Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2017–01–09; kl 08:30-12:30.

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A sapling of formulas will be attached to the exam sheet.
Each problem gives max 5p. Breakings: **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For GU students **G** :10-17p, **VG**: 18p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1617/index.html

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1. Determine the two-dimensional Fourier transform of the function

$$e^{-(|x_1-x_2|+|x_1+x_2|)}.$$

Hint: A change of coordinates might help.

2. Let H and G be the low-pass and high-pass filter functions in an orthogonal MRA. Assume that

$$H(\omega)=\Big(1+\omega^{M_1}h_1(\omega)\Big)+i\omega^{M_2}h_2(\omega), \quad \text{and} \quad G(\omega)=\omega^N g(\omega),$$

where $h_i(0)\neq 0$, $g(0)\neq 0$ and M_1, M_2, N are integeres ≥ 1. Prove that $M_1=2N$ and $M_2\geq N$.

Hint: For an orthogonal MRA we have that $|H(\omega)|^2+|G(\omega)|^2=1$.

3. Show that $(\cdot)^2 T=0,\ T\in\mathcal{S}'$, implies that $T=aD\delta+b\delta$.

4. To prove that, in an orthogonal MRA, $|\hat{\varphi}(0)|=1$ if $\hat{\varphi}$ is continuous, it suffices to show that

$$\sum_k |<f,\varphi_{j,k}>|^2 \to |\hat{\varphi}(0)|^2, \quad \text{as} \quad j\to+\infty,$$

for f with $\hat{f}(\omega)=1_{(-\pi,\pi)}(\omega)$. Prove the limit statement.

5. Prove the Poisson's summation formula:

$$\sum_{k\in\mathbb{Z}}\hat{\varphi}(k+\xi)=\sum_{k\in\mathbb{Z}}\varphi(k)e^{-2\pi ik\xi}, \qquad \varphi\in\mathcal{S}.$$

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Lösningar/Solutions.

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1. We have for

$$f(x_1, x_2) = e^{-\Big(|x_1-x_2|+|x_1+x_2|\Big)},$$

that

$$\hat{f}(u,v) = \iint_{\mathbb{R}} f(x_1,x_2)e^{-2\pi i(x_1u+x_2v)}\,dx_1\,dx_2.$$

Let now $x = x_1 - x_2,\ y = x_1 + x_2$, so that $x_1 = (x+y)/2,\ x_2 = (y-x)/2$ and

$$\frac{d(x_1,x_2)}{d(x,y)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}.$$

Then,

$$\begin{aligned}\hat{f}(u,v) =& \iint_{\mathbb{R}} e^{-(|x|+|y|)}e^{-2\pi i\Big(\frac{x+y}{2}u+\frac{y-x}{2}v\Big)}\frac{1}{2}\,dx\,dy \\ =&\frac{1}{2}\int_{\mathbb{R}} e^{-|x|}e^{-2\pi i\Big(\frac{u}{2}-\frac{v}{2}\Big)x}\,dx \cdot \int_{\mathbb{R}} e^{-|y|}e^{-2\pi i\Big(\frac{u}{2}+\frac{v}{2}\Big)y}\,dy \\ =&\frac{1}{2}\mathcal{F}_x\Big[e^{-|x|}\Big]\Big(\frac{u}{2}-\frac{v}{2}\Big)\cdot\mathcal{F}_y\Big[e^{-|y|}\Big]\Big(\frac{u}{2}+\frac{v}{2}\Big).\end{aligned}$$

Using the one dimensional Fourier transform

$$\mathcal{F}\Big[e^{-|x|}\Big](s) = \frac{2}{1+(2\pi s)^2},$$

we finally have that

$$\hat{f}(u,v) = \frac{1}{2}\cdot\frac{2}{1+\pi^2(u-v)^2}\cdot\frac{2}{1+\pi^2(u+v)^2} = \frac{2}{\Big(1+\pi^2(u-v)^2\Big)\Big(1+\pi^2(u+v)^2\Big)}.$$

2. Using the condition of the orthogonal MRA

$$|H(\omega)|^2 + |G(\omega)|^2 = 1,$$

for the data

$$H(\omega) = \Big(1+\omega^{M_1}h_1(\omega)\Big) + i\omega^{M_2}h_2(\omega), \quad \text{and} \quad G(\omega) = \omega^N g(\omega),$$

where $h_i(0) \neq 0$, $g(0) \neq 0$ and M_1, M_2, N are integeres ≥ 1, we have that

$$\Big[1+\omega^{M_1}h_1(\omega)\Big]^2 + \Big[\omega^{M_2}h_2(\omega)\Big]^2 + \omega^{2N}|g(\omega)|^2 = 1,$$

i.e.

$$1 + \omega^{2M_1}h_1^2(\omega) + 2\omega^{M_1}h_1(\omega) + \omega^{2M_2}h_2^2(\omega) + \omega^{2N}|g(\omega)|^2 = 1.$$

This gives that

$$\omega^{2M_1}h_1^2(\omega) + 2\omega^{M_1}h_1(\omega) + \omega^{2M_2}h_2^2(\omega) + \omega^{2N}|g(\omega)|^2 = 0, \quad \forall\omega,$$

which, dividing by ω^{M_1} can be written as

$$\omega^{M_1}h_1^2(\omega) + 2h_1(\omega) + \omega^{2M_2-M_1}h_2^2(\omega) + \omega^{2N-M_1}|g(\omega)|^2 = 0, \quad \forall\omega,$$

and yields the desired result.

3. Note that $\quad x^2T = 0, \quad x(xT) = 0, \quad \Longrightarrow \quad xT = a\delta.$

We have also $\quad (xD\delta)(\varphi) = (D\delta)(x\varphi) = -\delta\Big((x\varphi)'\Big) = -\delta(x\varphi' + \varphi) = -\varphi(0) = -\delta(\varphi),$

or $\quad 0 = D(x\delta) = xD\delta + \delta$, so that $x(T + aD\delta) = 0$. But

$$x(T + aD\delta) = 0, \quad \Longrightarrow \quad T + aD\delta = b\delta, \quad \Longrightarrow \quad T = -aD\delta + b\delta.$$

Alternatively:

$$\mathcal{F}\Big[x^2T\Big] = \Big(\frac{i}{2\pi}\Big)^2 D^2\hat{T} = 0, \quad \Longrightarrow \quad D^2\hat{T} = 0, \quad \Longrightarrow \quad \hat{T} = as + b, \quad \Longrightarrow \quad T = \frac{a}{2\pi i}\delta' + b\delta.$$

4. We assume that φ is a scaling function in an orthogonal MRA, and $\hat{\varphi}$ is continuous. Then, $\{\varphi(t-n)\}_{n\in\mathbb{Z}}$ is an orthonormal basis for V_0, $\varphi_{j,k}(t) = 2^{j/2}\varphi(2^jt - k)$ and $\{\varphi_{j,k}\}_{k\in\mathbb{Z}}$, is a basis for V_j. Further, $f_j = \sum_k < f, \varphi_{j,k} > \varphi_{j,k}$, is an orthogonal projection on V_j. $f_j \to f$ (in L_2) as $j \to \infty$ by the properties of an MRA.

$$\|f_j\|^2 = \sum_k | < f, \varphi_{j,k} > |^2 \to \|f\|^2 = \frac{1}{2\pi}\|\hat{f}\|^2.$$

For

$$(1) \qquad \hat{f}(\omega) = \begin{cases} 1, & |\omega| < \pi \\ 0, & |\omega| > \pi \end{cases} \qquad \text{we have } \|\hat{f}\|^2 = \int_{-\pi}^{\pi} d\omega = 2\pi, \quad \|f_j\|^2 \to 1.$$

Note that $\quad \hat{\varphi}_{j,k}(\omega) = 2^{j/2}2^{-j}e^{-ik\omega 2^{-j}}\hat{\varphi}(\omega 2^{-j})$. Thus

$$\sum_k | < f, \varphi_{j,k} > |^2 = \sum_k \frac{1}{(2\pi)^2} | < \hat{f}, \hat{\varphi}_{j,k} > |^2 = \frac{1}{4\pi^2}\sum_k \Big| \int_{-\pi}^{\pi} 2^{-j/2}e^{-ik\omega 2^{-j}}\hat{\varphi}(\omega 2^{-j})\, d\omega \Big|^2$$

$$= [2^{-j}\omega = \omega'] = \frac{1}{4\pi^2}\sum_k \Big| \int_{-\pi 2^{-j}}^{\pi 2^{-j}} 2^{-j/2}e^{-ik\omega'}\hat{\varphi}(\omega')2^j\, d\omega' \Big|^2 = 2^j \sum_k |C_{j,k}|^2,$$

where

$$C_{j,k} = \frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} \hat{\varphi}(\omega)e^{-ik\omega}\, d\omega.$$

Assume that $j \geq 0$ so that $\pi 2^{-j} \leq \pi$.

$$g(\omega) = \begin{cases} \hat{\varphi}(\omega), & |\omega| < \pi 2^{-j} \\ 0, & \pi 2^{-j} \leq |\omega| \leq \pi \end{cases}$$

and g is 2π periodic, then

$$C_{j,k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\omega)e^{-ik\omega}\, d\omega,$$

are the Fourier coefficients of g. By Parseval's relation

$$\sum_k |C_{j,k}|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |g(\omega)|^2\, d\omega = \frac{1}{2\pi}\int_{-\pi 2^{-j}}^{\pi 2^{-j}} |\hat{\varphi}(\omega)|^2\, d\omega.$$

Thus

$$\sum_k | < f, \varphi_{j,k} > |^2 = \frac{1}{2\pi}2^j \int_{-\pi 2^{-j}}^{\pi 2^{-j}} |\hat{\varphi}(\omega)|^2\, d\omega = |\hat{\varphi}(\xi_j)|^2,$$

for some ξ_j with $|\xi_j| \leq \pi 2^{-j}$. Thus

$$\|f_j\|^2 = \sum_k | < f, \varphi_{j,k} > |^2 \to |\hat{\varphi}(0)|^2, \quad \text{as } j \to \infty.$$

But this limit was also 1 (see(1)); hence $|\hat{\varphi}(0)| = 1$.

5. See lecture notes.

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