Mathematics Chalmers & GU

TMA462/MMA410: Fourier and Wavelet Analysis, 2013–04–04; kl 8:30-12:30.

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Course Books: Bergh et al AND Bracewell, Lecture Notes and Calculator are allowed.
Each problem gives max 5p. Breakings: **3**: 12-15p, **4**: 16-19p och **5**: 20p-
For GU **G** students :12p, **VG**: 18p- (if applicable)
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma462/1213/index.html

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1. What is the modulus of the frequency response of the filter with the only non-zero coefficients $h_0 = 1,\ h_2 = 2$?

2. Prove that the autocorrelation function is hermitian, that is, that $C(-u) = C^*(u)$, and hence that when the autocorrelation function is real it is even. Note that if the autocorrelation function is imaginary it is also odd.

3. Investigate the functional form of $\frac{\alpha^2}{x^2+\alpha^2} * \frac{\beta^2}{x^2+\beta^2}$ and its width in terms of the widths of the convolved functions.

4. Assume that $\hat{f}(s) = 0$ for all $|s| \geq 1/2$. What filter H must be used to obtain f, note that $(f \in \mathcal{S})$, in the form.

$$f(\cdot) = H * \sum_n f(n)\varphi(\cdot - n).$$

(Here $\hat{\varphi}(s) \neq 0$ for $|s| < 1/2$).

5. Show that, if H and G are (FIR) filters in a MRA with an orthonormal wavelet basis, scaling function φ, and H has a zero of order N at $\omega = \pi$ then $D^\alpha\hat{\varphi}(0) = 0$ for $0 < \alpha \leq N$, only if all $D^\alpha\hat{\varphi}$ for $0 < \alpha \leq N$, are 2π periodic.

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Lösningar/Solutions.

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1. What is the modulus of the frequency response of the filter with the only non-zero coefficients $h_0 = 1,\ h_2 = 2$?

Solution: We have that

$$H(\omega) = \sum_{-\infty}^{\infty} h_k e^{-ik\omega} = 1 + 2e^{-2i\omega}.$$

Hence

$$|H(\omega)|^2 = H(\omega)\overline{H(\omega)} = (1 + 2e^{-2i\omega})(1 + 2e^{2i\omega}) = 1 + 2(e^{2i\omega} + e^{-2i\omega}) + 4 = 5 + 4\cos 2\omega \le 9.$$

whence

$$|H(\omega)| \le 3.$$

2. Prove that the autocorrelation function is hermitian, that is, that $C(-u) = C^*(u)$, and hence that when the autocorrelation function is real it is even. Note that if the autocorrelation function is imaginary it is also odd.

Solution: We have that

$$C(x) := f \star f(x) = \int_{-\infty}^{\infty} f^*(u-x)f(u)\,du = [u - x = t] = \int_{-\infty}^{\infty} f^*(t)f(t+x)\,dt.$$

Thus,

$$C(-x) = \int_{-\infty}^{\infty} f^*(u+x)f(u)\,du = \Big[\int_{-\infty}^{\infty} f(u+x)f^*(u)\,du\Big]^* = C^*(x).$$

Now $C(x)$ real $\iff$ $C(-x) = C(x)$, i.e., $C(x)$ is even.

Further, since

$$2ReC(x) = C(x) + C^*(x) = C(x) + C(-x),$$

Therefore, $ReC(x) = 0 \iff C(-x) = -C(x)$, and hence in this case $C(x)$ is odd.

Let us assume that the function $f(x)$ has an odd autocorrelation, i.e., $f(x) = g(x) + ih(x)$ with g and h both real and $ReC(x) = 0$, where

$$\begin{aligned}
C(x) &= \int_{-\infty}^{\infty} \Big[g(u-x) - ih(u-x)\Big]\Big[g(u) + ih(u)\Big]\,du \\
&= \int_{-\infty}^{\infty} \Big[g(u-x)g(u) + h(u-x)h(u)\Big]\,du + i\int_{-\infty}^{\infty} \Big[g(u-x)h(u) - h(u-x)g(u)\Big]\,du.
\end{aligned}$$

Then $ReC(x) = g \star g(x) + h \star h(x) = 0, \quad \forall x$. Thus

$$\mathcal{F}[ReC(x)] = |\hat{g}(s)|^2 + |\hat{h}(s)|^2 = 0 \Longrightarrow \hat{g} = \hat{h} = 0 \quad \forall s, \quad \Longrightarrow \quad g = h = 0 \quad \forall x, \quad \Longrightarrow f = 0.$$

Thus, $f = 0$ is the only odd autocorrelation.

3. Investigate the functional form of $\frac{\alpha^2}{x^2+\alpha^2} * \frac{\beta^2}{x^2+\beta^2}$ and its width in terms of the widths of the convolved functions.

Solution: We have that

$$\mathcal{F}\Big[e^{-|x|}\Big] = \frac{2}{1+(2\pi s)^2} \quad \Longrightarrow \quad \mathcal{F}\mathcal{F}\Big[e^{-|x|}\Big] = \mathcal{F}\Big[\frac{2}{1+(2\pi s)^2}\Big] = e^{-|-s|} = e^{|-s|}.$$

Now changing x to $\frac{1}{2\pi\gamma}x$ yields

$$\mathcal{F}\Big[\frac{1}{1+(x/\gamma)^2}\Big] = \frac{1}{2}2\pi\gamma e^{-2\pi\gamma|s|} = \pi\gamma e^{-2\pi\gamma|s|}.$$

Note that $\frac{\gamma^2}{x^2+\gamma^2} = \frac{1}{1+(x/\gamma)^2}$, and hence

$$\begin{aligned}\mathcal{F}\Big[\frac{\alpha^2}{x^2+\alpha^2} * \frac{\beta^2}{x^2+\beta^2}\Big] &= \mathcal{F}\Big[\frac{1}{1+(x/\alpha)^2} * \frac{1}{1+(x/\beta)^2}\Big] = \pi\alpha e^{-2\pi\alpha|s|}\pi\beta e^{-2\pi\beta|s|}\\ &= \pi^2\alpha\beta e^{-2\pi(\alpha+\beta)|s|} = \pi^2\alpha\beta\frac{1}{\pi(\alpha+\beta)}\mathcal{F}\Big[\frac{1}{1+(\frac{x}{\alpha+\beta})^2}\Big].\end{aligned}$$

Due to the uniqueness of the Fourier transforms

$$\frac{1}{1+(x/\alpha)^2} * \frac{1}{1+(x/\beta)^2} = \frac{\pi\alpha\beta}{\alpha+\beta}\cdot\frac{1}{1+(\frac{x}{\alpha+\beta})^2}.$$

The equivalent width for $f(x) = \frac{1}{1+(x/\alpha)^2}$ is given by

$$W_f := \frac{\int_{-\infty}^{\infty} f(x)\,dx}{f(0)} = 2\int_0^{\infty}\frac{1}{1+(x/\alpha)^2}\,dx = \Big[2\alpha\arctan(x/\alpha)\Big]_0^{\infty} = 2\alpha\cdot\frac{\pi}{2} = \alpha\pi.$$

Thus the widths will add up.

4. Assume that $\hat{f}(s) = 0$ for all $|s| \geq 1/2$. What filter H must be used to obtain f, note that $(f \in \mathcal{S})$, in the form.

$$f(\cdot) = H * \sum_n f(n)\varphi(\cdot - n).$$

(Here $\hat{\varphi}(s) \neq 0$ for $|s| < 1/2$).

Solution: Instead of $\sum_{-\infty}^{\infty} f(n)\delta_n$ the sample value can be written as

$$f_s(x) = \sum_n f(n)\varphi(x-n) = \sum_n f(n)\,\varphi * \delta_n(x) = \Big(\varphi * \sum_n f(n)\delta_n\Big)(x) = \Big(\varphi * f\sum_n \delta_n\Big)(x).$$

Thus

$$\hat{f}_s = \hat{\varphi}(\cdot)\Big(\hat{f} * \mathcal{F}[\sum_n \delta_n]\Big) = [\text{ Poisson }] = \hat{\varphi}\cdot\hat{f} * \sum_n \delta_n,$$

i.e.,

$$\hat{f}_s(s) = \hat{\varphi}(s)\sum_n \hat{f}(s-n).$$

Multiplying by $\Pi(s) = \begin{cases} 1, & |s| < 1/2 \\ 0, & |s| \geq 1/2 \end{cases}$ to get

$$\Pi(s)\hat{f}_s(s) = \hat{\varphi}(s)\hat{f}(s), \quad \Longrightarrow \quad \hat{f}(s) = \frac{\Pi(s)}{\hat{\varphi}(s)}\hat{f}_s(s) = \hat{H}(s)\hat{f}_s(s).$$

Thus

$$f(x) = H * f_s.$$

5. Show that, if H and G are (FIR) filters in sa MRA with an orthonormal wavelet basis, scaling function φ, and H has a zero of order N at $\omega = \pi$ then $D^\alpha \hat{\varphi}(0) = 0$ for $0 < \alpha \leq N$, only if all $D^\alpha \hat{\varphi}$ for $0 < \alpha \leq N$, are 2π periodic.

Solution: Recall that we have

$$|H(\omega)|^2 + |G(\omega)|^2 = 1, \quad H(\omega)\overline{H(\omega+\pi)} + G(\omega)\overline{G(\omega+\pi)} = 0$$

and

$$\hat{\varphi}(\omega) = H(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2}), \quad G(\omega) = -e^{-i\omega}\overline{H(\omega+\pi)}, \quad H(0) = 1.$$

Thus

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1, \quad \text{and} \quad H(\omega) = \Big(\frac{1+e^{-i\omega}}{2}\Big)^N Q(\omega), \quad H(\omega+\pi) = \Big(\frac{1-e^{-i\omega}}{2}\Big)^N Q(\omega+\pi).$$

Thus both if $Q(\pi) \neq 0$ and $Q(2\pi) \neq 0$ then obviously H has a zero of odrer N at $\omega = \pi$. Further since $\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega)$. Thus, by the chain rule $D^\alpha \hat{\varphi}(0) = 0$, for $0 < \alpha \leq N$, only if all $D^\alpha \hat{\varphi}$ are 2π periodic.

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