Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2010–12–16; kl 8.30-13.30.

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Course Books: Bergh et al AND Bracewell, Lecture Notes and Calculator are allowed.
Each problem gives max 5p. Breakings: **3**: 12-15p, **4**: 16-19p och **5**: 20p-
For GU **G** students :12p, **VG**: 18p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1011/index.html

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1. Consider the dilation equation for the scaling function $\varphi(t)$, given in terms of the original lawpass filter coefficients h_k as:

$$\phi(t) = 2\sum_{k=0}^{N} h_k \phi(2t-k).$$

Show that

$$\int_{-\infty}^{\infty} \phi(t)\,dt = 1 \Longrightarrow \sum_{k=0}^{N} h_k = 1.$$

2. If $\varphi(t)$ has p vanishing moments, show that its Fourier transform

$$\hat{\varphi}(\omega) = \int \varphi(t)\, e^{-i\omega t}\,dt$$

has a pth order zero at $\omega = 0$.

3. Describe the operator $\tilde{x} = (\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2)x$,

i.e. give $\tilde{x}_n$ in terms of x_n. Also, in the frequency domain, describe $\tilde{X}(z)$ in terms of $X(z)$ and $X(-z)$.

4. Are the filters

$$(L_1 f)(t) = \int_{t-1}^{t} f(x)\,dx, \qquad \text{and} \quad (L_2 f)(t) = \int_0^1 x^2 f(t-x)\,dx,$$

a) linear? b) time-invariant? c) causal?

5. Let φ and ψ be the Haar scaling function and the Haar wavelet, respectively, and let the spaces V_j and W_j be defined as in the lectures. Let a function f be given by

$$f(x) = \begin{cases} -1 & 0 \le x < 1/4, \\ 4 & 1/4 \le x < 1/2, \\ 2 & 1/2 \le x < 3/4, \\ -3 & 3/4 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Express f in terms of the natural basis of V_2 and then decompose it into its V_0, W_0 and W_1 components.

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Lösningar/Solutions.

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1. Consider the dilation equation for the scaling function $\varphi(t)$, given in terms of the original lawpass filter coefficients h_k as:

$$\phi(t) = 2\sum_{k=0}^{N} h_k \phi(2t-k).$$

Show that

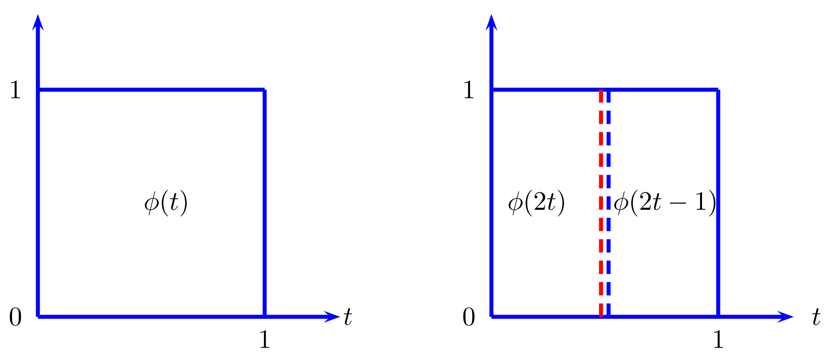
$$\int_{-\infty}^{\infty} \phi(t)\, dt = 1 \Longrightarrow \sum_{k=0}^{N} h_k = 1.$$

Solution: To illustrate, we start with $h_0 = h_1 = 1/2$, $h_k = 0, k = 2, \ldots, N$. Then, the dilation equation is

$$\phi(t) = \phi(2t) + \phi(2t-1).$$

The graph of $\phi(t)$ is compressed by 2, to give the graph of $\phi(2t)$. When that is shifted to the right by $1/2$, it becomes the graph of $\phi(2t-1)$. We ask the two compressed graphs to combine to the original graph. The Figure below shows that this occurs when $\phi(t)$ is a box function:

$$\phi(t) = \begin{cases} 1, & 0 \le t < 1, \\ 0, & \text{otherwise.} \end{cases}$$



The graphs $\phi(2t)$ and $\phi(2t-1)$ are half-size boxes. Their sum is the full-size box $\phi(t)$. The graphs of $\phi(2t)$ and every $\phi(2t-k)$ is compressed to area $1/2$:

$$2\int_{-\infty}^{\infty} \phi(2t-k)\, dt = \{ \text{ set } 2t-k = u\} = \int_{-\infty}^{\infty} \phi(u)\, du = 1.$$

So integrating both sides of the dilation equation $\phi(t) = 2\sum_{k=0}^{N} h_k \phi(2t-k)$ gives

$$h_0 + h_1 + \ldots + h_N = 1.$$

2. If $\varphi(t)$ has p vanishing moments, show that its Fourier transform

$$\hat{\varphi}(\omega) = \int \varphi(t)\, e^{-i\omega t}\, dt$$

has a pth order zero at $\omega = 0$.

Solution: By assumption

$$\int t^k \varphi(t)\, dt = 0, \qquad k = 0, \ldots, p-1.$$

Now, the Fourier transform of $\varphi(t)$ can be written as

$$\hat{\varphi}(\omega) = \int \varphi(t) e^{-i\omega t}\, dt.$$

We let $\alpha = -i\omega$ and expand the exponential term as

$$e^{-i\omega t} = e^{\alpha t} = \sum_{k=0}^{\infty} \frac{\alpha^k t^k}{k!}.$$

Hence

$$\hat{\varphi}(\omega) = \sum_{k=0}^{\infty} \frac{1}{k!} \alpha^k \int \varphi(t) t^k\, dt = \alpha^p \sum_{k=0}^{\infty} \frac{1}{(k+p)!} \alpha^k \int \varphi(t) t^{k+p}\, dt.$$

Note that interchanging the integral and the infinite sum can be justified by a careful application of Fubini's theorem. We can therefore observe that $\hat{\varphi}(\omega)$ has p zeros at $\omega = 0$ $(\alpha = -i\omega)$.

3. Describe the operator $\tilde{x} = (\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2)x$, i.e. give $\tilde{x}_n$ in terms of x_n. Also, in the frequency domain, describe $\tilde{X}(z)$ in terms of $X(z)$ and $X(-z)$.

Solution: The easiest approach in solving this problem is to observe that

$$(\downarrow 2)(\uparrow 2)x_n = x_n.$$

Hence

$$\begin{aligned}\tilde{x}_n &= (\uparrow 2)(\downarrow 2)(\uparrow 2)(\downarrow 2)x_n = (\uparrow 2)(\downarrow 2)x_n \\ &= \frac{1}{2}\Big(1 + (-1)^n\Big)x_n.\end{aligned}$$

In the frequency domain

$$\begin{aligned}\tilde{X}(z) &= \frac{1}{2} \sum_{n=-\infty}^{\infty} \Big(x_n\, z^{-n} + x_n(-z)^{-n}\Big) \\ &= \frac{1}{2}\Big(X(z) + X(-z)\Big).\end{aligned}$$

4. Are the filters

$$(L_1 f)(t) = \int_{t-1}^{t} f(x)\, dx, \qquad \text{and} \quad (L_2 f)(t) = \int_0^1 x^2 f(t-x)\, dx,$$

a) linear? b) time-invariant? c) causal?

Solution: Both filters are indeed linear, time-invariant and causal, because both are convolution filters, i.e.,

$$L_j f = g_j * f, \quad j = 1, 2,$$

where the functions g_j are supported on $\mathbb{R}_+$. The function g_2 is simply

$$g_2(x) = \begin{cases} x^2, & x \in [0,1] \\ 0, & \text{otherwise,} \end{cases}$$

and g_1 is the indicator function of the interval $[0, 1]$, since

$$(L_1 f) = \int_{-1}^{0} f(x+t)\, dx = \int_0^1 f(t-x)\, dx = \int_{-\infty}^{\infty} g_1(x) f(t-x)\, dx.$$

5. Let φ and ψ be the Haar scaling function and the Haar wavelet, respectively, and let the spaces V_j and W_j be defined as in the lectures. Let a function f be given by

$$f(x) = \begin{cases} -1 & 0 \le x < 1/4, \\ 4 & 1/4 \le x < 1/2, \\ 2 & 1/2 \le x < 3/4, \\ -3 & 3/4 \le x < 1, \\ 0 & \text{otherwise.} \end{cases}$$

Express f in terms of the natural basis of V_2 and then decompose it into its V_0, W_0 and W_1 components.

Solution: We start at the given representation

$$f(x) = -\varphi(2^2x) + 4\varphi(2^2x-1) + 2\varphi(2^2x-2) - 3\varphi(2^2x-3),$$

and run the wavelet decomposition algorithm. Using the relations

$$\varphi(2^{j+1}x-k) = \begin{cases} \frac{1}{2}\Big(\varphi(2^jx-\frac{k}{2}) + \psi(2^jx-\frac{k}{2})\Big), & k \text{ even,} \\ \frac{1}{2}\Big(\varphi(2^jx-\frac{k-1}{2}) - \psi(2^jx-\frac{k-1}{2})\Big), & k \text{ odd,} \end{cases}$$

we get

$$f(x) = \frac{3}{2}\varphi(2x) - \frac{1}{2}\varphi(2x-1) - \frac{5}{2}\psi(2x) - \frac{5}{2}\psi(2x-1).$$

We keep the part $-\frac{5}{2}\psi(2x) - \frac{5}{2}\psi(2x-1)$, since it is in W_1, and further decompose the part $\frac{3}{2}\varphi(2x) - \frac{1}{2}\varphi(2x-1)$, which is in V_1. This produces

$$f(x) = \underbrace{\frac{1}{2}\varphi(x)}_{\in V_0} + \underbrace{\psi(x)}_{\in W_0} \underbrace{- \frac{5}{2}\psi(2x) - \frac{5}{2}\psi(2x-1)}_{\in W_1}.$$

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