (a priori error estimate for the solution  $e = u - u_h$ ). *For a general mesh we have the following a priori error estimate for the solution of the Poisson equation (11.0.1),*

$$\|e\| = \|u - u_h\| \leq C^2 C_\Omega (\max_\Omega h) \|h D^2u\|, \quad (11.3.14)$$

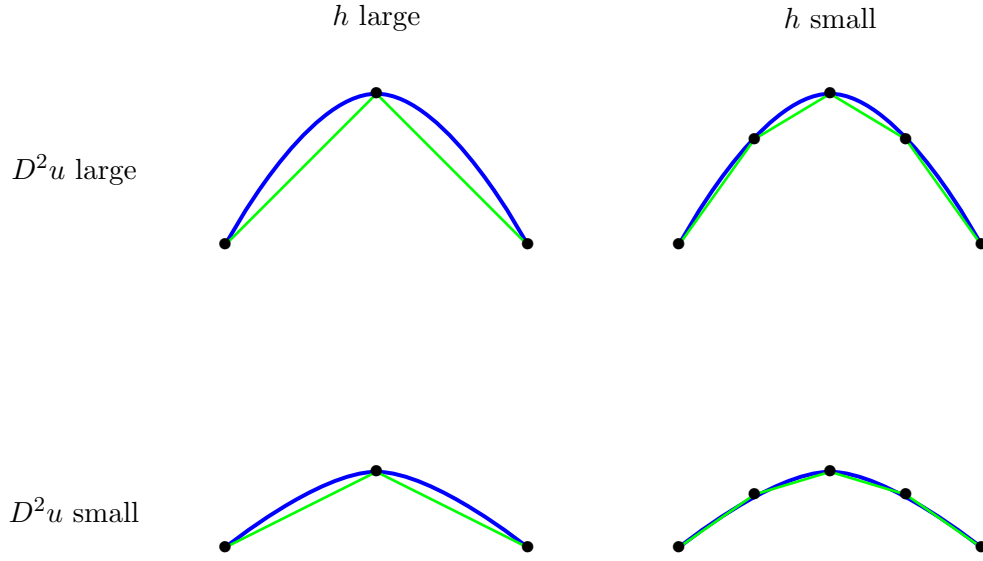
where  $C$  is a constant (generated twice).

*Proof.* Let  $\varphi$  be the solution of the dual problem for (11.0.1).

$$\begin{cases} -\Delta\varphi = e, & \text{in } \Omega \\ \varphi = 0, & \text{on } \partial\Omega. \end{cases} \quad (11.3.15)$$

Then, we have using Green's formula that

$$\begin{aligned} \|e\|^2 &= \int_\Omega e(-\Delta\varphi)dx = \int_\Omega \nabla e \cdot \nabla\varphi dx = \int_\Omega \nabla e \cdot \nabla(\varphi - v) dx \\ &\leq \|\nabla e\| \|\nabla(\varphi - v)\|, \quad \forall v \in V_h^0, \end{aligned} \quad (11.3.16)$$



**Figure 11.3:** The adaptivity principle: to refine mesh for large  $D^2u$

where in the last equality we have used the Galerkin orthogonality. We now choose  $v$  (e.g. as an interpolant of  $\varphi$ ) such that

$$\|\nabla(\varphi - v)\| \leq C \|h D^2\varphi\| \leq C(h) \|D^2\varphi\|. \quad (11.3.17)$$

Applying the lemma to  $\varphi$ , we get

$$\|D^2\varphi\| \leq C_\Omega \|\Delta\varphi\| = C_\Omega \|e\|. \quad (11.3.18)$$

Finally, combining (11.3.11)-(11.3.18) yields

$$\begin{aligned} \|e\|^2 &\leq \|\nabla e\| \|\nabla(\varphi - v)\| \leq \|\nabla e\| Ch \|D^2\varphi\| \\ &\leq \|\nabla e\| Ch C_\Omega \|e\| \leq C \|h D^2u\| Ch C_\Omega \|e\|. \end{aligned} \quad (11.3.19)$$

Thus we have obtained the desired a priori error estimate

$$\|e\| = \|u - u_h\| \leq C^2 C_\Omega h \|h D^2u\|. \quad (11.3.20)$$

□

**Corollary 11.1** (strong stability estimate). *Using the Lemma 10.2, for a piecewise linear approximation, the a priori error estimate (11.3.20) can be written as the following strong stability estimate,*

$$\|u - u_h\| \leq C^2 C_\Omega h^2 \|f\|. \quad (11.3.21)$$



**Theorem 11.5** (a posteriori error estimate). *Let  $u$  be the solution of the Poisson equation (11.0.1) and  $u_h$  its continuous piecewise linear finite element approximation. Then, there is a constant  $C$ , independent of  $u$  and  $h$ , such that*

$$\|u - u_h\| \leq C \|h^2 r\|, \quad (11.3.22)$$

where  $r = f + \Delta_h u_h$  is the residual with  $\Delta_h$  being the discrete Laplacian defined by

$$(\Delta_h u_h, v) = \sum_{K \in \mathcal{T}_h} (\nabla u_h, \nabla v)_K. \quad (11.3.23)$$

*Proof.* We start considering the following dual problem

$$\begin{cases} -\Delta \varphi(x) = e(x), & x \in \Omega, \\ \varphi(x) = 0, & x \in \partial\Omega, \end{cases} \quad e(x) = u(x) - u_h(x). \quad (11.3.24)$$

Then  $e(x) = 0, \forall x \in \partial\Omega$ , and using Green's formula we have that

$$\|e\|^2 = \int_{\Omega} e e \, dx = \int_{\Omega} e(-\Delta \varphi) \, dx = \int_{\Omega} \nabla e \cdot \nabla \varphi \, dx. \quad (11.3.25)$$

Thus by the Galerkin orthogonality (11.3.8) and using the boundary data:  $\varphi(x) = 0, \forall x \in \partial\Omega$ , we get

$$\begin{aligned} \|e\|^2 &= \int_{\Omega} \nabla e \cdot \nabla \varphi \, dx - \int_{\Omega} \nabla e \cdot \nabla v \, dx = \int_{\Omega} \nabla e \cdot \nabla(\varphi - v) \, dx \\ &= \int_{\Omega} (-\Delta e)(\varphi - v) \, dx \leq \|h^2 r\| \|h^{-2}(\varphi - v)\| \\ &\leq C \|h^2 r\| \|\Delta \varphi\| \leq C \|h^2 r\| \|e\|, \end{aligned} \quad (11.3.26)$$

where we use the fact that the  $-\Delta e = -\Delta u + \Delta_h u_h + [\frac{\partial u_h}{\partial \mathbf{n}}] = f + \Delta_h u_h = r$  is the residual and choose  $v$  as an interpolant of  $\varphi$ , so that

$$\|h^{-2}(\varphi - v)\| \leq C \|D^2 \varphi\| \leq C C_{\Omega} \|\Delta \varphi\|.$$

Thus, the final result reads as

$$\|u - u_h\| \leq C_{\Omega} \|h^2 r\|. \quad (11.3.27)$$

□

**Corollary 11.2** (strong stability estimate). *Observe that for piecewise linear approximations  $\Delta u_h = 0$  but  $\Delta_h u_h \neq 0$  on each element  $K$ . If  $\Delta_h u_h = 0$  on each element  $K$  then  $r \equiv f$  and our a posteriori error estimate above can be viewed as a strong stability estimate*

$$\|e\| \leq C \|h^2 f\|. \quad (11.3.28)$$

**Problem 11.2.** Show that  $\|\nabla(u - u_h)\| \leq C\|hr\|$ .

**Problem 11.3.** Verify that for  $v$  being the interpolant of  $\varphi$ , we have

$$\|e\| \leq C \|h^2 f\| \times \begin{cases} \|h^{-2}(\varphi - v)\| & \leq C \|\Delta\varphi\|, \quad \text{and} \\ \|h^{-1}(\varphi - v)\| & \leq C \|\nabla\varphi\|. \end{cases} \quad (11.3.29)$$

**Problem 11.4.** Derive the estimate corresponding to (11.3.27) for the one-dimensional case.

We now return to the proof of Lemma 10.2. One can show that, in the lemma for a convex  $\Omega$  the constant  $C_\Omega \leq 1$ . Otherwise  $C_\Omega > 1$  and increases with the degree of the singularity of a corner point at the boundary of the domain  $\Omega$ . With  $\Omega$  having a reentrant we have that  $C_\Omega = \infty$ .

### 11.3.1 Proof of the regularity Lemma

*Proof.* Let now  $\Omega$  be a rectangular domain and set  $u = 0$  on  $\partial\Omega$ . We have then

$$\|\Delta u\|^2 = \int_{\Omega} (u_{xx} + u_{yy})^2 dx dy = \int_{\Omega} (u_{xx}^2 + 2u_{xx}u_{yy} + u_{yy}^2) dx dy. \quad (11.3.30)$$

Applying Green's formula (in  $x$ -direction) for the rectangular domain  $\Omega$  we can write (note that partial derivatives of higher order appears),

$$\int_{\Omega} u_{xx}u_{yy} dx dy = \int_{\partial\Omega} (u_x \cdot n_x)u_{yy} ds - \int_{\Omega} u_x \underbrace{u_{yyx}}_{=u_{xyy}} dx dy. \quad (11.3.31)$$

Once again by Green's formula ( "with  $v = u_x$  and  $\Delta w = u_{xyy}$ " ),

$$\int_{\Omega} u_x u_{xyy} dx dy = \int_{\partial\Omega} u_x (u_{yx} \cdot n_y) ds - \int_{\Omega} u_{xy} u_{xy} dx dy. \quad (11.3.32)$$

Inserting (11.3.32) in (11.3.31) yields

$$\int_{\Omega} u_{xx}u_{yy} dx dy = \int_{\partial\Omega} (u_x u_{yy} n_x - u_x u_{yx} n_y) ds + \int_{\Omega} u_{xy} u_{xy} dx dy. \quad (11.3.33)$$

Now, as we can see from the figure, we have that  $(u_x u_{yy} n_x - u_x u_{yx} n_y) = 0$ , on  $\partial\Omega$  and hence

$$\int_{\Omega} u_{xx}u_{yy} dx dy = \int_{\Omega} u_{xy} u_{xy} dx dy = \int_{\Omega} u_{xy}^2 dx dy. \quad (11.3.34)$$

Thus, in this case,

$$\|\Delta u\|^2 = \int_{\Omega} (u_{xx} + u_{yy})^2 dx dy = \int_{\Omega} (u_{xx}^2 + 2u_{xy}^2 + u_{yy}^2) dx dy = \|D^2 u\|^2,$$

and the proof is complete by a constant  $C_\Omega \equiv 1$ . The general case of a polygonal domain, easily, follows from this proof.  $\square$

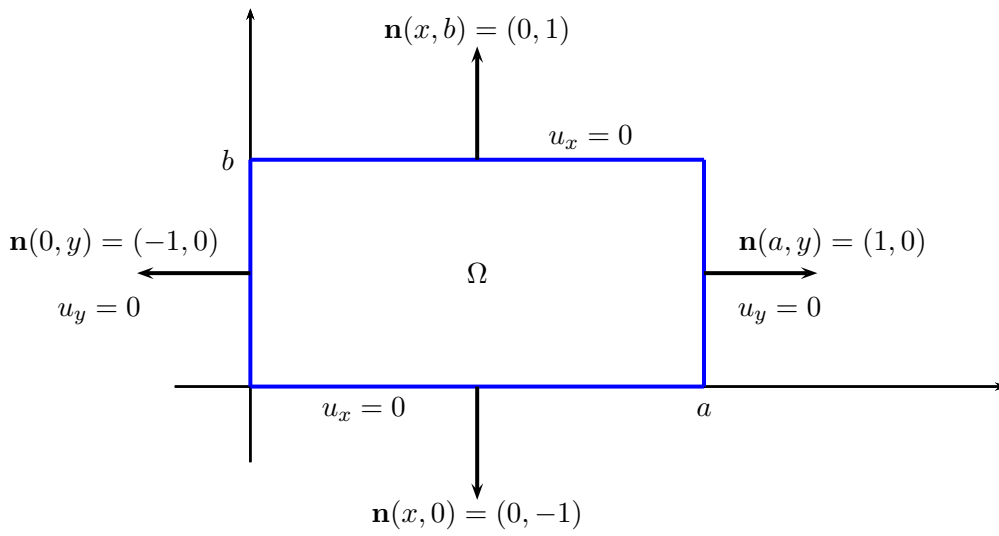


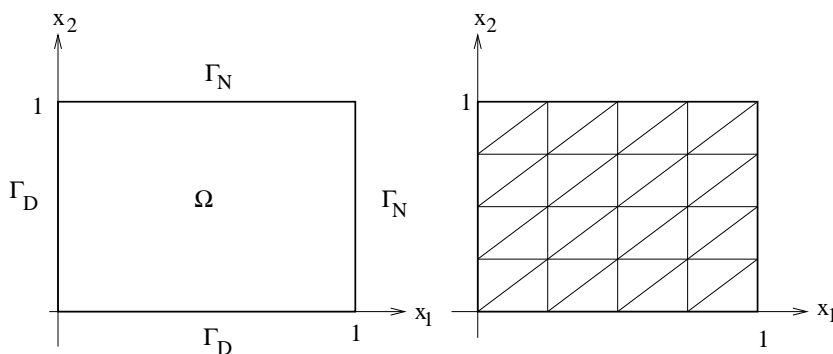
Figure 11.4: A rectangular domain  $\Omega$  with its outward unit normals

### 11.4 Exercises

**Problem 11.5.** Consider the following two dimensional problem:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_N \end{cases} \quad (11.4.1)$$

See figure below

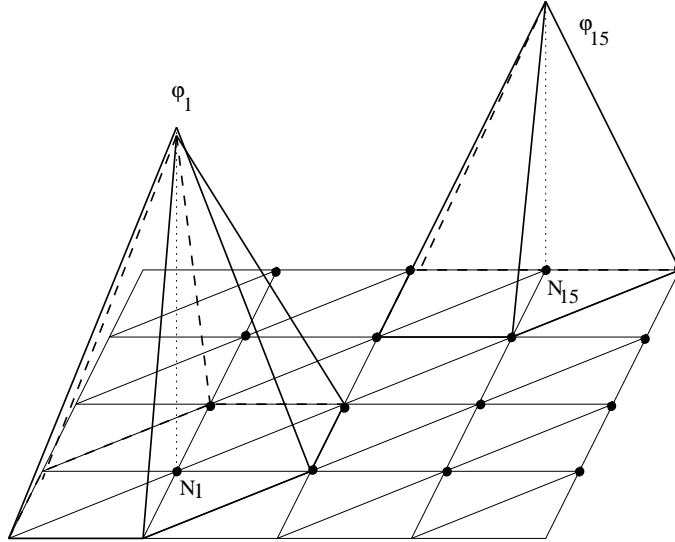


Triangulate  $\Omega$  as in the figure and let

$$U(x) = U_1\varphi_1(x) + \dots + U_{16}\varphi_{16}(x),$$

where  $x = (x_1, x_2)$  and  $\varphi_j$ ,  $j = 1, \dots, 16$  are the basis functions, see Figure below, and determine  $U_1, \dots, U_{16}$  so that

$$\int_{\Omega} \nabla U \cdot \nabla \varphi_j dx = \int_{\Omega} \varphi_j dx, \quad j = 1, 2, \dots, 16.$$



**Problem 11.6.** Generalize the procedure in the previous problem to the following case with mesh-size= $h$ .

$$\begin{cases} -\nabla(a\nabla u) = f, & \text{in } \Omega \\ u = 0, & \text{on } \Gamma_D \\ a \frac{\partial u}{\partial n} = 7, & \text{on } \Gamma_N \end{cases}, \text{ where } \begin{cases} a = 1 & \text{for } x_1 < \frac{1}{2} \\ a = 2 & \text{for } x_1 > \frac{1}{2} \\ f = x_2. \end{cases}$$

**Problem 11.7.** Consider the Dirichlet problem

$$-\nabla \cdot (a(x)\nabla u(x)) = f(x), \quad x \in \Omega \subset \mathbb{R}^2, \quad u(x) = 0, \text{ for } x \in \partial\Omega.$$

Assume that  $c_0$  and  $c_1$  are constants such that  $0 < c_0 \leq a(x) \leq c_1$ ,  $\forall x \in \Omega$  and let  $U$  be a Galerkin approximation of  $u$  in a subspace  $M$  of  $H_0^1(\Omega)$ . Prove the a priori error estimate

$$\|u - U\|_{H_0^1(\Omega)} \leq \frac{c_1}{c_0} \inf_{\chi \in M} \|u - \chi\|_{H_0^1(\Omega)}.$$

**Problem 11.8.** Determine the stiffness matrix and load vector if the  $cG(1)$  finite element method applied to the Poisson's equation on a triangulation

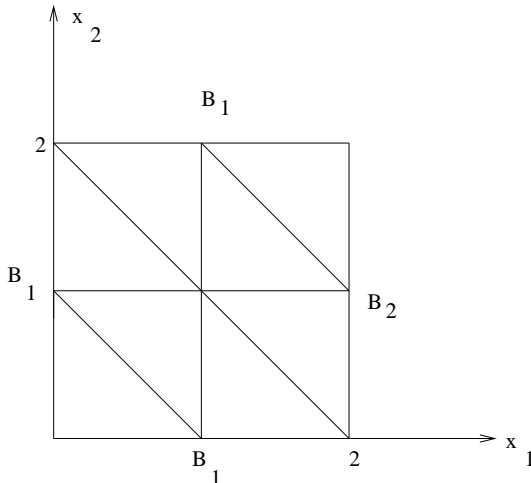
with triangles of side length  $1/2$  in both  $x_1$ - and  $x_2$ -directions:

$$\begin{cases} -\Delta u = 1, & \text{in } \Omega = \{(x_1, x_2) : 0 < x_1 < 2, 0 < x_2 < 1\}, \\ u = 0, & \text{on } \Gamma_1 = \{(0, x_2)\} \cup \{(x_1, 0)\} \cup \{(x_1, 1)\}, \\ \frac{\partial u}{\partial n} = 0, & \text{on } \Gamma_2 = \{(2, x_2) : 0 \leq x_2 \leq 1\}. \end{cases}$$

**Problem 11.9.** Let  $\Omega = (0, 2) \times (0, 2)$ ,  $B_1 = \partial\Omega \setminus B_2$  and  $B_2 = \{2\} \times (0, 2)$ . Determine the stiffness matrix and load vector in the  $cG(1)$  solution for the problem

$$\begin{cases} -\frac{\partial^2 u}{\partial x_1^2} - 2\frac{\partial^2 u}{\partial x_2^2} = 1, & \text{in } \Omega = (0, 2) \times (0, 2), \\ u = 0, & \text{on } B_1, \quad \frac{\partial u}{\partial x_1} = 0, & \text{on } B_2, \end{cases}$$

with piecewise linear approximation applied on the triangulation below:



**Problem 11.10.** Determine the stiffness matrix and load vector if the  $cG(1)$  finite element method with piecewise linear approximation is applied to the following Poisson's equation with mixed boundary conditions:

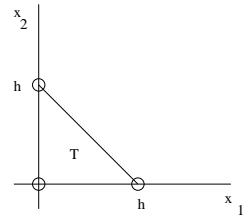
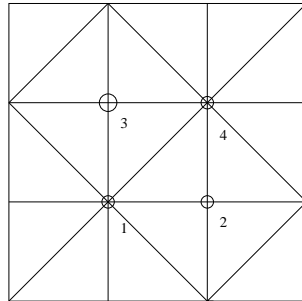
$$\begin{cases} -\Delta u = 1, & \text{on } \Omega = (0, 1) \times (0, 1), \\ \frac{\partial u}{\partial n} = 0, & \text{for } x_1 = 1, \\ u = 0, & \text{for } x \in \partial\Omega \setminus \{x_1 = 1\}, \end{cases}$$

on a triangulation with triangles of side length  $1/4$  in the  $x_1$ -direction and  $1/2$  in the  $x_2$ -direction.

**Problem 11.11.** Formulate the  $cG(1)$  method for the boundary value problem

$$-\Delta u + u = f, \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

Write down the matrix form of the resulting system of equations using the following uniform mesh:



## Chapter 12

# The Initial Boundary Value Problems in $\mathbb{R}^N$

This chapter is devoted to the study of time dependent problems formulated in spatial domain in  $\mathbb{R}^N$ ,  $N \geq 2$ . More specifically we shall consider the finite element methods for two fundamental equations: the heat equation and the wave equation. In this setting we have already studied time discretizations as well as a concise finite element analysis of the Laplace's equation in Chapters 8 and 11, respectively. Here we shall derive certain stability estimates for these equations and construct fully discrete schemes for temporal and spatial discretizations. We also formulate some convergence results that can be viewed as a combination of error estimates in Chapters 8 and 11. Detailed proofs for a priori and a posteriori error estimates are direct generalization of the results of Chapter 9 for one space dimension and can be found in, e.g., Thomeé [55] and Eriksson et al. in [20]

### 12.1 The heat equation in $\mathbb{R}^N$

In this section we shall study the stability of the heat equation in  $\mathbb{R}^N$ ,  $N \geq 2$ . Here our concern will be those aspects of the stability estimates for the higher dimensional case that are not a direct consequence of the study of the one-dimensional problem addressed in Chapter 9.

The initial value problem for the heat conduction can be formulated as

$$\begin{cases} \dot{u}(x, t) - \Delta u(x, t) = f(x, t), & \text{in } \mathbb{R}^d, d = 1, 2, 3 \quad (DE) \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R}^d. \quad (IC) \end{cases} \quad (12.1.1)$$

The equation (12.1.1) is of parabolic type with significant *smoothing* and *stability* properties. It can also be used as a model for a variety of physical phenomena involving *diffusion processes*. Some important physical properties of (12.1.1) are addressed in Chapter 2. Here, we shall focus on the

stability concept and construction of the fully discrete schemes, which is a combination of the discretization procedure in Chapters 7 and 11. For details in a priori and a posteriori analysis we refer the reader to Chapter 16 in [20].

### 12.1.1 The fundamental solution

We consider the homogeneous heat conduction in the general  $d$ -dimensional Euclidean space:

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = 0, & \text{for } (x, t) \in \mathbb{R}^d \times (0, \infty) \\ u(x, 0) = u_0(x), & \text{for } x \in \mathbb{R}^d. \end{cases} \quad (12.1.2)$$

First we seek the *fundamental solution* of (12.1.2) in the 2 dimensional case, i.e.,  $d = 2$ , i.e., and when the initial data is assumed to be the Dirac  $\delta$  function:  $u_0(x) = \delta_0(x)$ . We denote this solution by  $E(x, t)$ . One may easily verify that

$$E(x, t) = \frac{1}{(4\pi t)} \exp\left(-\frac{|x|^2}{4t}\right). \quad (12.1.3)$$

Then, for an arbitrary function  $u_0$  the solution for (12.1.2), with  $d = 2$ , is expressed in terms of fundamental solution and is obtained through convolution of  $u_0$  and  $E$ :

$$u(x, t) = \frac{1}{4\pi t} \int_{\mathbb{R}^2} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy. \quad (12.1.4)$$

For the general problem (12.1.2), using the  $d$ -dimensional Fourier transform, see [33], we have the  $d$ -dimensional version of (12.1.3) and (12.1.4), viz

$$E(x, t) = \frac{1}{(4\pi t)^{d/2}} \exp\left(-\frac{|x|^2}{4t}\right), \quad x \in \mathbb{R}^d, \quad t > 0, \quad (12.1.5)$$

and

$$u(x, t) =: (E(t)u_0)(x) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} u_0(y) \exp\left(-\frac{|x-y|^2}{4t}\right) dy. \quad (12.1.6)$$

Hence, if we let  $\eta = \frac{y-x}{\sqrt{4t}}$ , then

$$\frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} \exp\left(-\frac{|x-y|^2}{4t}\right) dy = \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-|\eta|^2} d\eta = 1, \quad (12.1.7)$$

thus, we have that

$$|u(x, t)| \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}, \quad \text{for } t > 0. \quad (12.1.8)$$



Now, we return to the original, non-homogeneous, heat equation (12.1.1) with  $f \neq 0$  and recall that  $E$ , defined as in (12.1.6), is interpreted as an operator acting on initial data  $u_0$ . This can be extended to  $f$  and using the *Duhamel's principle*, see [33], the solution to the problem (12.1.1) can be expressed as

$$u(t) = E(t)u_0 + \int_0^t E(t-s)f(x, s) ds. \quad (12.1.9)$$

Then as usual, it is easy to verify, (see below), that

$$\|u(t)\| \leq \|u_0\| + \int_0^t \|f(s)\| ds, \quad (12.1.10)$$

which yields the uniqueness in the standard way.

## 12.2 Stability

Here we consider the heat equation in a bounded domain  $\Omega \subset \mathbb{R}^2$ ,

$$\begin{cases} \dot{u}(x, t) - \Delta u(x, t) = 0, & \text{for } (x, t) \in \Omega \times (0, \infty) \\ u(x, t) = 0, & \text{for } x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega. \end{cases} \quad (12.2.1)$$

The stability estimates for the equation (12.2.1) are summarized in theorem below. It roughly states a control of certain energy norms of the solution  $u$  by the  $L_2$  norm of the initial data  $u_0$ , where some of the estimates below up in the vicinity of  $t = 0$ .

**Theorem 12.1** (Energy estimates). *The solution  $u$  of the problem (12.2.1) satisfies the following stability estimates*

$$\max \left( \|u(\cdot, t)\|^2, 2 \int_0^t \|\nabla u\|^2(s) ds \right) \leq \|u_0\|^2 \quad (12.2.2)$$

$$\|\nabla u(\cdot, t)\| \leq \frac{1}{\sqrt{2t}} \|u_0\| \quad (12.2.3)$$

$$\left( \int_0^t s \|\Delta u(\cdot, s)\|^2 ds \right)^{1/2} \leq \frac{1}{2} \|u_0\| \quad (12.2.4)$$

$$\|\Delta u(\cdot, t)\| \leq \frac{1}{\sqrt{2t}} \|u_0\| \quad (12.2.5)$$

$$\text{For } \varepsilon > 0 \text{ small, } \int_\varepsilon^t \|\dot{u}(\cdot, s)\| ds \leq \frac{1}{2} \sqrt{\ln \frac{t}{\varepsilon}} \|u_0\|. \quad (12.2.6)$$

Note that (12.2.6) indicates strong instabilities for  $\varepsilon$  (as well as  $t$ ) close to 0. This yields the splitting strategy used in error analysis below.

*Proof.* To derive the first two estimates in (12.2.2) we multiply (12.1.1) by  $u$  and integrate over  $\Omega$ , to get

$$\int_{\Omega} \dot{u}u \, dx - \int_{\Omega} (\Delta u)u \, dx = 0. \quad (12.2.7)$$

Note that  $\dot{u}u = \frac{1}{2} \frac{d}{dt} u^2$ . Further, using Green's formula with the Dirichlet boundary data,  $u = 0$  on  $\Gamma$ , we have that

$$- \int_{\Omega} (\Delta u)u \, dx = - \int_{\Gamma} (\nabla u \cdot \mathbf{n}) u \, ds + \int_{\Omega} \nabla u \cdot \nabla u \, dx = \int_{\Omega} |\nabla u|^2 \, dx. \quad (12.2.8)$$

Thus equation (12.2.7) can be written in the following two equivalent forms:

$$\frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx = 0 \iff \frac{1}{2} \frac{d}{dt} \|u\|^2 + \|\nabla u\|^2 = 0, \quad (12.2.9)$$

where  $\|\cdot\|$  denotes the  $L_2(\Omega)$  norm. We substitute  $t$  by  $s$  and integrate the equation (12.2.9) over  $s \in (0, t)$  to get

$$\frac{1}{2} \int_0^t \frac{d}{ds} \|u\|^2(s) \, ds + \int_0^t \|\nabla u\|^2(s) \, ds = \frac{1}{2} \|u\|^2(t) - \frac{1}{2} \|u\|^2(0) + \int_0^t \|\nabla u\|^2 \, ds = 0.$$

Hence, inserting the initial data  $u(0) = u_0$  we have

$$\|u\|^2(t) + 2 \int_0^t \|\nabla u\|^2(s) \, ds = \|u_0\|^2. \quad (12.2.10)$$

In particular, we have our first two stability estimates

$$\|u\|(t) \leq \|u_0\|, \quad \text{and} \quad \int_0^t \|\nabla u\|^2(s) \, ds \leq \frac{1}{2} \|u_0\|^2. \quad (12.2.11)$$

To derive (12.2.3) and (12.2.4), since  $\dot{u} = \Delta u$ , and we need to estimate the  $L_2$ -norm of  $\dot{u}$ , so we multiply the equation,  $\dot{u} - \Delta u = 0$ , by  $-t \Delta u$  and integrate over  $\Omega$  to obtain

$$-t \int_{\Omega} \dot{u} \Delta u \, dx + t \int_{\Omega} (\Delta u)^2 \, dx = 0. \quad (12.2.12)$$

Using Green's formula ( $u = 0$  on  $\Gamma \implies \dot{u} = 0$  on  $\Gamma$ ) yields

$$\int_{\Omega} \dot{u} \Delta u \, dx = - \int_{\Omega} \nabla \dot{u} \cdot \nabla u \, dx = - \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2, \quad (12.2.13)$$

so that (12.2.12) can be rewritten as

$$t \frac{1}{2} \frac{d}{dt} \|\nabla u\|^2 + t \|\Delta u\|^2 = 0. \quad (12.2.14)$$

Now using the relation  $t \frac{d}{dt} \|\nabla u\|^2 = \frac{d}{dt} (t \|\nabla u\|^2) - \|\nabla u\|^2$ , (12.2.14) can be rewritten as

$$\frac{d}{dt} (t \|\nabla u\|^2) + 2t \|\Delta u\|^2 = \|\nabla u\|^2. \quad (12.2.15)$$

Once again we substitute  $t$  by  $s$  and integrate over  $(0, t)$  to get

$$\int_0^t \frac{d}{ds} (s \|\nabla u\|^2(s)) ds + 2 \int_0^t s \|\Delta u\|^2(s) ds = \int_0^t \|\nabla u\|^2(s) ds.$$

Using (12.2.11), the last equality is estimated as

$$t \|\nabla u\|^2(t) + 2 \int_0^t s \|\Delta u\|^2(s) ds \leq \frac{1}{2} \|u_0\|^2. \quad (12.2.16)$$

In particular, we have

$$\|\nabla u\|(t) \leq \frac{1}{\sqrt{2t}} \|u_0\| \quad \text{and} \quad \left( \int_0^t s \|\Delta u\|^2(s) ds \right)^{1/2} \leq \frac{1}{2} \|u_0\|,$$

which are our third and fourth stability estimates (12.2.3) and (12.2.4). The stability estimate (12.2.5) is proved analogously. Now using (12.1.1): ( $\dot{u} = \Delta u$ ) and (12.2.5) we may write

$$\int_\varepsilon^t \|\dot{u}\|(s) ds \leq \frac{1}{\sqrt{2}} \|u_0\| \int_\varepsilon^t \frac{1}{s} ds = \frac{1}{\sqrt{2}} \ln \frac{t}{\varepsilon} \|u_0\| \quad (12.2.17)$$

or more carefully

$$\begin{aligned} \int_\varepsilon^t \|\dot{u}\|(s) ds &= \int_\varepsilon^t \|\Delta u\|(s) ds = \int_\varepsilon^t 1 \|\Delta u\|(s) ds = \int_\varepsilon^t \frac{1}{\sqrt{s}} \sqrt{s} \|\Delta u\|(s) ds \\ &\leq \left( \int_\varepsilon^t s^{-1} ds \right)^{1/2} \left( \int_\varepsilon^t s \|\Delta u\|^2(s) ds \right)^{1/2} \\ &\leq \frac{1}{2} \sqrt{\ln \frac{t}{\varepsilon}} \|u_0\|, \end{aligned}$$

where in the last two inequalities we use Cauchy Schwarz inequality and (12.2.4), respectively. This yields (12.2.6) and the proof is complete.  $\square$

**Problem 12.1.** Show that  $\|\nabla u(t)\| \leq \|\nabla u_0\|$  (the stability estimate for the gradient). Hint: Assume  $f = 0$ , multiply (12.1.1) by  $-\Delta u$  and integrate over  $\Omega$ .

Is this inequality valid for  $u_0 = \text{constant}$ ?

**Problem 12.2.** Derive the corresponding estimate for Neumann boundary condition:

$$\frac{\partial u}{\partial n} = 0. \quad (12.2.18)$$

## 12.3 The finite element for heat equation

### 12.3.1 The semidiscrete problem

Here we consider only the space discretization, hence the problem is continuous in time, and is referred to as a *spatially semidiscrete* problem. We focus on a 2-dimensional heat conduction formulated for a bounded convex domain  $\Omega \subset \mathbb{R}^2$  with smooth boundary  $\Gamma$ . Hence we consider the *non-homogeneous* initial-boundary value problem (12.2.1).

$$\begin{cases} \dot{u}(x, t) - \Delta u(x, t) = f(x, t), & \text{for } x \in \Omega \subset \mathbb{R}^d, d = 1, 2, 3 & (DE) \\ u(x, t) = 0, & x \in \Gamma := \partial\Omega, \quad 0 < t \leq T, & (BC) \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega. & (IC) \end{cases} \quad (12.3.1)$$

Using Green's formula we have the following weak formulation:

$$(u_t, \varphi) + (\nabla u, \nabla \varphi) = (f, \varphi), \quad \forall \varphi \in H_0^1(\Omega), \quad t > 0. \quad (12.3.2)$$

Let now  $\mathcal{T}_h$  be a triangulation of  $\Omega$  of the types introduced in Chapter 10 (see Tables 1-3), with the interior nodes  $\{x_j\}_{j=1}^{M_h}$ . Further, let  $V_h$  be the continuous piecewise linear functions on  $\mathcal{T}_h$  vanishing on  $\partial\Omega$ . Let  $\{\psi_j\}_{j=1}^{M_h}$  be the standard basis of  $V_h$  corresponding to the nodes  $\{x_j\}_{j=1}^{M_h}$ . Further we recall the interpolant  $\mathcal{I}_h : C_0(\bar{\Omega}) \rightarrow V_h$  and the  $L_2(\Omega)$  version of the error bound stated in Theorem 10.2. Then, we formulate an approximate problem as: for each  $t$  find  $u_h(t) = u_h(\cdot, t) \in V_h$  such that

$$\begin{aligned} (u_{h,t}, \chi) + (\nabla u_h, \nabla \chi) &= (f, \chi), \quad \forall \chi \in V_h, \quad t > 0, \\ u_h(0) &= u_{0,h}, \end{aligned} \quad (12.3.3)$$

where  $u_{0,h}$  is an approximation of  $u_0$ .

Thus the problem (12.3.3) is equivalent to find the coefficients  $\xi_j(t)$  in the representation

$$u_h(x, t) = \sum_{j=1}^M \xi_j(t) \psi_j(x),$$

such that

$$\sum_{j=1}^M \xi_j'(t) (\psi_j, \psi_i) + \sum_{j=1}^M \xi_j(t) (\nabla \psi_j, \nabla \psi_i) = (f(t), \psi_i), \quad i = 1, \dots, M_h,$$

and, with  $\eta_j$  denoting the nodal values of the initial approximation  $u_{0,h}$ :

$$\xi_j(0) = \eta_j, \quad j = 1, \dots, M_h.$$

In the matrix form this can be written as

$$M\xi'(t) + S\xi(t) = b(t), \quad t > 0, \quad \text{and} \quad \xi(0) = \eta, \quad (12.3.4)$$

where  $M = (m_{ij}) = ((\psi_j, \psi_i))$  is the mass matrix,  $S = (s_{ij}) = ((\nabla\psi_j, \nabla\psi_i))$  is the stiffness matrix, and  $b = (b_i) = ((f, \psi_i))$  is the load vector. Recall that both mass and stiffness matrices are symmetric and positive definite, in particular  $M$  is invertible and hence (12.3.4) can be written as a system of ordinary differential equations, viz

$$\xi'(t) + M^{-1}S\xi(t) = M^{-1}b(t), \quad t > 0, \quad \text{and} \quad \xi(0) = \eta, \quad (12.3.5)$$

which has a unique solution for  $t > 0$ .

We can easily derive the counterpart of the stability estimates for the semi-discrete problem (12.3.3) with  $\chi$  replaced by  $u_h$ :

$$\begin{aligned} (u_{h,t}, u_h) + (\nabla u_h, \nabla u_h) &= (f, u_h), \quad \forall \chi \in S_h, \quad t > 0, \\ u_h(0) &= u_{0,h}, \end{aligned} \quad (12.3.6)$$

and end up with the  $L_2(\Omega)$  estimate

$$\|u_h(t)\| \leq \|u_{0,h}\| + \int_0^t \|f\| ds. \quad (12.3.7)$$

To proceed, we introduce the *discrete Laplacian* operator  $\Delta_h : V_h^0 \rightarrow V_h^0$ , as in a discrete analogue of Green's formula:

$$(-\Delta_h \psi, \chi) = (\nabla \psi, \nabla \chi), \quad \forall \psi, \chi \in V_h^0. \quad (12.3.8)$$

One can show that the operator  $\Delta_h$  is self-adjoint and  $-\Delta_h$  is positive definite with respect to the  $L_2$ -inner product. Let now  $P_h$  denote the  $L_2$ -projection onto  $V_h^0$ , then the equation (12.3.3) can be written as

$$(u_{h,t} - \Delta_h u_h - P_h f, \chi) = 0, \quad \forall \chi \in V_h^0.$$

Now since  $u_{h,t} - \Delta_h u_h - P_h f \in V_h^0$  taking  $\chi = u_{h,t} - \Delta_h u_h - P_h f$ , we thus obtain

$$u_{h,t} - \Delta_h u_h = P_h f, \quad t > 0, \quad u_h(0) = u_{0,h}. \quad (12.3.9)$$

Now considering  $E_h(t)$  as the solution operator of homogeneous (12.3.9), with  $f = 0$ , we have that  $u_h(t) = E_h(t)u_{0,h}$ , and the Duhamel's principle reads as

$$u_h(t) = E_h(t)u_{0,h} + \int_0^t E_h(t-s)P_h f(s) ds. \quad (12.3.10)$$

We can now prove the following error estimate:

**Theorem 12.2.** *Let  $u_h$  and  $u$  be the solutions of (12.3.3) and (12.3.1) respectively. Then we have the following  $L_2(\Omega)$  estimate*

$$\|u_h(t) - u(t)\| \leq \|u_{0,h} - u_0\| + Ch^2 \left( \|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right).$$

*Proof.* We recall the common estimate for interpolation  $\mathcal{I}_h$ ,  $L_2$ -projection  $P_h$  and elliptic (Ritz) projection  $R_h$ ,

$$\|v - v_h\| \leq Ch^2 \|v\|, \quad \text{where } v_h = \mathcal{I}_h v, v_h = P_h v, \text{ or } v_h = R_h v. \quad (12.3.11)$$

Below we shall use the Ritz projection and rewrite the error as

$$u - u_h = (u - R_h u) + (R_h u - u_h) = \rho + \theta. \quad (12.3.12)$$

Now, recalling (10.4.22), the first term is estimated as

$$\begin{aligned} \|\rho(t)\| &\leq Ch^2 \|u(t)\| = Ch^2 \|u_0 + \int_0^t u_t ds\|_2 \\ &\leq Ch^2 \left( \|u_0\|_2 + \int_0^t \|u_t\|_2 ds \right). \end{aligned} \quad (12.3.13)$$

As for a bound for  $\theta$ , we note that

$$\begin{aligned} (\theta_t, \varphi) + a(\theta, \varphi) &= -(u_{h,t}, \varphi) - a(u_h, \varphi) + (R_h u_{h,t}, \varphi) + a(R_h u_h, \varphi) \\ &= -(f, \varphi) + (R_h u_t, \varphi) + a(u, \varphi) = (-u_t + R_h u_t, \varphi), \end{aligned} \quad (12.3.14)$$

which, using the fact that  $R_h$  commutes with time differentiation, i.e.,  $R_h u_t = (R_h u)_t$ , yields

$$(\theta_t, \varphi) + a(\theta, \varphi) = -(\rho_t, \varphi), \quad \forall \varphi \in V_h. \quad (12.3.15)$$

Now using the variational formulation (12.3.2) and the finite element formulation for the discrete problem (12.3.6) together with the estimate (12.3.7) we get

$$\|\theta(t)\| \leq \|\theta(0)\| + \int_0^t \|\rho_t\| ds \quad (12.3.16)$$

Hence

$$\begin{aligned} \|\theta(0)\| &= \|R_h u_0 - u_{0,h}\| \leq \|R_h u_0 - u_0\| + \|u_0 - u_{h,0}\| \\ &\leq \|u_0 - u_{h,0}\| + Ch^2 \|u_0\|_2. \end{aligned} \quad (12.3.17)$$

Further

$$\|\rho_t\| = \|u_t - R_h u_t\| \leq Ch^2 \|u\|_2, \quad (12.3.18)$$

Combining the above estimates we get the desired result.  $\square$

The gradient estimate counterpart has a similar form, viz:

**Theorem 12.3.** *Let  $u_h$  and  $u$  be the solutions of (12.3.3) and (12.3.1) respectively. Then we have the following  $L_2(\Omega)$  estimate for the gradient of the solution*

$$|u_h(t) - u(t)|_1 \leq |u_{0,h} - u_0|_1 + Ch \left[ \|u_0\|_2 + \|u(t)\|_2 + \left( \int_0^t \|u_t\|_1^2 ds \right)^{1/2} \right].$$

*Proof.* Let  $\rho(t)$  be defined as in (12.3.12). Then using (10.4.22) we have

$$|\rho(t)|_1 = |u - R_h u|_1 \leq Ch \|u(t)\|_2. \quad (12.3.19)$$

To estimate  $\nabla \theta$  we use (12.3.15) with  $\varphi = \theta_t$ . Then

$$\|\theta_t\|^2 + \frac{1}{2} \frac{d}{dt} |\theta|_1^2 = -(\rho_t, \theta_t) \leq \frac{1}{2} (\|\rho_t\|^2 + \|\theta_t\|^2), \quad (12.3.20)$$

and hence

$$\frac{d}{dt} |\theta|_1^2 \leq \|\rho_t\|^2, \quad (12.3.21)$$

which integrating in  $t$ , yields

$$|\theta(t)|_1^2 \leq |\theta(0)|_1^2 + \int_0^t \|\rho_t\|^2 ds \leq (|R_h u_0 - u_0|_1^2 + |u_0 - u_{0,h}|_1^2) + \int_0^t \|\rho_t\|^2 ds.$$

Now recalling the elliptic projection estimate:

$$\|R_h w - w\| + h |R_h w - w|_1 \leq Ch^s \|w\|_a, \quad s = 1, 2, \quad (12.3.22)$$

we get

$$|\theta(t)|_1 \leq |u_0 - u_{0,h}|_1 + Ch \left[ \|u_0\|_2 + \left( \int_0^t \|u_t\|^2 ds \right)^{1/2} \right], \quad (12.3.23)$$

which completes the proof.  $\square$

### 12.3.2 A fully discrete algorithm

For the time discretization of (12.1.1) we shall consider a partition of the time interval  $I = [0, T] : 0 = t_0 < t_1 < \dots < t_N = T$  into sub-intervals  $I_n = (t_{n-1}, t_n)$  of length  $k_n = t_n - t_{n-1}$ . Then, we divide the *space-time slab*  $S_n = \Omega \times I_n$  into prisms  $K \times I_n$ , where  $K \in \mathcal{T}_n$  and  $\mathcal{T}_n$  is a triangulation of  $\Omega$  with the mesh function  $h_n$ . We construct a finite element mesh based on approximations by continuous piecewise linear functions in space and discontinuous polynomials of degree  $r$  in time. We shall refer to this method as  $cG(1)dG(r)$  method. We define the trial space as,

$$W_k^{(r)} = \{v(x, t) : v|_{S_n} \in W_{kn}^{(r)}\}, \quad n = 1, 2, \dots, N,$$

where

$$W_{kn}^{(r)} = \{v(x, t) : v(x, t) = \sum_{j=0}^r t^j \psi_j(x), \psi_j \in V_n, (x, t) \in S_n\},$$

with  $V_n = V_{h_n}$  is the space of continuous piecewise linear functions vanishing on  $\Gamma$  associated to  $\mathcal{T}_n$ . Generally, the functions in  $W_k^{(r)}$  are discontinuous across the discrete time levels  $t_n$  with jump discontinuities  $[w_n] = w_n^+ - w_n^-$  and  $w_m^\pm = \lim_{s \rightarrow 0^\pm} w(t_n + s)$ .

The  $cG(1)dG(r)$  method is based on a variational formulation for (12.2.1), viz: find  $U \in W_k^{(r)}$  such that for  $n = 1, 2, \dots, N$ ,

$$\int_{I_n} \left( (\dot{U}, v) + (\nabla U, \nabla v) \right) dt + \left( [U_{n-1}], v_{n-1}^+ \right) = \int_{I_n} (f, v) dt, \quad \forall v \in W_{kn}^{(r)}, \quad (12.3.24)$$

where  $U_0^- = u_0$  and  $(\cdot, \cdot)$  is the  $L_2(\Omega)$  inner product.

Using the discrete Laplacian  $\Delta_{h_n}$ , in the case of  $r = 0$ , we may write (12.3.24) as follows: find  $U_n \in V_n$  such that

$$(I - k_n \Delta_n) U_n = P_n U_{n-1} + \int_{I_n} P_n f dt, \quad (12.3.25)$$

where  $U_n = U_n^- = U|_{I_n} \in V_n$ , and  $P_n$  is the  $L_2(\Omega)$  projection onto  $V_n$ . Note that the *initial data*  $U_{n-1} \in V_{n-1}$  from the previous time interval  $I_{n-1}$  is projected into the space  $V_n$ . If  $V_{n-1} \subset V_n$ , then  $P_n U_{n-1} = U_{n-1}$ . For the case  $r = 1$ , writing  $U(t) = \Phi_n + (t - t_{n-1})\Psi_n$  on  $I_n$  with  $\Phi_n, \Psi_n \in V_n$ , the variational formulation (12.3.24) takes the following form

$$\begin{cases} (I - k_n \Delta_n) \Phi_n + (I - \frac{k_n}{2} \Delta_n) \Psi_n = P_n U_{n-1} + \int_{I_n} P_n f dt, \\ (\frac{1}{2} I - \frac{k_n}{3} \Delta_n) \Psi_n - \frac{k_n}{2} \Delta_n \Phi_n = \int_{I_n} \frac{t - t_{n-1}}{k_n} P_n f dt, \end{cases} \quad (12.3.26)$$

and gives a system of equations for  $\Phi_n$  and  $\Psi_n$ .

### 12.3.3 Constructing the discrete equations

To construct the matrix equation that determines  $U_n$  in the case  $r = 0$  according (12.3.25), we let  $\varphi_{n,j}$  denote the nodal basis for  $V_n$  associated with  $J_n$  interior nodes of  $\mathcal{T}_n$ , then we can write  $U_n$  as

$$U_n = \sum_{j=1}^{J_n} \xi_{n,j} \varphi_{n,j}.$$

Here  $\xi_{n,j}$  are the nodal values of  $U_n$ . Let us denote the vector coefficients by  $\xi_n = (\xi_{n,j})$ . We denote the  $J_n \times J_n$  mass matrix by  $M_n$ , stiffness matrix



by  $S_n$  and the  $J_n \times 1$  data vector by  $b_n$ , with the associated elements

$$(M_n)_{i,j} = (\varphi_{n,j}, \varphi_{n,i}), \quad (S_n)_{i,j} = (\nabla \varphi_{n,j}, \nabla \varphi_{n,i}), \quad \text{and } (b_n)_i = \int_{t_{n-1}}^{t_n} (f, \varphi_{n,i}),$$

for  $1 \leq i, j \leq J_n$ . Finally, we denote the  $J_n \times J_{n-1}$  matrix  $M_{n-1,n}$  with entries

$$(M_{n-1,n})_{ij} = (\varphi_{n,j}, \varphi_{n-1,i}), \quad 1 \leq i \leq J_n, \quad 1 \leq j \leq J_{n-1}. \quad (12.3.27)$$

Now, the discrete equation for  $cG(1)dG(0)$  approximation on  $I_n$  reads as follows: find  $\xi_n$  such that

$$(M_n + k_n S_n) \xi_n = M_{n-1,n} \xi_{n-1} + b_n. \quad (12.3.28)$$

One can easily verify that the coefficient matrix of the system (12.3.28) is sparse, symmetric, and positive definite, which can be solved by a direct or an iterative algorithm.

### 12.3.4 An a priori error estimate: Fully discrete problem

To derive a priori (or a posteriori) error estimates for the heat equation in higher dimensions is a rather involved procedure. Below we outline the prerequisites and formulate an a priori error estimate theorem without a proof. Detailed studies of the error analysis, for both heat and wave equations, can be found in [20].

We assume that there are positive constants  $\alpha_i$ ,  $i = 1, 2$ , with  $\alpha_2$  sufficiently small, such that for  $n = 1, \dots, N$ ,

$$\alpha_1 k_n \leq k_{n+1} \leq \alpha_1^{-1} k_n, \quad (12.3.29)$$

$$\alpha_1 h_n(x) \leq h_{n+1}(x) \leq \alpha_1^{-1} h_n, \quad x \in \Omega \quad (12.3.30)$$

$$\left( \max_{x \in \bar{\Omega}} h(x) \right)^2 \leq \alpha_2 k_n, \quad (12.3.31)$$

The relation (12.3.31) enters if  $V_{n-1} \not\subseteq V_n$ . An a priori error estimate is then given as:

**Theorem 12.4.** *If  $\Omega$  is convex and  $\alpha_2$  sufficiently small, then there is a constant  $C_i$  depending only on  $\tau$  (which is the minimum angle in the triangulation of  $\Omega$ ) and  $\alpha_i$ ,  $i = 1, 2$ , such that for  $N \geq 1$ , and under the assumptions (12.3.29)-(12.3.31),*

$$\|u(t_N) - U_N\| \leq C_i L_N \max_{1 \leq n \leq N} (k_n \|\dot{u}\|_{I_n} + \|h_n^2 D^2 u\|_{I_n}), \quad (12.3.32)$$

where

$$L_N = 2 + \max_{1 \leq n \leq N} \max \left\{ \left( \log \left( \frac{t_n}{k_n} \right) \right)^{1/2}, \log \left( \frac{t_n}{k_n} \right) \right\}.$$

## 12.4 Exercises

**Problem 12.3.** Consider the following general form of the heat equation

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t), & \text{for } x \in \Omega, \quad 0 < t \leq T, \\ u(x, t) = 0, & \text{for } x \in \Gamma, \quad 0 < t \leq T, \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \quad (12.4.1)$$

where  $\Omega \in \mathbb{R}^2$  with boundary  $\Gamma$ . Let  $\tilde{u}$  be the solution of (12.4.1) with a modified initial data  $\tilde{u}_0(x) = u_0(x)\varepsilon(x)$ .

- Show that  $w := \tilde{u} - u$  solves (12.4.1) with initial data  $w_0(x) = \varepsilon(x)$ .
- Give estimates for the difference between  $u$  and  $\tilde{u}$ .
- Prove that the solution of (12.4.1) is unique.

**Problem 12.4.** Formulate the equation for  $cG(1)dG(1)$  for the two-dimensional heat equation using the discrete Laplacian.

**Problem 12.5.** In two dimensions the heat equation, in the case of radial symmetry, can be formulated as  $r\ddot{u} - (ru_r)' = rf$ , where  $r = |x|$  and  $w_r' = \frac{\partial w}{\partial r}$ .

- Verify that  $u = \frac{1}{4\pi t} \exp(-\frac{r^2}{4t})$  is a solution for the homogeneous equation ( $f = 0$ ) with the initial data being the Dirac  $\delta$  function  $u(r, 0) = \delta(r)$ .
- Sketching  $u(r, t)$  for  $t = 1$  and  $t = 0.01$ , deduce that  $u(r, t) \rightarrow 0$  as  $t \rightarrow 0$  for  $r > 0$ .
- Show that  $\int_{\mathbb{R}^2} u(x, t) dx = 2\pi \int_0^\infty u(r, t) r dr = 1$  for all  $t$ .
- Determine a stationary solution to the heat equation with data

$$f = \begin{cases} 1/(\pi\varepsilon)^2, & \text{for } r < \varepsilon, \\ 0, & \text{otherwise.} \end{cases}$$

- Determine the fundamental solution corresponding to  $f = \delta$ , letting  $\varepsilon \rightarrow 0$ .

**Problem 12.6.** Consider the Schrödinger equation

$$iu_t - \Delta u = 0, \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

where  $i = \sqrt{-1}$  and  $u = u_1 + iu_2$ .

- a) Show that the total probability  $\int_{\Omega} |u|^2$  is independent of the time.  
 Hint: Multiplying by  $\bar{u} = u_1 - iu_2$ , and consider the imaginary part
- b) Consider the corresponding eigenvalue problem, i.e, find the eigenvalue  $\lambda$  and the corresponding eigenfunction  $u \neq 0$  such that

$$-\Delta u = \lambda u \quad \text{in } \Omega, \quad u = 0, \quad \text{on } \partial\Omega.$$

Show that  $\lambda > 0$  and give the relationship between the norms  $\|u\|$  and  $\|\nabla u\|$  for the corresponding eigenfunction  $u$ .

- c) Determine (in terms of the smallest eigenvalue  $\lambda_1$ ), the smallest possible value for the constant  $C$  in the Poincare estimate

$$\|u\| \leq C \|\nabla u\|,$$

derived for all solutions  $u$  vanishing at the boundary ( $u = 0$ , on  $\partial\Omega$ ).

**Problem 12.7.** Consider the initial-boundary value problem

$$\begin{cases} u_t(x, t) - \Delta u(x, t) = f(x, t), & \text{for } x \in \Omega, \quad t > 0, \\ u(x, t) = 0, & \text{for } x \in \Gamma, \quad t > 0, \\ u(x, 0) = u_0(x), & \text{for } x \in \Omega, \end{cases} \quad (12.4.2)$$

- a) Prove (with  $\|u\| = (\int_{\Omega} u^2 dx)^{1/2}$ ) that

$$\begin{aligned} \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds &\leq \|u_0\|^2 + \int_0^t \|f(s)\|^2 ds \\ \|\nabla u(t)\|^2 + \int_0^t \|\Delta u(s)\|^2 ds &\leq \|\nabla u_0\|^2 + \int_0^t \|f(s)\|^2 ds \end{aligned}$$

- b) Formulate  $dG(0) - cG(1)$  method for this problem.

**Problem 12.8.** Formulate and prove  $dG(0) - cG(1)$  a priori and a posteriori error estimates for the two dimensional heat equation (cf. the previous problem) that uses lumped mass and midpoint quadrature rule.

**Problem 12.9.** Consider the convection problem

$$\beta \cdot \nabla u + \alpha u = f, \quad x \in \Omega, \quad u = g, \quad x \in \Gamma_-, \quad (12.4.3)$$

Define the outflow  $\Gamma_+$  and inflow  $\Gamma_-$  boundaries. Assume that  $\alpha - \frac{1}{2}\nabla \cdot \beta \geq c > 0$ . Show the following stability estimate

$$c\|u\|^2 \int_{\Gamma_+} n \cdot \beta u^2 ds dt \leq \|u_0\|^2 + \frac{1}{c}\|f\|^2 + \int_{\Gamma_-} |n \cdot \beta| g^2 ds. \quad (12.4.4)$$

*Hint: Show first that*

$$2(\beta \cdot \nabla u, u) = \int_{\Gamma_+} n \cdot \beta u^2 ds - \int_{\Gamma_-} \|n \cdot \beta\| u^2 ds - ((\nabla \cdot \beta)u, u).$$

*Formulate the streamline diffusion for this problem.*

**Problem 12.10.** *Consider the convection problem*

$$\begin{aligned} \dot{u} + \beta \cdot \nabla u + \alpha u &= f, & x \in \Omega, \quad t > 0, \\ u &= g, & x \in \Gamma_-, \quad t > 0, \\ u(x, 0) &= u_0(x), & x \in \Omega, \end{aligned} \tag{12.4.5}$$

where  $\Gamma_+$  and  $\Gamma_-$  are defined as above. Assume that  $\alpha - \frac{1}{2}\nabla \cdot \beta \geq c > 0$ . Show the following stability estimate

$$\begin{aligned} \|u(\cdot, T)\|^2 + c \int_0^T \|u(\cdot, t)\|^2 dt + \int_0^T \int_{\Gamma_+} n \cdot \beta u^2 ds dt \\ \leq \|u_0\|^2 + \frac{1}{c} \int_0^T \|f(\cdot, t)\|^2 dt + \int_0^T \int_{\Gamma_-} |n \cdot \beta| g^2 ds dt, \end{aligned} \tag{12.4.6}$$

where  $\|u(\cdot, T)\|^2 = \int_{\Omega} u(x, T)^2 dx$ .

## 12.5 The wave equation in $\mathbb{R}^N$

The study of the wave equation in  $\mathbb{R}^N$ ,  $N \geq 2$  is an extension of the results in the one-dimensional case introduced in Chapter 8. Some additional properties of the wave equation are introduced in Chapter 2. For a full finite element study of wave propagation in higher dimensions we refer to [31] and [20]. [?]. In this section we shall prove the *law of conservation of energy*, derive spatial and temporal discretizations and give a stability estimate for the wave equation in  $\mathbb{R}^N$ ,  $N \geq 2$ .

**Theorem 12.5** (Conservation of energy). *For the wave equation given by*

$$\left\{ \begin{array}{ll} \ddot{u} - \Delta u = 0, & \text{in } \Omega \subset \mathbb{R}^N \quad (DE) \\ u = 0, & \text{on } \partial\Omega = \Gamma \quad (BC) \\ (u = u_0) \text{ and } (\dot{u} = v_0) & \text{in } \Omega, \text{ for } t = 0, \quad (IC) \end{array} \right. \quad (12.5.1)$$

where  $\ddot{u} = \partial^2 u / \partial t^2$ , with  $t \in [0, T]$ , or  $t \in (0, \infty)$ , we have that

$$\frac{1}{2} \|\dot{u}(\cdot, t)\|^2 + \frac{1}{2} \|\nabla u(\cdot, t)\|^2 = \text{constant, independent of } t, \quad (12.5.2)$$

*i.e., the total energy is conserved, where  $\frac{1}{2} \|\dot{u}\|^2$  is the kinetic energy, and  $\frac{1}{2} \|\nabla u\|^2$  is the potential (elastic) energy.*

*Proof.* We multiply the equation by  $\dot{u}$  and integrate over  $\Omega$  to obtain

$$\int_{\Omega} \ddot{u} \dot{u} \, dx - \int_{\Omega} \Delta u \cdot \dot{u} \, dx = 0. \quad (12.5.3)$$

Using Green's formula:

$$- \int_{\Omega} (\Delta u) \dot{u} \, dx = - \int_{\Gamma} (\nabla u \cdot n) \dot{u} \, ds + \int_{\Omega} \nabla u \cdot \nabla \dot{u} \, dx, \quad (12.5.4)$$

and the boundary condition  $u = 0$  on  $\Gamma$ , (which implies  $\dot{u} = 0$  on  $\Gamma$ ), we get

$$\int_{\Omega} \ddot{u} \cdot \dot{u} \, dx + \int_{\Omega} \nabla u \cdot \nabla \dot{u} \, dx = 0. \quad (12.5.5)$$

Consequently we have that

$$\int_{\Omega} \frac{1}{2} \frac{d}{dt} (\dot{u}^2) \, dx + \int_{\Omega} \frac{1}{2} \frac{d}{dt} (|\nabla u|^2) \, dx = 0 \iff \frac{1}{2} \frac{d}{dt} (\|\dot{u}\|^2 + \|\nabla u\|^2) = 0,$$

and hence

$$\frac{1}{2} \|\dot{u}\|^2 + \frac{1}{2} \|\nabla u\|^2 = \frac{1}{2} \|v_0\|^2 + \frac{1}{2} \|\nabla u_0\|^2,$$

which is the desired result and the proof is complete.  $\square$

### 12.5.1 The weak formulation

We multiply the equation  $\ddot{u} - \Delta u = f$  by a test function  $v \in H_0^1(\Omega)$  and integrate over  $\Omega$  to obtain

$$\begin{aligned} \int_{\Omega} f v \, dx &= \int_{\Omega} \ddot{u} v \, dx - \int_{\Omega} \Delta u v \, dx \\ &= \int_{\Omega} \ddot{u} v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx - \int_{\partial\Omega} (n \cdot \nabla u) v \, ds \\ &= \int_{\Omega} \ddot{u} v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx. \end{aligned} \quad (12.5.6)$$

where we used Green's formula and boundary data on  $v$ . Thus we have the following *variational formulation* for the wave equation: Find  $u \in H_0^1(\Omega)$  such that for every fixed  $t \in [0, T]$ ,

$$\int_{\Omega} \ddot{u} v \, dx + \int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in H_0^1(\Omega). \quad (12.5.7)$$

### 12.5.2 The semi-discrete problem

Let  $V_h^0$  be a subspace of  $H_0^1(\Omega)$  consisting of continuous piecewise linear functions on a partition  $\mathcal{T}_h$  of  $\Omega$ . The semi-discrete (spatial discretization) counterpart of (12.5.7) reads as follows: find  $U \in V_h^0$ , such that for every fixed  $t \in (0, T]$ ,

$$\int_{\Omega} \ddot{U} v \, dx + \int_{\Omega} \nabla U \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in V_h^0. \quad (12.5.8)$$

Now let  $\{\varphi_j\}_{j=1}^N$  be a set of basis functions for  $V_h^0$ , thus in (12.5.8) we may choose  $v = \varphi_i$ ,  $i = 1, \dots, N$ . Next, we make the ansatz

$$U(x, t) = \sum_{j=1}^N \xi_j(t) \varphi_j(x), \quad (12.5.9)$$

where  $\xi_j$ ,  $j = 1, \dots, N$  are  $N$  time dependent unknown coefficients. Hence, with  $i = 1, \dots, N$  and  $t \in I = (0, T]$ , (12.5.8) can be written as

$$\sum_{j=1}^N \ddot{\xi}_j(t) \int_{\Omega} \varphi_j \varphi_i \, dx + \sum_{j=1}^N \xi_j(t) \int_{\Omega} \nabla \varphi_j \cdot \nabla \varphi_i \, dx = \int_{\Omega} f \varphi_i \, dx. \quad (12.5.10)$$

In the matrix form (12.5.10) is written as the  $N \times N$  linear system of ODEs,

$$M \ddot{\xi}(t) + S \xi(t) = b(t), \quad t \in I = (0, T], \quad (12.5.11)$$

where  $M$  and  $S$  are the usual mass and stiffness matrices, respectively, and  $b$  is the load vector.

### 12.5.3 A priori error estimates for the semi-discrete problem

**Theorem 12.6.** *The spatial discrete solution  $u_h$  given by (12.5.9) satisfies the a priori error estimate*

$$\|u(t) - u_h(t)\| \leq Ch^2 \left( \|\Delta u(\cdot, t)\| + \int_0^t \|\ddot{u}(\cdot, s)\| ds \right). \quad (12.5.12)$$

*Proof.* The proof follows the general path of a priori error estimates combined with conservation of energy Theorem 12.2.  $\square$

Below we state and prove a general semi-discrete error estimate that contains the proof of the above theorem as well.

**Theorem 12.7.** *Let  $u$  and  $u_h$  be the solution of (12.5.1) and (12.5.8), respectively. Then there is nondecreasing time dependent constant  $C(t)$  such that for  $t > 0$ ,*

$$\begin{aligned} & \|u(t) - u_h(t)\| + |u(t) - u_h(t)|_1 + \|u_t(t) - u_{h,t}(t)\| \\ & \leq C \left( |u_{0,h} - R_h u_0|_1 + \|v_h - R_h v\| \right) \\ & \quad + C(t) h^2 \left[ \|u(t)\|_2 + \|u_t(t)\|_2 + \left( \int_0^t \|u_{tt}\|_2^2 ds \right)^{1/2} \right]. \end{aligned} \quad (12.5.13)$$

*Proof.* We start with the, Ritz projection, split

$$u - u_h = (u - R_h u) + (R_h u - u_h) = \rho + \theta.$$

We may bound  $\rho$  and  $\rho_t$  as in the case of heat equation:

$$\|\rho(t)\| + h|\rho(t)|_1 \leq Ch^2 \|u(t)\|_2, \quad \|\rho_t(t)\| \leq Ch^2 \|u_t(t)\|_2.$$

As for  $\theta(t)$ , we note that

$$(\theta_{tt}, \psi) + a(\theta, \psi) = -(\rho_{tt}, \psi), \quad \forall \psi \in V_h^0, \quad t > 0. \quad (12.5.14)$$

To separate the effects from the initial values and the data, we let  $\theta = \eta + \zeta$  where

$$\begin{cases} (\eta_{tt}, \psi) + a(\eta, \psi) = 0, & \forall \psi \in V_h^0, \quad t > 0 \\ \eta(0) = \theta(0), & \eta_t(0) = \theta_t(0). \end{cases}$$

Then  $\eta$  and  $\eta_t$  are bounded (as in conservation of energy).

As for the contribution from the  $\zeta$ , we note that by the initial data for  $\eta$ ,  $\zeta$  satisfies (12.5.14) with the data  $\zeta(0) = \zeta_t(0) = 0$ , where with  $\psi = \zeta_t$ , we end up with

$$\frac{1}{2} \frac{d}{dt} (\|\zeta_t\|^2 + |\zeta|_1^2) = -(\rho_{tt}, \zeta_t) \leq \frac{1}{2} \|\rho_{tt}\|^2 + \leq \frac{1}{2} \|\zeta_t\|^2.$$

Then, integrating with respect to  $t$ , we have that

$$\|\zeta_t(t)\|^2 + |\zeta(t)|_1^2 \leq \int_0^t \|\rho_{tt}\|^2 ds + \int_0^t \|\zeta_t\|^2 ds.$$

Finally, using Gronwall's lemma (with  $C(t) = e^t$ ), we get

$$\|\zeta_t(t)\|^2 + |\zeta(t)|_1^2 \leq C(t) \int_0^t \|\rho_{tt}\|^2 ds \leq C(t) h^4 \int_0^t \|u_{tt}\|_2^2 ds,$$

where we used the bound for  $\rho_{tt}$ . This bound the first two terms on the left hand side in theorem. The third term is estimated in a similar way.  $\square$

#### 12.5.4 The fully-discrete problem

Here we shall discretize the ODE system (12.5.11) in the time variable. To this end we let  $\eta = \dot{\xi}$  and rewrite (12.5.11) as two first order systems:

$$\begin{aligned} M\dot{\xi}(t) &= M\eta(t) \\ M\dot{\eta}(t) + S\xi(t) &= b(t). \end{aligned} \tag{12.5.15}$$

We apply the Crank-Nicolson method to each system, and as in Chapter 7, we end up with

$$M \frac{\xi_n - \xi_{n-1}}{k_n} = M \frac{\eta_n + \eta_{n-1}}{2}, \tag{12.5.16}$$

$$M \frac{\eta_n - \eta_{n-1}}{k_n} + S \frac{\xi_n + \xi_{n-1}}{2} = \frac{b_n + b_{n-1}}{2}. \tag{12.5.17}$$

In block matrix form this can be written as

$$\begin{bmatrix} M & -\frac{k_n}{2}M \\ \frac{k_n}{2}S & M \end{bmatrix} \begin{bmatrix} \xi_n \\ \eta_n \end{bmatrix} = \begin{bmatrix} M & \frac{k_n}{2}M \\ -\frac{k_n}{2}S & M \end{bmatrix} \begin{bmatrix} \xi_{n-1} \\ \eta_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_n}{2}(b_n + b_{n-1}) \end{bmatrix}.$$

Here we may, e.g. choose the starting iterates  $\xi_0$  and  $\eta_0$  as the nodal interpolations of  $u_0$  and  $v_0$ , respectively.

Note that, as we mentioned earlier in Chapter 7, the Crank-Nicolson method is more accurate than the Euler methods and it has the *energy norm preserving property*: it conserves energy. Therefore it is a more suitable method to the numerical study of the wave equation.

## 12.6 Exercises

**Problem 12.11.** *Derive a total conservation of energy relation using the Robin type boundary condition:  $\frac{\partial u}{\partial n} + u = 0$ .*



**Problem 12.12.** Determine a solution for the following equation

$$\ddot{u} - \Delta u = e^{it}\delta(x),$$

where  $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ ,  $i = \sqrt{-1}$ ,  $x = (x_1, x_2, x_3)$  and  $\delta$  is the Dirac-delta function.

*Hint:* Let  $u = e^{it}v(x)$ ,  $v(x) = w(r)/r$  where  $r = |x|$ . Further  $rv = w \rightarrow \frac{1}{4\pi}$  as  $r \rightarrow 0$ . Use spherical coordinates.

**Problem 12.13.** Consider the initial boundary value problem

$$\begin{cases} \ddot{u} - \Delta u + u = 0, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (12.6.1)$$

Rewrite the problem as a system of two equations with a time derivative of order at most 1. Why this modification?

**Problem 12.14.** Consider the initial boundary value problem

$$\begin{cases} \ddot{u} - \Delta u = 0, & x \in \Omega, \quad t > 0, \\ u = 0, & x \in \partial\Omega, \quad t > 0, \\ u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = u_1(x), & x \in \Omega. \end{cases} \quad (12.6.2)$$

Formulate the  $cG(1)cG(1)$  method for this problem. Show that the energy is conserved.

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