Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2015–01–17; kl 14:00-18:00.

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Course Books: Bergh et al, Bracewell, Lecture Notes and Calculator are allowed (no solutions).
Each problem gives max 6p. Breakings: **3**: 12-17p, **4**: 18-23p och **5**: 24p-
For GU students **G** :12-20p, **VG**: 21p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1415/index.html

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1. Prove that, in an orthonormal MRA,

$$|\hat{\psi}(\omega)|^2 + |\hat{\varphi}(\omega)|^2 = |\hat{\varphi}(\omega/2)|^2$$

2. Show that the Fourier transform of the function of two variables $f(\mathbf{x}) = \sqrt{\pi}(x_1+x_2)\exp(-\pi|\mathbf{x}|^2)$ is the same function multiplied by some number, and determine this number. Here $\mathbf{x} = (x_1, x_2)$.

3. Set $H(\omega) = \sum_{k=0}^{N-1} h_k e^{-i\omega k}$ and put $G(\omega) = -e^{-i\omega}\overline{H(\omega+\pi)}$. Show that $g_k = (-1)^k h_{1-k}$, for real-valued h_k.

4. Assume that $\varphi \in \mathcal{S}$, $|\hat{\varphi}(\xi)| < \varepsilon$ for $1/2 < |\xi| < 2$ and $|\hat{\varphi}(\xi)| = 0$ for $|\xi| \geq 2$. Estimate the error

$$\Big|\hat{\varphi}(\xi) - \sum_n \varphi(n)e^{-2\pi i n\xi}\Big|, \qquad \text{for} \quad |\xi| < 1/2.$$

5. Show that if ψ is a wavelet and ϕ is a bounded integrable function, then the convolusion $\psi * \phi$ is a wavelet.

Hint: To Solve this problem you may use the relation (ii) from the following two equivalent definitions for the wavelets.
(i) Let $0 \neq \psi \in L_2(\mathbb{R})$ have a compact support. Then

(1) $$\int_{\mathbb{R}} \psi(t)\,dt = 0 \quad \Longleftrightarrow \quad \psi \quad \text{is a wavelet}$$

(ii) A wavelet is a function $0 \neq \psi \in L_2(\mathbb{R})$ which satisfies the condition

(2) $$C_\psi = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\,d\omega < \infty.$$

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Lösningar/Solutions.

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1. Prove that, in an orthonormal MRA,

$$|\hat{\psi}(\omega)|^2 + |\hat{\varphi}(\omega)|^2 = |\hat{\varphi}(\omega/2)|^2.$$

Solution: We know that

$$\hat{\varphi}(\omega) = H(\omega/2)\,\hat{\varphi}(\omega/2), \qquad \hat{\psi}(\omega) = G(\omega/2)\,\hat{\varphi}(\omega/2),$$

and

$$|H(\omega)|^2 + |G(\omega)|^2 = 1.$$

Thus

$$|\hat{\psi}(\omega)|^2 + |\hat{\varphi}(\omega)|^2 = \Big[|H(\omega/2)|^2 + |G(\omega/2)|^2\Big]|\hat{\varphi}(\omega/2)|^2 = |\hat{\varphi}(\omega/2)|^2.$$

2. Show that the Fourier transform of the function of two variables $f(\mathbf{x}) = \sqrt{\pi}(x_1+x_2)\exp(-\pi|\mathbf{x}|^2)$ is the same function multiplied by some number, and determine this number. Here $\mathbf{x} = (x_1, x_2)$.

Solution: The function

$$f(\mathbf{x}) = \sqrt{\pi}(x_1 + x_2)e^{-\pi|\mathbf{x}|^2} = \sqrt{\pi}(x_1 + x_2)\,e^{-\pi(x_1+x_2)^2},$$

has its multidimensional Fourier transform as

$$\hat{f}(\xi) := \hat{f}(\xi_1, \xi_2) = \int\int_{\mathbb{R}^2} f(x_1, x_2)\,e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)}\,dx_1 dx_2.$$

For the Gausian $g(\mathbf{x}) = e^{-\pi|\mathbf{x}|^2}$ we have for $\xi = (\xi_1, \xi_2)$ that

$$\begin{aligned}
\hat{g}(\xi) &= \int\int_{\mathbb{R}^2} e^{-\pi x_1^2} \cdot e^{-\pi x_2^2}\,e^{-2\pi i \xi_1 x_1} \cdot e^{-2\pi i \xi_2 x_2} \\
&= \int_{-\infty}^{\infty} e^{-\pi x_1^2}\,e^{-2\pi i \xi_1 x_1}\,dx_1 \cdot \int_{-\infty}^{\infty} e^{-\pi x_2^2}\,e^{-2\pi i \xi_2 x_2}\,dx_2 \\
&= \mathcal{F}[e^{-\pi x_1^2}](\xi_1) \cdot \mathcal{F}[e^{-\pi x_2^2}](\xi_2) = e^{-\pi \xi_1^2} \cdot e^{-\pi \xi_2^2} = e^{-\pi(\xi_1^2+\xi_2^2)} = e^{-\pi|\xi|^2}.
\end{aligned}$$

Further, since

$$\frac{\partial \hat{g}}{\partial \xi_1} = -2\pi i \mathcal{F}_2[x_1 e^{-\pi|\mathbf{x}|^2}], \qquad \frac{\partial \hat{g}}{\partial \xi_2} = -2\pi i \mathcal{F}_2[x_2 e^{-\pi|\mathbf{x}|^2}],$$

we have that

$$\begin{aligned}
\hat{f}(\xi) &= \sqrt{\pi}\Big(\frac{i}{2\pi}\frac{\partial \hat{g}}{\partial \xi_1} + \frac{i}{2\pi}\frac{\partial \hat{g}}{\partial \xi_2}\Big) = \frac{i}{2\pi}\sqrt{\pi}\Big(-2\pi\xi_1 e^{-\pi|\xi|^2} - 2\pi\xi_2 e^{-\pi|\xi|^2}\Big) \\
&= -i\sqrt{\pi}(\xi_1 + \xi_2)e^{-\pi|\xi|^2} = -if(\xi).
\end{aligned}$$

Hence $\hat{f}(\xi) = -if(\xi)$ and the constant to multiply is $-i$.

3. Set $H(\omega) = \sum_{k=0}^{N-1} h_k e^{-i\omega k}$ and put $G(\omega) = -e^{-i\omega}\overline{H(\omega+\pi)}$. Show that $g_k = (-1)^k h_{1-k}$, for real-valued h_k.

Solution: We have that

$$\begin{aligned} G(\omega) &= -e^{-i\omega}\overline{H(\omega+\pi)} = -e^{-i\omega}\overline{\sum_{k=0}^{N-1} h_k e^{-i(\omega+\pi)k}} = -e^{-i\omega}\sum_{k=0}^{N-1} h_k e^{i\omega k}\, e^{ik\pi} \\ &= \sum_{k=0}^{N-1} h_k\,(-1)^k e^{-i(1-k)\omega} = [1-k=j] \\ &= \sum_{j=2-N}^{1} h_{1-j}(-1)^j e^{-ij\omega} = \sum_k g_k e^{-i\omega k}, \qquad g_k = (-1)^k h_{1-k}. \end{aligned}$$

4. Assume that $\varphi \in \mathcal{S}$, $|\hat{\varphi}(\xi)| < \varepsilon$ for $1/2 < |\xi| < 2$ and $|\hat{\varphi}(\xi)| = 0$ for $|\xi| \geq 2$. Estimate the error

$$\Big|\hat{\varphi}(\xi) - \sum_n \varphi(n)e^{-2\pi i n\xi}\Big|, \qquad \text{for} \quad |\xi| < 1/2.$$

Solution: By the Poisson summation formula:

$$\sum_n \varphi(n)e^{-2\pi i n\xi} = \sum_n \hat{\varphi}(\xi+n)$$

Thus we can write

$$\Big|\hat{\varphi}(\xi) - \sum_n \varphi(n)e^{-2\pi i n\xi}\Big| = \Big|\hat{\varphi}(\xi) - \sum_n \hat{\varphi}(\xi+n)\Big| = \Big|\sum_{n\neq 0} \hat{\varphi}(\xi+n)\Big|.$$

Now assume that $0 \leq \xi < 1/2$ then since $|\hat{\varphi}(\xi)| = 0$ for $|\xi| \geq 2$, we get that

$$\begin{aligned} \Big|\sum_{n\neq 0} \hat{\varphi}(\xi+n)\Big| &= \Big|\hat{\varphi}(\xi+1) + \hat{\varphi}(\xi-1) + \hat{\varphi}(\xi-2)\Big| \\ &\leq |\hat{\varphi}(\xi+1)| + |\hat{\varphi}(\xi-1)| + |\hat{\varphi}(\xi-2)| < 3\varepsilon. \end{aligned}$$

The case with $-1/2 \leq \xi \leq 0$ is estimated analogeously, giving the same bound.

5. Show that if ψ is a wavelet and ϕ is a bounded integrable function, then the convolution $\psi * \phi$ is a wavelet.

Solution: Since

$$\begin{aligned}
\int_{-\infty}^{\infty} |\psi * \phi(x)|^2\, dx &= \int_{-\infty}^{\infty} \Big| \int_{-\infty}^{\infty} \psi(x-u)\phi(u)\, du \Big|^2\, dx \\
&\le \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} |\psi(x-u)||\phi(u)|\, du \Big]^2\, dx \\
&= \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} |\psi(x-u)||\phi(u)|^{1/2}||\phi(u)|^{1/2}\, du \Big]^2\, dx \\
&\le \int_{-\infty}^{\infty} \Big(\int_{-\infty}^{\infty} |\psi(x-u)|^2|\phi(u)|\, du \cdot \int_{-\infty}^{\infty} |\phi(u)|\, du \Big)\, dx \\
&\le \int_{-\infty}^{\infty} |\phi(u)|\, du \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\psi(x-u)|^2|\phi(u)|\, dx\, du \\
&= \Big(\int_{-\infty}^{\infty} |\phi(u)|\, du \Big)^2 \int_{-\infty}^{\infty} |\psi(x)|^2\, dx < \infty.
\end{aligned}$$

We have $\psi * \phi \in L_2(\mathbb{R})$. Moreover

$$\begin{aligned}
C_{\psi*\phi} &= \int_{-\infty}^{\infty} \frac{|\mathcal{F}(\psi * \phi)|^2}{|\omega|}\, d\omega = \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)\hat{\phi}(\omega)|^2}{|\omega|}\, d\omega \\
&= \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|} |\hat{\phi}(\omega)|^2\, d\omega \le \int_{-\infty}^{\infty} \frac{|\hat{\psi}(\omega)|^2}{|\omega|}\, d\omega \cdot \sup |\hat{\phi}(\omega)|^2 < \infty.
\end{aligned}$$

Thus the convolution $\psi * \phi$ is a wavelet.

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