Fourier and Wavelet analysis

Fourier transform:

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\, dx$$

[The formula works equally well in $\mathbb{R}^n$, if $\int_{-\infty}^{\infty}$ is replaced by $\int_{\mathbb{R}^n}$, and $x\xi$ denotes the scalar product $\langle x, \xi \rangle$]

The course

- The Fourier transform
- Distribution theory
- Transforms related to the Fourier transform (Radon, Hankel, Z ...)
- The wavelet transforms
- Multi resolution analysis
- Discrete transforms, sampling
- Applications (three Lab assignments)

Notation

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\,dx$$

$$\hat{f}(\xi) = (\mathcal{F} f)(\xi)$$

$$f(x) \overset{\mathcal{F}}{\supset} \hat{f}(\xi)$$

Basic properties

Linearity: $\begin{cases} f+g \supset \hat{f}+\hat{g} \\ \alpha f \supset \alpha \hat{f} \end{cases}$ $(\alpha \in \mathbb{R}, \mathbb{C})$

Scaling: $f \supset \hat{f} \iff \frac{1}{a} f\left(\frac{x}{a}\right) \supset \hat{f}(a\xi)$, $(a>0)$

$\left(\text{notation: } \frac{1}{a} f\left(\frac{\cdot}{a}\right) \supset \hat{f}(a\xi)\right)$

"If f is reasonably "regular"/smooth so is $\hat{f}$"

(This will be made precise later)

Fourier tranform in L^2

$L^2(\mathbb{R}) := \{ f : f \text{ measurable, such that } \int_{\mathbb{R}} |f(x)|^2 dx < \infty \}$

Scalar product: $\langle f, g \rangle = \int_{\mathbb{R}} f(x)\,\overline{g(x)}\,dx$

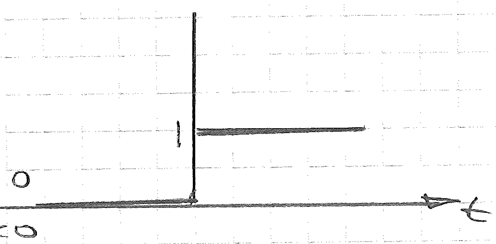
parsevals: $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$

By "regular/smooth" we will mean Schwartz class $\mathcal{S}$

Often: $f(t)$, a signal

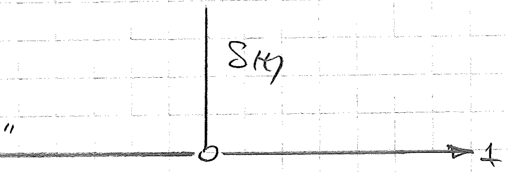
$f(t) = \sin(\omega t)$

$f(t) = H(t) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



"$f(t) = \delta(t)$"

the dirac "δ-function"



These are not in L^2, nor in $\mathcal{S}$.

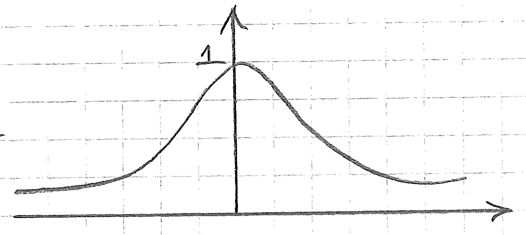
Fundamental: The Fourier inversion formula

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\, dx$$

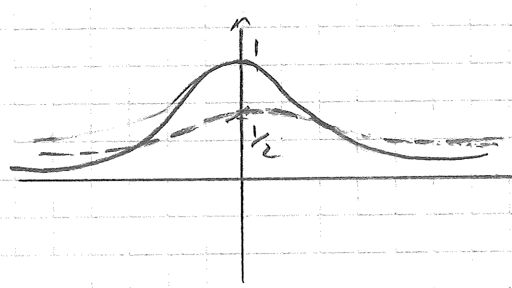
$$\Longleftrightarrow \quad f(x) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi .$$

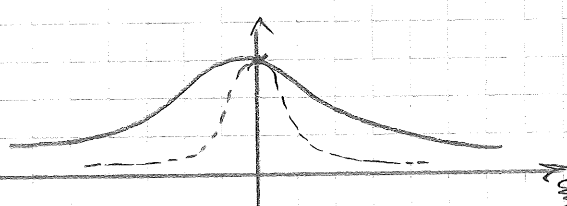
Example (Important)

$$\mathcal{F}\left(e^{-\pi x^2}\right) = e^{-\pi \xi^2}$$

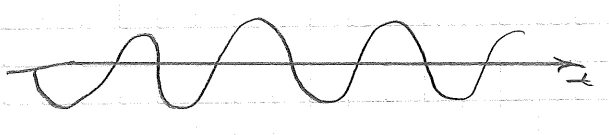


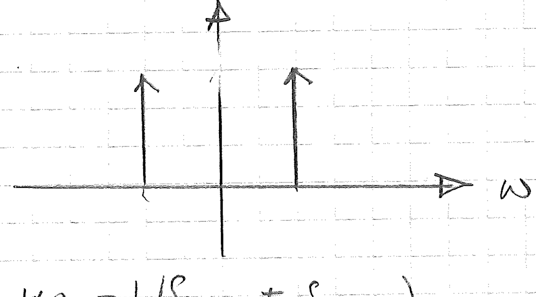
(Scaling $a = 2$: $\frac{1}{2} e^{-\pi \left(\frac{x}{2}\right)^2} \Rightarrow e^{-\pi (2\xi)^2}$





Example $f(t) = \sin(\omega t)$





$$\hat{f}(t) = \frac{1}{2}\left(\delta_{-\frac{\omega}{2\pi}} + \delta_{\omega/2\pi}\right)$$

The Fourier transform does not distinguish between what happened a long time ago, now, and in the future.

One remedy: Wavelets (another: Windowed Fourier Transf)

Other transforms:

Hankel: $\hat{f}(\xi_1, \xi_2) = \iint_{\mathbb{R}^2} e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)} f(x_1, x_2)\, dx_1\, dx_2$

Let $f(x_1, x_2) = F(\sqrt{x_1^2 + x_2^2}) = F(r)$ $\quad \begin{cases} x_1 = r\cos\theta \\ x_2 = r\sin\theta \end{cases}$

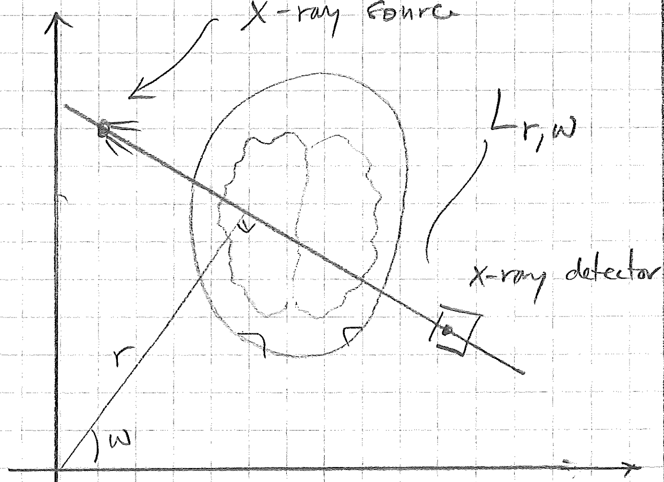
Then $\hat{f}(\xi_1, \xi_2) = \tilde{F}(\sqrt{\xi_1^2 + \xi_2^2}) = \tilde{F}(\rho)$ $\quad \begin{cases} \xi_1 = \rho\cos\alpha \\ \xi_2 = \rho\sin\alpha \end{cases}$

$F(r) \longmapsto \tilde{F}(\rho)$ is the

Hankel transform.

Radon

If $f(x_1, x_2)$ is the density of the head at point (x_1, x_2), then



$\int_{L_{r,\omega}} f(x_1, x_2)\, dl$ gives a measure of the damping along $L_{r,\omega}$. This can be measured.

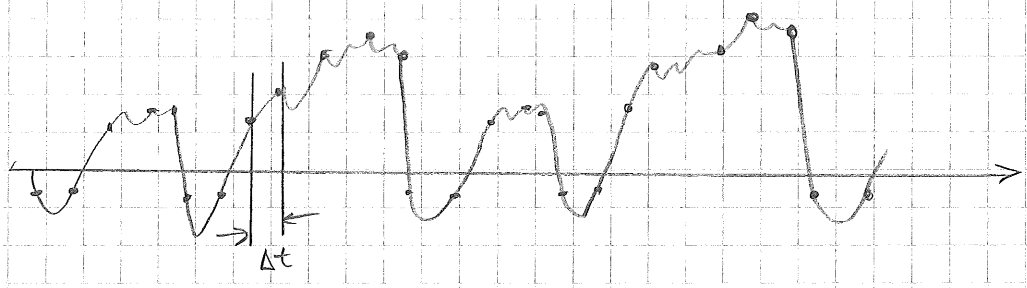
Notation: $R_\omega f(r) = \int_{L_{r,\omega}} f(x_1, x_2)\, dl,$

is the Radon transform.

Important because there is a formula for recovering f from $R_\omega f(r)$.

(Computer tomography).

Discrete transforms, Sampling, filters

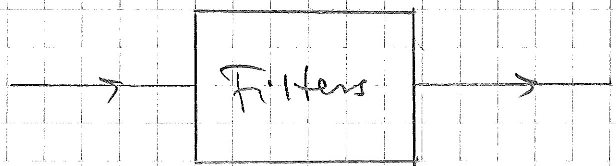


$$f(t) \longrightarrow \{f_n\}_{n=-\infty}^{\infty} = \{f(n\Delta t)\}_{n=-\infty}^{\infty}$$

$\{f_n\}$ is a discrete signal.

How often do we need to sample?

How do we construct digital filters?



Low pass, High pass ..

Fast Fourier transforms.

What functions do have a Fourier transform?

$L^2(\mathbb{R})$

S (smooth functions with rapid decay)

$L^1(\mathbb{R}) = \{f : \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$

Note! $f(x) \in L^1(\mathbb{R}) \Rightarrow |\hat{f}(\xi)| = \left|\int_{\mathbb{R}} e^{-2\pi i \xi x} f(x)\,dx\right|$

$$\leq \int_{\mathbb{R}} |e^{-2\pi i \xi x} f(x)|\,dx = \int_{\mathbb{R}} |f(x)|\,dx < \infty$$

$(f \in L^1 \Rightarrow \hat{f} \in L^\infty)$.

Hausdorff Young's inequality

$f \in L^p, \quad (1 \leq p \leq 2) \Rightarrow \hat{f} \in L^q, \quad \frac{1}{p} + \frac{1}{q} = 1$

Convolution theorem

Suppose that $f(x) \supset \hat{f}(\xi)$, $g(x) \supset \hat{g}(\xi)$, Then

$$f * g\,(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \supset \hat{f}(\xi)\, \hat{g}(\xi).$$

Other properties of convolution:

commutative : $f * g = g * f$

associative : $f * (g * h) = (f * g) * h$

distributive : $f * (g + h) = f * g + f * h.$

The Fourier transform [Further properties]

1) $f \supset \hat{f} \Rightarrow$ $Df \supset 2\pi i \xi\, \hat{f}(\xi)$

$-2\pi i (\cdot) f \supset D\hat{f}$

Pf $\hat{f}(\xi) = \int e^{-2\pi i x \xi} f(x)\, dx$

$D\hat{f}(\xi) = \int e^{-2\pi i x \xi} (-2\pi i x f(x))\, dx = \mathcal{F}(-2\pi i (\cdot) f)$

$\mathcal{F}(Df) = \int e^{-2\pi i x \xi} Df(x)\, dx \overset{PI}{=} - \int D(e^{-2\pi i x \xi}) f(x)\, dx$

$= 2\pi i \xi \int e^{-2\pi i x \xi} f(x)\, dx = 2\pi i \xi\, \hat{f}(\xi).$

2) Translation : $\quad \tau_a : f(\cdot) \longmapsto f(\cdot - a)$

$$\mathcal{F}(\tau_a f)(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(\underbrace{x-a}_{y})\, dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i (y+a)\xi} f(y)\, dy = e^{-2\pi i a \xi} \hat{f}(\xi).$$

$$\tau_a \hat{f}(\xi) = \int e^{-2\pi i x(\xi - a)} f(x)\, dx$$

$$= \int e^{-2\pi i x \xi}\left[e^{2\pi i a x} f(x)\right] dx = \mathcal{F}\left(e^{2\pi i a(\cdot)} f\right)$$

Summary :

1a) $\quad -2\pi i(\cdot) f \supset D\hat{f}$

1b) $\quad Df \supset 2\pi i(\cdot)\hat{f}$

2a) $\quad \tau_a f \supset e^{-2\pi i a(\cdot)} \hat{f}$

2b) $\quad e^{2\pi i a(\cdot)} f \supset \tau_a \hat{f}$

The class S and S'

Distributions (generalized functions)
were introduced by Laurent Schwartz (~ 1940)
and S Sobolev (~ 1935), to give a mathematically
rigorous theory of mathematical objects like the Dirac δ-function.

[note Schwarz in Cauchy-Schwarz was German
H A Schwarz].

<u>Definition</u> The function class $\mathcal{S}$ are complex valued functions f of a real variabel, such that

$f : \mathbb{R} \to \mathbb{C}$ satisfies

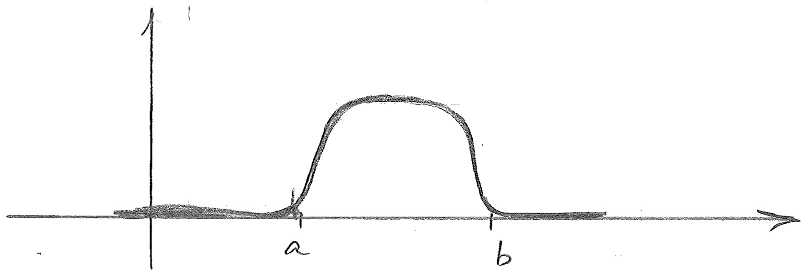
$$\sup_{x\in\mathbb{R}} \left| |x|^{\alpha} D^{\beta} f(x) \right| < \infty$$

for any choice of $\alpha \geq 0$ and $\beta \geq 0$.

<u>Ex</u>. $f(x) = e^{-ax^2}$ $(a>0)$

(what if $a = i$? $a<0$?)

<u>Ex</u>. $f(x) = \begin{cases} 0 & (x \leq a) \\ e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}}, & a<x<b \\ 0 & (x \geq b). \end{cases}$



This function has "Compact Support" and is C^{∞}, but <u>not</u> (real) analytic (i.e., its power series is not convergent everywhere).

<u>Ex</u>. Let $g \in L^1(\mathbb{R})$. Then $(a=-b?)$

$$g_{\varepsilon}(x) = \int_{\mathbb{R}} g(y) \frac{1}{\varepsilon} f\left(\frac{x-y}{\varepsilon}\right) dy \longrightarrow g(x) \quad \text{as} \quad \varepsilon \to 0.$$

(in other words,

$g * \frac{1}{\varepsilon} f\left(\frac{\circ}{\varepsilon}\right) \longrightarrow g(x)$ almost everywhere as $\varepsilon \to 0$)

Note that the Fourier transform is well defined for $f \in \mathcal{S}$.

Properties of S

Lemma I) If $f \in S$ and if $g(x) = x^\alpha D^\beta f(x)$, $(\alpha, \beta \in \mathbb{Z}^+)$

Then $g \in S$

II) $f \in S \Rightarrow \hat{f} \in S$.

Pf. I): Just calculate

II): Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$.

Then $$D^\beta \hat{f}(\xi) = \int_{\mathbb{R}} f(x) \left(\frac{d}{d\xi}\right)^\beta \left(e^{-2\pi i x \xi}\right) dx$$

(This is ok. why?)

$$= \int_{\mathbb{R}} \underbrace{f(x) \, (-2\pi i x)^\beta}_{\text{decays rapidly}} e^{-2\pi i x \xi} dx$$

$$\Rightarrow |D^\beta \hat{f}(\xi)| \le \int |f(x)| \, |2\pi x|^\beta dx < \infty \qquad (*)$$

Also $$2\pi i \xi \hat{f}(\xi) = \int_{-\infty}^{\infty} f(x) \underbrace{(2\pi i \xi) e^{-2\pi i x \xi}}_{= -\frac{d}{dx} e^{-2\pi i x \xi}} dx = PI =$$

$$= \int_{-\infty}^{\infty} \left(\frac{d}{dx} f(x)\right) \cdot e^{-2\pi i x \xi} dx = \mathcal{F}(Df)$$

$$\Rightarrow |\xi \hat{f}(\xi)| \le \frac{1}{2\pi} \int |D f(x)| \, dx < \infty.$$

This can be repeated for any power ξ^α which together with $(*)$ concludes the result.

Ex $f(x) = e^{-\pi x^2} \Rightarrow \hat{f} = f$

[Note if $f = \mathcal{F} f$, then f is a "fix point" for $\mathcal{F}$.
Are there any other fix points?]

Pf $f \in S$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi} dx$

$$D\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2}(-2\pi i x) e^{-2\pi i x \xi} dx = i\int_{\mathbb{R}} D(e^{-\pi x^2}) e^{-2\pi i x \xi} dx$$

$$= i\, \mathcal{F}(Df) \underset{\uparrow}{=} -2\pi \xi \hat{f}(\xi)$$

$$\boxed{\mathcal{F}(Df) = 2\pi i (\cdot) \hat{f}}$$

Also $\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2} dx = 1$ (see any calculus book).

Hence $\begin{cases} \hat{f}'(\xi) + 2\pi\xi \hat{f}(\xi) = 0 \\ \hat{f}(0) = 1 \end{cases} \Rightarrow \hat{f}(\xi) = e^{-\pi \xi^2}$. ☒

The Fourier Inversion formula

Suppose that $f \in S$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x) dx$. Then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi) d\xi .$$

Pf. Suppose we can check that $f(0) = \int_{\mathbb{R}} \hat{f}(\xi) d\xi$ for all $f \in S$; then we done, because then

$$f(a) = \tau_a f(0) = \int_{\mathbb{R}} \mathcal{F}(\tau_a f) d\xi \overset{(7)}{=} \int_{\mathbb{R}} e^{2\pi i a \xi} \hat{f}(\xi) d\xi.$$

I) Assume that $f(0) = 0$. Then $g(x) = \frac{f(x)}{x} \in S$, and $f(x) = x\, g(x)$. If $g(x) \supset \hat{g}(\xi)$ then $-2\pi i x g(x) \overset{(6)}{\supset} D\hat{g}$

That means that $\hat{f}(\xi) = -\frac{1}{2\pi i} \hat{g}'(\xi)$ $\Rightarrow$

$$\int_{-\infty}^{\infty} \hat{f}(\xi) d\xi = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{g}'(\xi) d\xi = 0,$$ because $\hat{g}(\xi) \in S$.

II) If $f(0) \neq 0$, consider

$$\tilde{f}(x) \equiv f(x) - f(0) e^{-\pi x^2} \supset \hat{f}(\xi) - f(0) e^{-\pi \xi^2} = \hat{\tilde{f}}(\xi).$$ Then

$$\tilde{f}(0) = 0 = \int_{-\infty}^{\infty} \hat{\tilde{f}}(\xi) d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) - \int_{-\infty}^{\infty} f(0) e^{-\pi \xi^2} d\xi$$

$$= \int_{-\infty}^{\infty} \hat{f}(\xi) d\xi - f(0). \quad ☒$$

The class $S'(\mathbb{R})$

<u>Notation</u> $\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{R}} \varphi_1(x)\, \varphi_2(x)\, dx$

<u>note</u> This is a scalar product only if φ_1, φ_2 are real-valued.

<u>Def</u>. A linear mapping $T: S \longrightarrow \mathbb{C}$ must satisfy:

$$\begin{cases} T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2, & \varphi_1, \varphi_2 \in S \\ T(\alpha\varphi_1) = \alpha T\varphi_1 . \end{cases}$$

<u>Def</u>. A linear mapping $T: S \longrightarrow \mathbb{C}$, $\varphi \longmapsto T(\varphi)$ is called a <u>tempered distribution</u>,

if for any sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n \in S$, such that for all $\alpha, \beta \in \mathbb{Z}$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} \left| |x|^{\alpha} D^{\beta} \varphi_n(x) \right| = 0$$

we have

$$\lim_{n \to \infty} T(\varphi_n) = 0.$$

<u>Example</u> Take f such that $\frac{f(x)}{(1+x^2)^{\alpha}}$ is integrable for some $\alpha \geq 0$, and let

$$T(\varphi) = \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\varphi(x)\, dx,$$

Then T is a tempered distribution.

<u>Note</u> We may identify f with T. (write $f(\varphi)$ for $T(\varphi)$) This must not be confused with $f(x)$, where we think of $f: \mathbb{R} \longrightarrow \mathbb{C}$.

Note. $\mathcal{S}$ is a linear space: $\varphi_1, \varphi_2 \in \mathcal{S}$, $\alpha_1, \alpha_2 \in \mathbb{C}$

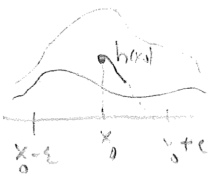
$$\Rightarrow \alpha_1\varphi_1 + \alpha_2\varphi_2 \in \mathcal{S}, \qquad (\varphi_1 \equiv 0 \in \mathcal{S}).$$

To say that $\varphi_n \longrightarrow \varphi$ in $\mathcal{S}$ is equivalent to saying that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| |x|^\alpha D^\beta(\varphi_n(x) - \varphi(x)) \right| = 0 \qquad *$$

The family of limits $*$ (indexed by α and β) define a topology in $\mathcal{S}$.

If $\mathfrak{X}$ and $\mathfrak{Y}$ are topological spaces, and $f : \mathfrak{X} \longrightarrow \mathfrak{Y}$, we say that f is continuous if $f(x_n) \longrightarrow f(x)$ in $\mathfrak{Y}$, whenever $x_n \longrightarrow x$ in $\mathfrak{X}$.



We have $T : \mathcal{S} \longrightarrow \mathbb{C}$. so a tempered distribution is a continuous linear map $\mathcal{S} \longrightarrow \mathbb{C}$.

Exercise If $f \in C(\mathbb{R})$, $g \in C(\mathbb{R})$ and $f = g$ in $\mathcal{S}'$ then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

Proof. $f = g$ in $\mathcal{S}'$ means that for all $\varphi \in \mathcal{S}$,

$$\langle f, \varphi\rangle = \langle g, \varphi\rangle \iff \int_{\mathbb{R}} (f(x) - g(x))\varphi(x)\,dx = 0.$$

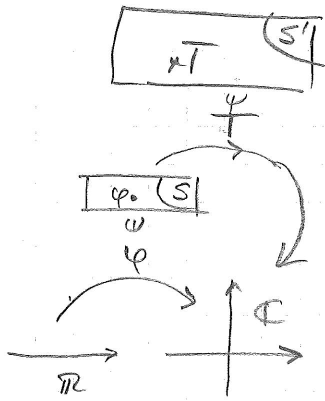
Let $f(x) - g(x) = h(x)$, and assume that there is $x_0 \in \mathbb{R}$ such that $h(x_0) > 0$. Because $h \in C(\mathbb{R})$, there is an interval $]x_0-\varepsilon, x_0+\varepsilon[$ such that $h(x) > \frac{1}{2}h(x_0)$ in that intervall. Take $\varphi \in \mathcal{S}$ such that $\varphi(x) = 0$ when $x \notin]x_0-\varepsilon, x_0+\varepsilon[$ and $\varphi(x) > 0$ when $x \in]x_0-\varepsilon, x_0+\varepsilon[$. Then

$$\langle h, \varphi\rangle = \int_{-b}^{b} h(x)\varphi(x)\,dx = \int_{x_0-\varepsilon}^{x_0+\varepsilon} h(x)\varphi(x)\,dx \geq \frac{1}{2}h(x_0)\int_{x_0-\varepsilon}^{x_0+\varepsilon}\varphi(x) > 0.$$

But this contradicts the statement $\langle f, \varphi\rangle = \langle g, \varphi\rangle$ for all φ.

<u>Example</u> $T: S \to \mathbb{C}$

$$\varphi \mapsto \varphi(a), \quad a \in \mathbb{R}$$



(i.e., φ is evaluated at the point a)

This is the Dirac "δ-function" at a

1) T is linear

2) Let $\varphi_n \in S$, $\varphi_n \to 0$ in S,

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta \varphi_n(x)| \to 0. \quad (n \to \infty) \qquad (* \text{ page } 12)$$

$$\sup_{x\in\mathbb{R}} |x^\alpha D^\beta (\varphi_n(x) - 0)| \to 0$$

In particular $\varphi_n(a) \to 0$, so $T(\varphi_n) = \varphi_n(a) \to 0$.

<u>Note</u>, In this case only φ and no derivatives need to be evaluated.

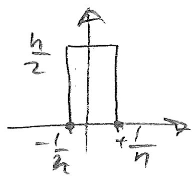
<u>Example</u> Let $f_n(x) = \sqrt{n}\, e^{-n\pi x^2}$, $n = 1, \dots$

Then $f_n \in S'$ and for all $\varphi \in S$,

$$f_n(\varphi) = \langle f_n, \varphi \rangle \to \varphi(0) = \delta_0(\varphi) \quad \text{when } n \to \infty.$$

We say that $f_n \to \delta_0$ in S'

(other choice of f_n with similar properties



<u>Pf</u>. $\langle f_n, \varphi \rangle = \int_{-\infty}^{\infty} \sqrt{n}\, e^{-n\pi x^2} \varphi(x)\,dx = \{ x = \frac{y}{\sqrt{n}} \Rightarrow dx = \frac{dy}{\sqrt{n}} \}$

$$= \int_{-\infty}^{\infty} e^{-\pi y^2} \varphi\left(\frac{y}{\sqrt{n}}\right) dy \to \int_{-\infty}^{\infty} e^{-\pi y^2} \varphi(0)\, dy = \varphi(0)$$

by uniform convergence (or dominated convergence).

<u>Ex</u>. e^x is <u>not</u> a tempered distribution, because it is growing too fast as $x \to \infty$.

Take $\varphi(x) = e^{-\sqrt{1+x^2}} \in S$ (prove that).

Then $\langle e^{(\cdot)}, \varphi \rangle = \int_{-\infty}^{\infty} e^x e^{-\sqrt{1+x^2}}\,dx$.

which is divergent.

Ex. Let $\delta_n : \varphi \longmapsto \varphi(n)$, and let

$$Ш = \sum_{n=-\infty}^{\infty} \delta_n, \quad \text{so,} \quad \langle Ш, \varphi \rangle = \sum_{n=-\infty}^{\infty} \varphi(n)$$

This is a tempered distribution (prove that!)

Next we wish to prove that tempered distributions share many properties with ordinary functions. They can be differentiated, Fourier transformed etc.

Differentiation of distributions

Let $f \in C^1(\mathbb{R})$, and assume that it is not growing very fast (it may be bounded for example). Then

$$\langle f', \varphi \rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx$$

which is well defined for all $\varphi \in \mathcal{S}$.

Definition Let T be a tempered distribution. Then we define DT by

$$\langle DT, \varphi \rangle = -\langle T, D\varphi \rangle \qquad \text{for all } \varphi \in \mathcal{S}.$$

Multiplication by a function.

Let $f(x) \in C(\mathbb{R})$ and $g(x) \in C^{\infty}(\mathbb{R})$, and assume that there is an $\alpha \in \mathbb{Z}^+$ such that $\frac{|g(x)|}{(1+x^2)^\alpha}$ is bounded. Then we have

$$\langle fg, \varphi \rangle = \int_{\mathbb{R}} f(x)\, g(x)\varphi(x)\,dx = \langle f, g\varphi \rangle.$$

Definition Let $T \in \mathcal{S}'$, and let g be as above. Then we define gT by

$\langle gT, \varphi \rangle = \langle T, g\varphi \rangle$. This is o.k. because $g\varphi \in \mathcal{S}$ if $\varphi \in \mathcal{S}$.

Translation. Let $f \in C(\mathbb{R})$ and write $f_\tau(x) = \tau f(x) = f(x-\tau)$.

Then $\int_{\mathbb{R}} f_\tau(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x-\tau)\varphi(x)\,dx = \{y = x-\tau\}$

$$= \int_{\mathbb{R}} f(x)\varphi(x+\tau)\,dx\;dx = \langle f, \varphi_{-\tau}\rangle .$$

Def. For $T \in S'$, we define T_τ by

$$\langle T_\tau, \varphi\rangle = \langle T, \varphi_{-\tau}\rangle .$$

But these definitions are only useful if DT, gT and T_τ satisfy some good properties.

Proposition If $T \in S'$, then also DT, gT and T_τ belong to S'.

Proof (for DT)

We have $DT : \varphi \longmapsto -\langle T, D\varphi\rangle$.

Then DT is obviously linear (why?)

Take $\{\varphi_n\}$ with $\varphi_n \in S$, and $\varphi_n \to 0$ in S;

Then $\sup_x |x^\alpha D^\beta \underbrace{D\varphi_n}(x)| = \sup_x |x^\alpha D^{\beta+1}\varphi_n(x)| \to 0$

as $n \to \infty$, and so, because $T \in S'$,

$$\langle T, D\varphi_n\rangle \to 0 \quad \text{as } n \to \infty .$$

The structure theorem. Let $T \in S'$. Then there exists functions $f_j \in C(\mathbb{R})$ such that

$$T = \sum_j D^{\beta_j} f_j$$

That means that any tempered distribution can be written as linear combination of (distributional) derivatives of continuous functions.

Repetation #2

$f\in L^2(\mathbb{R}),\ f\in L^1(\mathbb{R}),\ f\in S$ (Schwartz class) : $\begin{cases}\hat f(\xi)=\int_{-\infty}^{\infty} e^{-2\pi i x\xi} f(x)\,dx\\ f(x)=\int_{\mathbb R} e^{2\pi i x\xi}\hat f(\xi)\,d\xi\end{cases}$

$f(x) \overset{\mathcal F}{\supset} \hat f(\xi)$, $\quad f\in S \Rightarrow \hat f\in S$

$f\mapsto \hat f$ linear $\quad f\in L^1 \Rightarrow \hat f\in L^\infty$.

$$\frac1a f\left(\frac{x}{a}\right) \supset \hat f(a\xi) \quad a>0$$

$$e^{-\pi x^2}\supset e^{-\pi\xi^2}$$

Hausdorff-Young's: $f\in L^p,\ 1\le p\le 2 \Rightarrow \hat f\in L^q,\ \frac1p+\frac1q=1$

Convolution: $f*g(x)=\int_{\mathbb R} f(x-y)g(y)\,dy \supset \hat f(\xi)\hat g(\xi)$

$Df \supset 2\pi i\xi \hat f(\xi)$, $\quad -2\pi i(x)f\supset D\hat f$

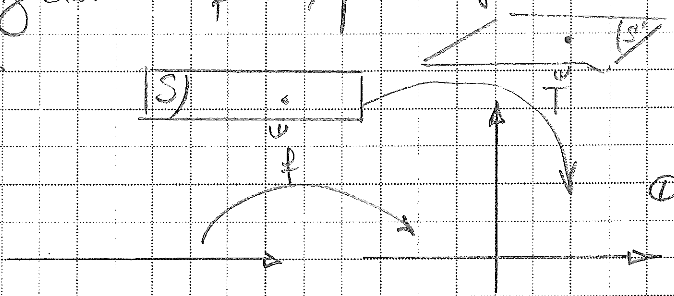
$\tau_a f\supset e^{-2\pi i a\xi}\hat f(\xi)$, $\quad e^{2\pi i a x}f(x)\supset \tau_a\hat f$

Def: $f:\mathbb R\to\mathbb C$ in Schwartz class S if $\forall \alpha\ge 0,\ \beta\ge 0$ integer

$$\sup_{x\in\mathbb R}|x^\alpha D^\beta f(x)|<\infty$$

Ex. $e^{-x^2}\in S$

$f\in S\Rightarrow \hat f\in S$,



$T\in S'$: i) $T: S\to\mathbb C$, ii) T linear

$\{\varphi_n\}_1^\infty\subset S$, $\varphi_n\to 0$ (i.e. $\sup_x |x^\alpha D^\beta \varphi_n(x)|\to 0$ as $n\to\infty$) $\Rightarrow T\varphi_n\to 0$ as $n\to\infty$

Def $\varphi_n\to\varphi\in S$: $\lim_{n\to\infty}\sup_x|x^\alpha D^\beta(\varphi_n-\varphi(x))|=0$

$f\in C(\mathbb R),\ g\in C(\mathbb R)$, if $f=g$ in S' $\left(\langle f,\varphi\rangle=\langle g,\varphi\rangle\ \forall\varphi\in S\right)$

then $f(x)=g(x)\ \forall x\in\mathbb R$

Ex. $f_n=\sqrt n\, e^{-n\pi x^2}$, $n=1,\dots$ $\Rightarrow f_n\in S'$ & $\langle f_n,\varphi\rangle \xrightarrow[n\to\infty]{} \varphi(0)=\delta_0(\varphi)$, $\varphi\in S$

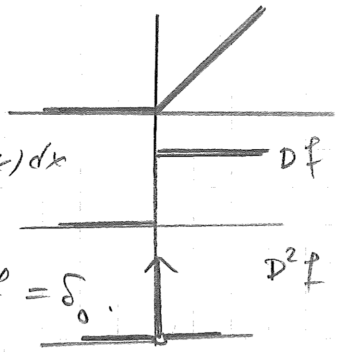
Defs. $T\in S'$, $\langle DT,\varphi\rangle=-\langle T,D\varphi\rangle\ \forall\varphi\in S$. $\quad DT\in S'$

$\langle gT,\varphi\rangle=\langle T,g\varphi\rangle$ " $\quad gT\in S'$

$\langle T_\tau,\varphi\rangle=\langle T,\varphi_{-\tau}\rangle$ " $\quad T_\tau\in S'$

Structure thm: $T\in S'$, $\exists f_j\in C(\mathbb R)$: $T=\sum_j D^{\beta_j} f_j$.

<u>Ex</u>. Let $f(x) = \begin{cases} x, & \text{when } x > 0 \\ 0 & \text{when } x \le 0. \end{cases}$



Then $\langle D^2 f, \varphi \rangle = \int_{\mathbb{R}} f(x)\, D^2\varphi(x)\, dx = \int_0^\infty x\, D^2\varphi(x)\, dx$

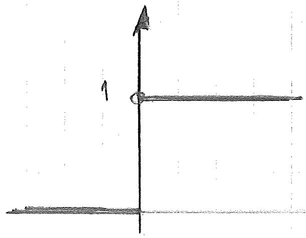
$= -\int_0^\infty D\varphi(x)\, dx = \varphi(0)$, so $D^2 f = \delta_0$.

The proof of the structure theorem is rather difficult.

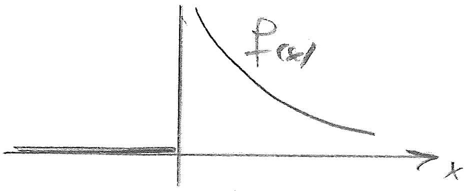
<u>Example</u>

Let $f(x) = \frac{1}{2} x^{-3/2} H(x)$

where $H(x) = \begin{cases} 0 & \text{when } x \le 0 \\ 1 & \text{when } x > 0 \end{cases}$



[Note: Usually it does not matter if $H(0)$ is defined differently]

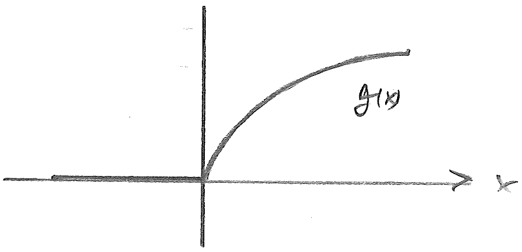


Define a distribution T by

$$\langle T, \varphi \rangle = \lim_{\varepsilon \to 0+} \left(-\frac{1}{2} \int_\varepsilon^\infty x^{-3/2} \varphi(x)\, dx + \frac{1}{\varepsilon^{1/2}} \varphi(0) \right)$$

Then T is a tempered distribution.

In fact $T = D^2 g$, where $g(x) = 2x^{1/2} H(x) \in S'$
(and $\in C(x)$)



(But as a function, $\overset{f}{g}(x)$ is not in S', because

$\int_{-\infty}^{\infty} \overset{f}{g}(x)\, \varphi(x)\, dx$ is divergent, in general.)

Let $\varphi_n \to 0$ in $\mathcal{S}$.

Then $\langle g, \varphi_n \rangle = \int_0^\infty x^{1/2} \varphi_n(x)\, dx$

$$= \int_0^\infty \frac{x^{1/2}}{1+x^2} (1+x^2)\varphi_n(x)\, dx \to 0 \quad \text{when } n \to \infty,$$

because if $\varphi_n \to 0$ in $\mathcal{S}$, so does $(1+x^2)\varphi_n$,

and $\int_0^\infty \frac{x^{1/2}}{1+x^2} dx$ is convergent.

Next, $\langle D^2 g, \varphi \rangle = -\langle Dg, D\varphi \rangle = \langle g, D^2\varphi \rangle$

$$= \int_0^\infty 2x^{1/2} \varphi''(x)\, dx = \underbrace{\int_\varepsilon^\infty 2x^{1/2} \varphi''(x)\, dx}\, + \underbrace{2\int_0^\varepsilon x^{1/2} \varphi''(x)\, dx}_{\to 0, \text{ as } \varepsilon \to 0}$$

$$= -\int_\varepsilon^\infty x^{-1/2} \varphi'(x)\, dx + \left[2x^{1/2}\varphi'(x)\right]_\varepsilon^\infty$$

$$= -\left[x^{-1/2}\varphi(x)\right]_\varepsilon^\infty + \int_\varepsilon^\infty -\tfrac{1}{2} x^{-3/2} \varphi(x)\, dx - 2\varepsilon^{\frac{1}{2}}\varphi'(\varepsilon)$$

$$= -\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\, dx + \varepsilon^{-1/2}\varphi(0) + \underbrace{\varepsilon^{-1/2}(\varphi(\varepsilon) - \varphi(0))}_{\to 0, \text{ when } \varepsilon \to 0} \underbrace{-2\varepsilon^{\frac{1}{2}}\varphi'(\varepsilon)}_{\to 0 \text{ as } \varepsilon \to 0}$$

In conclusion

$$\langle g, D^2\varphi \rangle = -\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\, dx + \varepsilon^{-1/2}\varphi(0) + \varepsilon^{-1/2}(\varphi(\varepsilon) - \varphi(0)) + 2\varepsilon\int_0^\varepsilon x^{1/2}\varphi''(x)\,dx \; -2\varepsilon^{\frac{1}{2}}\varphi'(\varepsilon)$$

$$= \lim_{\varepsilon \to 0} \left(-\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\, dx + \varepsilon^{-1/2}\varphi(0) \right)$$

T is called the finite part of f.

Proposition

The plancherel formula: Let $f, \varphi \in \mathcal{S}$. Then

$$\int_{\mathbb{R}} \hat{f}(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\hat{\varphi}(x)\,dx$$

Proof.

$$\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i x y} f(y)\,dy\ \varphi(x)\,dx =$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-2\pi i x y} f(y)\varphi(x)\,dx\,dy = \int_{\mathbb{R}} f(y)\int_{\mathbb{R}} e^{-2\pi i x y}\varphi(x)\,dx\,dy.$$

The changes of order of integration are justified because $f, \varphi \in \mathcal{S}$.

The Fourier transform of distributions

Because $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$ for $f \in \mathcal{S}$, we define the Fourier transform of $T \in \mathcal{S}'$ by

$$\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle \qquad \text{for all } \varphi \in \mathcal{S}.$$

Proposition $\hat{T} \in \mathcal{S}'$ (this we have to prove!).

1) Linear: Clear because $\varphi \mapsto \hat{\varphi}$ is linear.

2) Take $\varphi_n \to 0$ in $\mathcal{S}$. Then $\hat{\varphi}_n \to 0$ in $\mathcal{S}$,

so $\langle \hat{T}, \varphi_n\rangle = \langle T, \hat{\varphi}_n\rangle \to 0$ as $n \to \infty$.

[Note: that $\hat{\varphi}_n \to 0$ in $\mathcal{S}$ follows from the calculations needed to prove that $\hat{\varphi}_n \in \mathcal{S}$.]

Example Let $\beta \in \mathbb{Z}^+$, Compute $\mathcal{F}(D^\beta \delta_0)$

$$\langle \mathcal{F}(D^\beta\delta_0), \varphi\rangle = \langle D^\beta\delta_0, \hat{\varphi}\rangle = (-1)^\beta\langle \delta_0, D^\beta\hat{\varphi}\rangle = \langle \delta_0, \mathcal{F}((2\pi i\cdot)^\beta\varphi)\rangle$$

$$= \int e^{-i2\pi\xi x}(2\pi i x)^\beta \varphi(x)\,dx\Big|_{\xi=0} = \int_{\mathbb{R}} (2\pi i x)^\beta\varphi(x)\,dx$$

Conclusion: $\mathcal{F}(D^\beta\delta_0) = (2\pi i\cdot)^\beta$

$\beta = 0$ $\quad \mathcal{F}(\delta_0) = 1$

$\delta_0 \notin L_1$

$\hat{\delta}_0 = 1 \in L_\infty$

Repetation #1

For $f \in L^2, L^1(\mathbb{R}), S$ (Schwartz class) we define

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx \,, \qquad f(x) \supset \hat{f}(\xi)$$

<u>properties</u> $f \longmapsto \hat{f}$ linear

$$\frac{1}{a} f\left(\frac{\cdot}{a}\right) \supset \hat{f}(a\xi) \,; \quad a > 0.$$

$$e^{-\pi x^2} \supset e^{-\pi \xi^2}$$

Hausdorff-Young's $\leq$

$$f \in L^p \ (1 \le p \le 2) \Rightarrow \hat{f} \in L^q \,, \quad \frac{1}{p} + \frac{1}{q} = 1 .$$

<u>Convolution</u>: $f * g\,(x) = \int_{\mathbb{R}} f(x-y) g(y)\,dy \supset \hat{f}(\xi)\hat{g}(\xi)$.

[is commutative, associative & distributive]

$$Df \supset 2\pi i \xi \hat{f}(\xi)$$

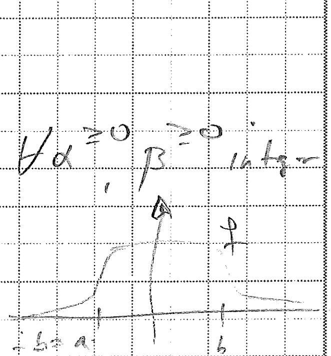
$$-2\pi i (\cdot) f \supset D\hat{f}$$

$$\tau_a f \supset e^{-2\pi i a \xi} \hat{f}(\xi)$$

$$e^{2\pi i a x} f(x) \supset \tau_a \hat{f}$$

<u>Def</u>. $f : \mathbb{R} \to \mathbb{C}$ is Schwartz class S if $\forall \alpha \ge 0, \beta \ge 0$ integer

$$\sup_{x \in \mathbb{R}} |x^\alpha D^\beta f(x)| < \infty .$$



Ex. $\underline{e^{-x^2}}$, $\left(e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}}\right)\left[\theta(x-b) - \theta(x-a)\right]$

Ex. Let $g \in L^1$, ~~cont~~ at $x = \cancel{b}$, Then

$$f_\varepsilon(x) := \int_{\mathbb{R}} g(y) \frac{1}{\varepsilon} f\left(\frac{x-y}{\varepsilon}\right) dx \longrightarrow g(x) \text{ as } \varepsilon \to 0$$

i.e. $g * \frac{1}{\varepsilon} f\left(\frac{\cdot}{\varepsilon}\right) \longrightarrow g(x)$ a.e. as $\varepsilon \to 0$

Theorem The Fourier inversion formula for tempered distributions:

Let $\check{\varphi}(x) = \varphi(-x)$ when $\varphi \in S$, and let $\check{T}$ be defined by $\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle$ for $T \in S'$. Then for all $T \in S'$

$$\mathcal{F}\mathcal{F}T = \check{T}.$$

Pf. $\check{\varphi}(x) = \varphi(-x) = \int_{\mathbb{R}} e^{2\pi i \xi(-x)} \hat{\varphi}(\xi)\, d\xi = \mathcal{F}\hat{\varphi}(x) = \mathcal{F}\mathcal{F}\varphi(x)$.

Hence

$$\langle \mathcal{F}\mathcal{F}T, \varphi \rangle = \langle \mathcal{F}T, \mathcal{F}\varphi \rangle = \langle T, \mathcal{F}\mathcal{F}\varphi \rangle = \langle T, \check{\varphi} \rangle = \langle \check{T}, \varphi \rangle.$$

Further properties of the Fourier transform of tempered distribution

Proposition

1) The Fourier transform is linear

$$\mathcal{F}(T_1 + T_2) = \mathcal{F}(T_1) + \mathcal{F}(T_2)$$

$$\mathcal{F}(cT_1) = c\mathcal{F}(T_1).$$

2) Let $T \in S'$ and $f \in C^\infty$, such that for all $\beta > 0$ there is α such that

$$\sup_{x \in \mathbb{R}} \left((1+x^2)^{-\alpha} |D^\beta f(x)| \right) < \infty$$

Then

- $\mathcal{F}(DT) = 2\pi i(\cdot)\mathcal{F}(T)$
- $\mathcal{F}(-2\pi i(\cdot))T) = D\hat{T}$
- $\mathcal{F}(fT) = \hat{f} * \hat{T}$ (This is a definition?)
- $\mathcal{F}(\hat{f} * T) = \check{f}\mathcal{F}(T)$
- $\mathcal{F}(\tau_s T) = e^{-2\pi i s(\cdot)}\hat{T}$
- $\mathcal{F}(e^{2\pi i(\cdot)s} T) = \tau_s \hat{T}$

and moreover

$$D(\hat{f} * T) = (D\hat{f}) * T = \hat{f} * DT$$

Pf. All follows directly from the defs. except that we have not yet defined the convolution for distributions.

Recall that if $\varphi_1, \varphi_2 \in S$. Then

$$\mathcal{F}(\varphi_1 * \varphi_2) = \hat{\varphi}_1 \hat{\varphi}_2 .$$

If $\hat{T} \in S'$ and $\hat{f} \in S'$ and $f \in C^\infty$ and has moderate growth, then $fT \in S'$ (because we can multiply distributions and functions)

Hence we can <u>define</u> $\hat{f} * \hat{T}$ by

$$\hat{f} * \hat{T} = \mathcal{F}(fT), \text{ i.e.}$$

$$\langle \hat{f} * \hat{T}, \varphi \rangle = \langle \mathcal{F}(fT), \varphi \rangle = \langle fT, \hat{\varphi} \rangle .$$

Taking Fourier transforms, we find

$$\check{f}\check{T} = \mathcal{F}(\hat{f} * \hat{T}) \iff f\hat{T} = \mathcal{F}(\hat{f} * T) \qquad (**)$$

Also

$$\mathcal{F}(D(\hat{f} * T)) = \underbrace{(2\pi i(\cdot)) f \hat{T}}_{= \mathcal{F}(D\hat{f})} = \mathcal{F}(D\hat{f} * T) = \mathcal{F}(\hat{f} * DT).$$

<u>Differential Equations</u>

Recall that if $f \in C^1(\mathbb{R})$ and $f' \equiv 0$

then f is a constant.

What can we say if $T \in S'(\mathbb{R})$ and

$DT = 0$ in $S'(\mathbb{R})$?

$(**)$ $\cancel{(\check{f}\hat{T}, \varphi) \neq (\hat{T}, \check{f}\varphi) = (T, (\check{f}\varphi)^\wedge) = (T, \hat{\check{f}} * \hat{\varphi})}$

Lemma Let $T \in S'$, and assume that $(\cdot)T = 0$ in S'. Then there is $a \in \mathbb{C}$ such that $T = a\delta$.

Proof. Let $\psi \in S$ with $\psi(0) = 0$, and let $\varphi(x) = \frac{\psi(x)}{x}$.

Then $\varphi \in S(\mathbb{R})$. For $T \in S'(\mathbb{R})$

(by assumption)

$$T(\psi) = T(\underset{\uparrow x}{(\cdot)}\varphi) = \underbrace{\underset{\uparrow x}{(\cdot)}\, T(\varphi)}_{} = 0 \qquad (*)$$

note: This should be interpreted as

$$[(\cdot)T](\varphi) = \langle (\cdot)T, \varphi \rangle.$$

Next we take $\varphi \in S$ arbitrary, and fix $\varphi_1 \in S$ with $\varphi_1(0) = 1$. We can write

$$\varphi(x) = \underbrace{\varphi(x) - \varphi(0)\varphi_1(x)}_{\equiv \psi(x) \;\leftarrow\; \psi(0) = 0} + \varphi(0)\varphi_1(x)$$

$a \in \mathbb{C}$
$(\cdot)T = 0 \Rightarrow T = a\delta$
$DT = 0 \Rightarrow \begin{cases} \hat{T} = a\delta \\ T = a \end{cases}$

Then

$$T(\varphi) = T(\psi) + T(\varphi(0)\varphi_1) \overset{(*)}{=} T(\varphi_1)\,\varphi(0) \quad \text{so}$$

$$T = T(\varphi_1)\delta.$$

Note that the result depends on the choice of φ_1, but in the C^1 case, $f' \equiv 0$ implies that f is a constant but it does not say which constant.

Corollary Let $T \in S'(\mathbb{R})$ and assume that $DT = 0$. Then $\hat{T} = a\delta$ for some $a \in \mathbb{C}$.

proof:

$$0 = \mathcal{F}(DT) = (2\pi i\,(\cdot))\,\hat{T}$$

$\Rightarrow \hat{T} = a\delta$ for some $a \in \mathbb{C}$

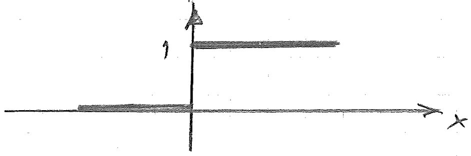
$\Rightarrow T = a$, a constant.

(just think "evil"
$2\pi i(x)\delta(x) = 0$.
$2\pi i(x)\, a\delta(x) = 0$
$\hat{T}$ should be ~~unique~~ $a\delta$ for some a)

$\langle T, \varphi\rangle = \langle \hat{T}, \check{\varphi}\rangle = \langle a\delta, \check{\varphi}\rangle = a\check{\varphi}(0) = a\int\varphi = \langle a, \varphi\rangle$

<u>Example</u> Let H be the "Heaviside" step function

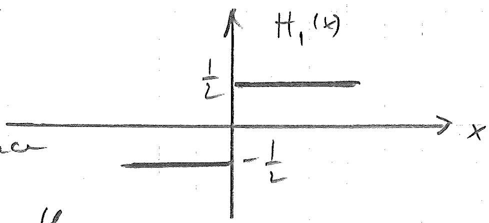
$$H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases}$$



Then $\mathcal{F} H = \dfrac{1}{2\pi i(\cdot)} + \dfrac{1}{2}\delta$, where

$$\left\langle \frac{1}{\cdot}, \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} \frac{1}{2\pi i x} \varphi(x)\, dx \quad \text{(The Cauchy principal value)}$$

<u>Pf.</u> Let $H_1 = H - \frac{1}{2}$



Then $DH_1 = DH = \delta$, and hence

$2\pi i(\cdot)\hat{H}_1 = 1$ and consequently

$$\hat{H}_1 = \frac{1}{2\pi i(\cdot)} + a\delta$$

(why do you have to add $a\delta$? because $DT = 0 \Rightarrow \hat{T} = a\delta$)

But H_1 is odd $\Rightarrow \mathcal{F} H_1$ is odd.

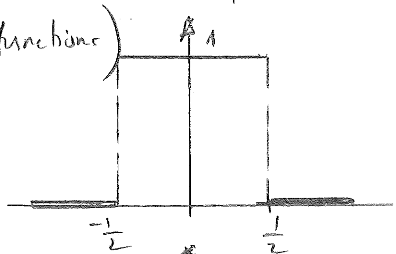
And $\frac{1}{(\cdot)}$ is odd, δ is even, and hence for $a = 0$.

Then $\hat{H} = \mathcal{F}(H_1 + \frac{1}{2}) = \dfrac{1}{2\pi i(\cdot)} + \dfrac{1}{2}\delta$.

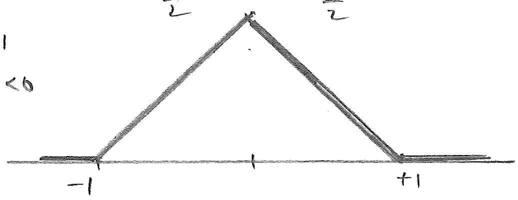
qed.

<u>Common notation</u> (Generalized functions)

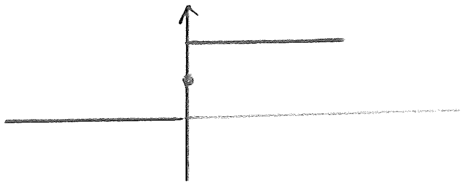
$$\Pi(x) = \begin{cases} 1 & \text{when } |x| < 1/2 \\ 0 & \text{otherwise} \end{cases}$$



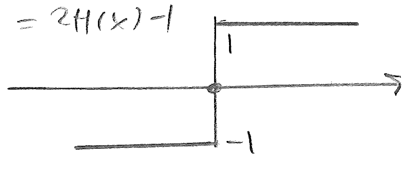
$$\Lambda(x) = \begin{cases} 1 - x & \text{when } 0 \le x < 1 \\ x + 1 & \text{when } -1 < x < 0 \\ 0 & \text{else} \end{cases}$$



$$H(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x < 0 \end{cases}$$



$$\operatorname{sgn}(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases} = 2H(x) - 1$$



$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

Note check that $\int_{-\infty}^{\infty} \operatorname{sinc} x \, dx = 1$,
if the divergent integral is properly defined.

$$\mathcal{F}(\Pi) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-2\pi i x \xi} \, dx = \frac{e^{\pi i \xi} - e^{-\pi i \xi}}{2\pi i \xi} = \frac{\sin \pi \xi}{\pi \xi} = \operatorname{sinc} \xi$$

Both Π and sinc are even, so

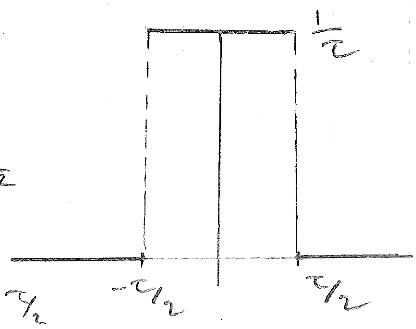
$\mathcal{F}\mathcal{F}(\Pi) = \mathcal{F}(\operatorname{sinc})$

$\check{\Pi} = \mathcal{F}(\operatorname{sinc})$

Π ← even

$$\Pi = \mathcal{F}(\operatorname{sinc}(\cdot)), \text{ and}$$

$$\Pi(0) = \int_{-\infty}^{\infty} \mathcal{F}(\Pi) \, d\xi = \int_{-\infty}^{\infty} \frac{\sin \pi \xi}{\pi \xi} \, d\xi .$$



Distributions and generalized functions

Let $\Pi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{elsewhere} \end{cases}$

Then $\frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right) = \begin{cases} \frac{1}{\tau} & \text{when } |x| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$

Then $\int_{-\infty}^{\infty} \frac{1}{\tau} \Pi\left(\frac{x}{\tau}\right) dx = \int_{-\infty}^{\infty} \Pi(x) \, dx = 1 \qquad \left(\frac{x}{\tau} \leadsto x\right)$

and for $\varphi \in C(\mathbb{R})$.

On the other hand

$$\int_{-\infty}^{\infty} \frac{1}{\tau} \Pi\left(\frac{x}{\tau}\right) \varphi(x) \, dx = \int_{-\infty}^{\infty} \Pi(x) \varphi(\tau x) \, dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\tau x) \, dx \to \varphi(0)$$

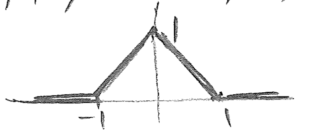
when $\tau \to 0$.

because $\varphi(\tau x) \to \varphi(0)$ uniformly in $-\frac{1}{2} \le x \le \frac{1}{2}$.

Hence we can see δ as the limit of $\frac{1}{\tau}\Pi\left(\frac{\cdot}{\tau}\right)$ when $\tau \to 0$.

Def. Let T_n be a family of distributions, $T_n \in S'(\mathbb{R})$.
we say that $T_n \to T$ in S if for every $\varphi \in S(\mathbb{R})$,

$$\lim_{n \to \infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle .$$

Let $\Lambda(x) =$  , Then as before

$$\int_{-\infty}^{\infty} \frac{1}{\tau} \Lambda\left(\frac{x}{\tau}\right) \varphi(x) \, dx = \int_{-\infty}^{\infty} \Lambda(x) \varphi(\tau x) \, dx \to \varphi(0) \text{ as } \tau \to 0.$$

There are many other examples

$$\int_{-\infty}^{\infty} \frac{1}{\tau} e^{-\pi\left(\frac{x}{\tau}\right)^2} \varphi(x)\,dx \longrightarrow \varphi(0)$$

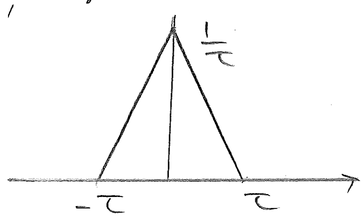
$$\int_{-\infty}^{\infty} \frac{1}{\tau} \operatorname{sinc}\left(\frac{x}{\tau}\right) \varphi(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\tau} \frac{\sin(\pi x/\tau)}{\pi x/\tau} \varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \varphi(\tau x)\,dx .$$

The last one is more difficult to prove, because $\operatorname{sinc}(x)$ is <u>not</u> absolutely integrable, but rather it is necessary to rely on cancellations due to the oscillatory behaviour of $\sin(\pi x)$.

<u>What about derivatives of generalized functions?</u>

With $\delta = \lim_{\tau \to 0} \frac{1}{\tau} \Pi\left(\frac{\cdot}{\tau}\right)$ in S', you can't do much

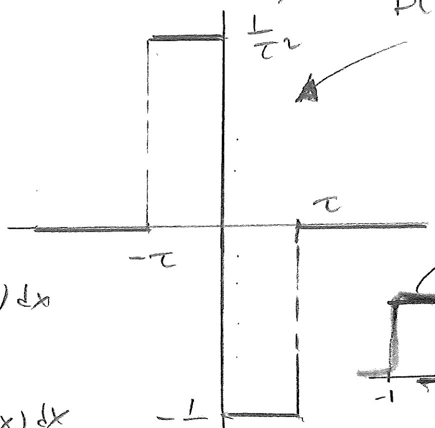


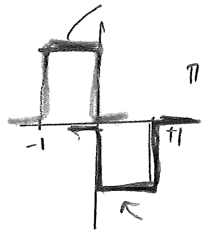
but for

$$\delta = \lim_{\tau \to 0} \frac{1}{\tau} \Lambda\left(\frac{\cdot}{\tau}\right)$$

you have

$$D\left(\frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\right) = \frac{1}{\tau^2}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right) - \Pi\left(\frac{x-\tau/2}{\tau}\right)\right)$$





Hence

$$\int_{-\infty}^{\infty} D\left(\frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\right)\varphi(x)\,dx =$$

$$= \int_{-\infty}^{\infty}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right) - \Pi\left(\frac{x-\tau/2}{\tau}\right)\right)\frac{1}{\tau^2}\varphi(x)\,dx$$

$$x \to \tau x \quad = \int_{-\infty}^{\infty}\left(\Pi\left(x+\tfrac{1}{2}\right) - \Pi\left(x-\tfrac{1}{2}\right)\right)\frac{1}{\tau}\varphi(\tau x)\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\varphi(\tau(x-\frac{1}{2})) - \varphi(\tau(x+\frac{1}{2}))}{\tau}\,dx \underset{\uparrow}{=} \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{\tau}\left(\varphi'(0)\left(\tau\left(x-\tfrac{1}{2}\right) - \tau\left(x+\tfrac{1}{2}\right)\right) + O(\tau^2)\right)dx$$

$$= \varphi'(0)\int_{-1/2}^{1/2}(-1+O(\tau))\,dx \xrightarrow{\text{Taylor}} -\varphi'(0), \quad \text{when } \tau \to 0.$$

But we also have $\langle D\delta, \varphi \rangle = -\varphi'(0)$, so

it seems that $D\left(\frac{1}{\tau}\Lambda\left(\frac{\cdot}{\tau}\right)\right) \longrightarrow \delta'$ in S'.

Is this always true? Could you compute

$$\lim_{\tau \to 0} D\left(\frac{1}{\tau}\operatorname{sinc}\left(\frac{x}{\tau}\right)\right) ?$$

The great advantage with studying S and S' is that you move all difficult operations to $\varphi \in S$.

The poisson summation formula

Let $\varphi \in S$. Then $\displaystyle\sum_{k\in\mathbb{Z}} \hat{\varphi}(k+\xi) = \sum_{k\in\mathbb{Z}} \varphi(k) e^{-2\pi i k \xi}$ $(*)$

Note The left hand side is periodic with period 1, and this is true also for the RHS, because $e^{-2\pi i k \xi}$ is 1 periodic.

Proof: Recall that $\mathcal{F}\left(e^{2\pi i (\cdot) s}\varphi\right) = \tau_s \hat{\varphi}$ and

that $e^{-2\pi i s(\cdot)}\hat{T} = \mathcal{F}(\tau_s T)$.

Here $\varphi \in S$ and $T \in S'$

Hence $\displaystyle\sum_{k\in\mathbb{Z}} \hat{\varphi}(k+\xi) = \sum_{k\in\mathbb{Z}} \tau_{-\xi}\hat{\varphi}(k) = \sum_{k\in\mathbb{Z}} \mathcal{F}\left(e^{-2\pi \xi (\cdot)}\varphi\right)(k)$

$$= \sum_{k\in\mathbb{Z}} \left\langle \delta_k, \mathcal{F}\left(e^{-2\pi i \xi(\cdot)}\varphi\right)\right\rangle = \left\langle \sum_{k\in\mathbb{Z}} \delta_k, \mathcal{F}\left(e^{-2\pi i \xi(\cdot)}\varphi\right)\right\rangle$$

$$= \sum_{k\in\mathbb{Z}} \varphi(k) e^{-2\pi i k \xi}$$

$\langle T, \hat{W}\rangle = \langle \mathcal{F}T, W\rangle$

So the left hand side of $(*)$ is: $\left\langle \mathcal{F}\left(\sum_{k\in\mathbb{Z}}\delta_k\right), e^{-2\pi i k(\cdot)}\varphi\right\rangle$

and the right hand side is $\left\langle \sum_k \delta_k, e^{-2\pi i k(\cdot)}\varphi\right\rangle$.

We need to prove that $\mathcal{F}\left(\sum_k \delta_k\right) = \sum_k \delta_k$,

i.e. that $\sum_k \delta_k$ is a fixed point of $\mathcal{F}$.

But this follows from

1) $\tau_{\pm 1}\left(\sum_{k\in\mathbb{Z}} \delta_k\right) = \sum_{k\in\mathbb{Z}} \delta_k$ (This is a translation invariant distribution, for integer translates)

2) $e^{2\pi i(\cdot)} \sum_{k\in\mathbb{Z}} \delta_k = \sum_{k\in\mathbb{Z}} \delta_k$ (because the exponential is equal to 1 at all integer points)

Taking Fourier transform gives

integer

$\hat{1})$ $\mathcal{F}\left(\tau_1\left(\sum_{k\in\mathbb{Z}} \delta_k\right)\right) = e^{-2\pi i(\cdot)} \mathcal{F}\left(\sum_{k\in\mathbb{Z}} \delta_k\right) = \mathcal{F}\left(\sum_k \delta_k\right)$

$\hat{2})$ $\mathcal{F}\left(e^{2\pi i(\cdot)} \sum_{k\in\mathbb{Z}} \delta_k\right) = \tau_1\left(\mathcal{F}\left(\sum_k \delta_k\right)\right) = \mathcal{F}\left(\sum_k \delta_k\right)$

From $\hat{1})$ we find that

$$0 = \left(e^{-2\pi i(\cdot)} - 1\right) \mathcal{F}\left(\sum_k \delta_k\right) = \frac{e^{-2\pi i(\cdot)} - 1}{(\cdot)} (\cdot)\, \mathcal{F}\left(\sum_k \delta_k\right).$$

But $\dfrac{e^{-2\pi i\xi} - 1}{\xi} \longrightarrow -2\pi i \neq 0$ when $\xi \to 0$, (L'Hôpital). So

$$(\cdot)\,\mathcal{F}\left(\sum_k \delta_k\right) = 0 \quad \Rightarrow \quad \mathcal{F}\left(\sum_k \delta_k\right) = a\delta \quad \text{for some } a \in \mathbb{C}.$$

From $\hat{2})$ we have that $\tau_{\pm 1}\left(\mathcal{F}\left(\sum_k \delta_k\right)\right) = \mathcal{F}\left(\sum_k \delta_k\right)$, i.e. the expression can be translated by integers. But then

$$\mathcal{F}\left(\sum_k \delta_k\right) = a \sum_k \delta_k .$$

What is the constant ?

$$\left\langle \mathcal{F}\left(\sum_k \delta_k\right), \varphi \right\rangle = a \left\langle \sum \delta_k, \varphi \right\rangle \quad \text{and}$$

$$= \left\langle \sum_k \delta_k, {}^t\mathcal{F}\varphi \right\rangle, \qquad \text{with } \varphi = e^{-\pi x^2}, \quad \varphi = \hat{\varphi},$$

we have that $a = 1$. qed.

Example

If $\varphi \in \mathcal{S}$ and $\hat{\varphi}(\xi) = 0$ when $|\xi| > 1$, then $\sum_k \varphi(k) = \int_{\mathbb{R}} \varphi(x)\, dx$,

because

$$\sum_k \varphi(k) = \left\langle \sum \delta_k, \varphi \right\rangle = \left\langle \mathcal{F}\left(\sum \delta_k\right), \varphi \right\rangle = \left\langle \sum \delta_k, \mathcal{F}\varphi \right\rangle$$

$$\underset{\uparrow}{=} \left\langle \delta_0, \hat{\varphi} \right\rangle = \hat{\varphi}(0) = \langle 1, \varphi \rangle = \int_{\mathbb{R}} \varphi(x)\, dx .$$

because $\hat{\varphi}(\xi) = 0$ for $|\xi| > 1$