Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2017–04–18; kl 08:30-12:30.

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A sapling of formulas will be attached to the exam sheet.
Each problem gives max 5p. Breakings: **3**: 10-14p, **4**: 15-19p och **5**: 20p-
For GU students **G** :10-17p, **VG**: 18p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1617/index.html

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1. Let $v = (\downarrow 3)x$, i.e. $v(k) = x(3k)$. Write down $V(\omega)$ in terms of X.

2. Show that the convolution $\phi_1(t) * \phi_2(t)$ satisfies a dilation equation with coefficients from $H_1 * H_2$.

3. If $w(t)$ has p vanishing moments, show that its Fourier transform has a p-th order zero at $\omega = 0$. Then factoring $(i\omega)^p$ from $\hat{w}$ gives the transform $\hat{I}_p$ of the p-fold integral of $w(t)$.

4. Let $f \in \mathcal{S}'$ be a function such that $|f(T)| = f(0)$ and $|f(x)| < f(0)$ for $0 < x < T$. Show that $\mathcal{F}f = \sum_n c_n \delta_{(n+\alpha)/T}$ for some $\alpha \in [0, 1]$.

5. Prove the following version of "Heisenberg inequality":

$$\Big(\int_{\mathbb{R}} x^2 |f|^2 \, dx \Big/ ||f||^2 \Big) \Big(\int_{\mathbb{R}} \xi^2 |\hat{f}|^2 \, dx \Big/ ||\hat{f}||^2 \Big) \geq \frac{1}{16\pi^2}.$$

Hint: You may use the Cauchy-Schwarts inequality:

$$\Big| \int f g^\star \, dx \Big|^2 \leq \int |f(x)|^2 \, dx \int |g(x)|^2 \, dx.$$

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Lösningar/Solutions.

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1. Following the procedure for $(\downarrow 2)x$ let

(1)
$$u(n) = \begin{cases} x(n/3), & n = 3k,\ k \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$
$$= \frac{1}{3}\Big(1 + e^{-2\pi i n/3} + e^{-4\pi i n/3}\Big)x(n).$$

Hence
$$U(\omega) = \frac{1}{3}\Big(X(\omega) + X(\omega + 2\pi/3) + X(\omega + 4\pi/3)\Big),$$
and
$$V(\omega) = U(\omega/3) = \frac{1}{3}\Big(X(\omega/3) + X((\omega + 2\pi/3)/3) + X((\omega + 4\pi/3)/3)\Big).$$

2. Let the two scaling functions satisfy refinement equations of the form:

(2)
$$\phi_1(x) = 2\sum_k h_1(k)\phi_1(2x-k)$$
$$\phi_2(x) = 2\sum_l h_2(l)\phi_1(2x-l),$$

then
$$(\phi_1 * \phi_2)(x) = 4\int \phi_1(X)\phi_2(x-X)\,dX = 4\sum_k\sum_l h_1(k)h_2(l)\int \phi_1(2X-k)\phi_2(2x-2X-l)\,dX.$$

Let $2X - k = Y$ and $k + l = n$. The above expression can now be simplified as:
$$(\phi_1 * \phi_2)(x) = 2\sum_k\sum_l h_1/n-l)h_2(l)\int \phi_1(Y)\phi_2(2x-Y-n)\,dY.$$

Hense the filter in the refinement relation is $H_1 * H_2$.
This problem can also be solved taking the Fourier transforms of both sides of equation (2).

3. By assumption
$$\int t^k w(t)\,dt = 0 \qquad k = 0, 1, \dots, p-1.$$

The Fourier transform of $w(t)$ can be written as
$$W(\omega) = \int w(t)e^{-i\omega t}\,dt.$$

Let $\theta = -i\omega$ and expand the exponential term as
$$e^{-i\omega t} = e^{\theta t} = \sum_{k=0}^{\infty}\frac{\theta^k t^k}{k!}.$$

Hence
$$W(\omega) = \sum_{k=0}^{\infty}\frac{1}{k!}\theta^k\int w(t)t^k\,dt = \theta^p\sum_{k=0}^{\infty}\frac{1}{(k+p)!}\theta^k\int w(t)t^{k+p}\,dt.$$

Note that interchanging the integration and the infinite sum can be justified by a careful application of Fubini's theorem. We can therefore observe that $W(\omega)$ has p zeros at $\omega = 0$.

4. We have that $f \in \mathcal{S}'$ is a function such that $|f(T)| = f(0)$ and $|f(x)| < f(0)$, for $0 < x < T$. Thus we can put $f(T) = e^{2\pi i\alpha} f(0)$ for some $\alpha,\ 0 \le \alpha < 1$.
Now consider

$$g(x) = e^{-2\pi i \alpha x/T} f(x).$$

Then

$$g(0) = f(0),\ g(T) = e^{-2\pi i\alpha} f(T) = f(0) = g(0).$$

Further,

$$|g(x)| = |f(x)| < f(0) = g(0) \quad \text{for} \quad 0 < x < T.$$

This means that g is T-periodic. In that case

$$\hat{g} = \sum_n c_n \delta_{n/T},$$

and $f(x) = e^{2\pi i\alpha x/T} g(x)$ gives

$$\hat{f} = \hat{g}_{\alpha/T} = \sum_n c_n \delta_{\frac{n}{T}+\frac{\alpha}{T}} = \sum_n c_n \delta_{(n+\alpha)/T}.$$

5. See lecture notes.

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