Sampling

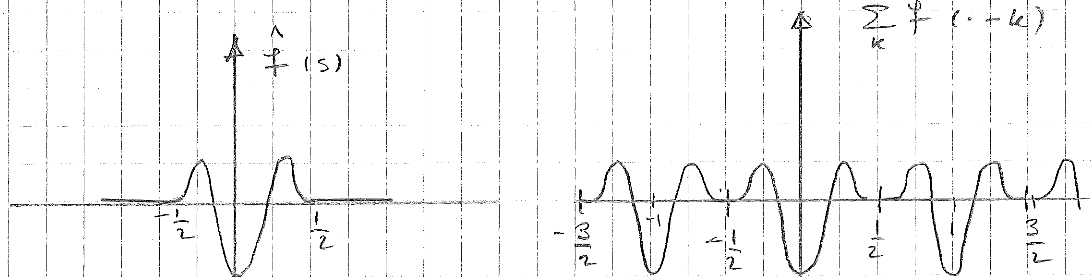
Definition
$$\operatorname{Sinc} x = \begin{cases} \frac{\sin \pi x}{\pi x}, & \text{for } x \neq 0 \\ 1, & x = 0 \end{cases}$$

$$\Pi(x) = 1\!\!1_{]-\frac{1}{2},\frac{1}{2}[} = \begin{cases} 1, & \text{for } |x| < \frac{1}{2} \\ 0, & \text{else.} \end{cases}$$

Note: $\mathcal{F}(\Pi) = \operatorname{Sinc}(\cdot)$

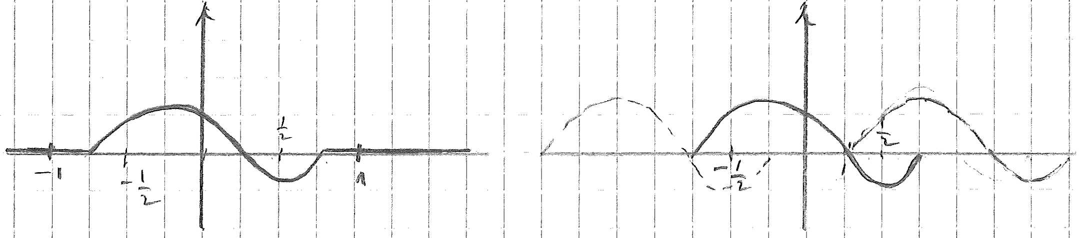
Theorem Let $f \in C^\infty$ be a function with moderate growth (i.e. $\forall \beta \in \mathbb{N}, \exists \alpha:\ \sup_x |(1+|x|)^{-\alpha} D^\beta f(x)| < \infty$).
Assume in addition that $\hat{f}(s) = 0$ for $|s| \geq \frac{1}{2}$. Then
$$f = \operatorname{Sinc} * \left(\sum_{k=-\infty}^{\infty} f(k)\delta_k\right) = \sum_{k=-\infty}^{\infty} f(k)\operatorname{sinc}(\cdot - k).$$

Pf. Note that $\sum_{k=-\infty}^{\infty} f(k)\delta_k = f \sum_{k=-\infty}^{\infty} \delta_k$ in $\mathcal{S}'$. By $\mathcal{F}$-transf we then get
$$\mathcal{F}\left(\sum_k f(k)\delta_k\right) = \hat{f} * \mathcal{F}\left(\sum_k \delta_k\right) = \hat{f} * \sum_k \delta_k = \sum_k \hat{f}(\cdot - k).$$
The right hand side is periodic with period 1, because it is the sum of all translated copies of $\hat{f}$



Note that because $\hat{f}(s) = 0$ outside $|s| \leq \frac{1}{2}$, the translated copies do not overlap, and hence $\hat{f}(s) = \left(\sum_k \hat{f}(s-k)\right)\Pi(s)$.
So then
$$\begin{aligned}\hat{f}(s) &= \left(\sum_k \hat{f}(s-k)\right)\Pi(s) \\ &= \mathcal{F}\left(\sum_k f(k)\delta_k\right)(s) \cdot \mathcal{F}(\operatorname{Sinc})(s) = \mathcal{F}\left(\operatorname{Sinc} * \sum_k f(k)\delta_k\right)(s)\end{aligned}$$
By inverse Fourier transform we get the result.
$$f = \sum_{k=-\infty}^{\infty} f(k)\operatorname{Sinc}(\cdot - k). \quad \square$$

Aliasing If $\hat{f}(s) \neq 0$ for $|s| \geq \frac{1}{2}$. The translated copies will overlap.



Then $\hat{f}(s) \neq \left(\sum \hat{f}(s-k)\right)\Pi(s)$.

This gives an error if one tries to recover a sampled signal by convolution with sinc: It is called aliasing.

The discrete Fourier transform

Proposition. Let $\hat{T} \in S'$, and assume $\hat{T}_1 = \hat{T}$, i.e. $\hat{T}$ is periodic with period 1. Assume that also T is periodic. Then T has integer period, $T_N = T$ and

$$T = \sum_k t_k \delta_k \qquad \hat{T} = \sum_k c_k \delta_{k/N}$$

The two sequences t_k and c_k are periodic with period N, and are related by the discrete Fourier transform:

$$Nc_k = \sum_{l=1}^{N} t_l e^{-2\pi i k l/N} \qquad t_l = \sum_{k=1}^{N} c_k e^{2\pi i k l/N}$$

Note The discrete Fourier transform can be defined independently of T & $\hat{T}$.

Pf. $\hat{T}_1 = \hat{T}$ implies that $T = \sum_{k=-\infty}^{\infty} t_k \delta_k$

for some $t_k \in \mathbb{C}$ (by a previous lemma).

If T is periodic, $T_a = T$, a must be an integer $a = N \in \mathbb{N}$.

it follows that

$$\hat{T} = \sum_{k=-\infty}^{\infty} c_k \delta_{k/N}$$

[by the same argument as before: $T_a = T \Rightarrow e^{-2\pi i a \sigma} \hat{T} = \hat{T}$

$\Rightarrow \left(e^{-2\pi i a \sigma} - 1\right)\hat{T} = 0$.

and therefore $\hat{T}$ must be concentrated on the point k/a, $k \in \mathbb{Z}$.]

Then $\mathcal{F}(T) = \mathcal{F}\left(\sum_k t_k \delta_k\right) = \tilde{\mathcal{F}}\left(\sum_{k=1}^{N} t_k \sum_{l=-\infty}^{\infty} \delta_{k+lN}\right)$ (because the sequence $\{t_k\}$ is periodic)

$$= \sum_{k=1}^{N} t_k \mathcal{F}\left(\sum_{l=-\infty}^{\infty} \delta_{k+lN}\right) = \sum_{k=1}^{N} t_k e^{-2\pi i k(\cdot)} \mathcal{F}\left(\sum_{l=-\infty}^{\infty} \delta_{lN}\right)$$

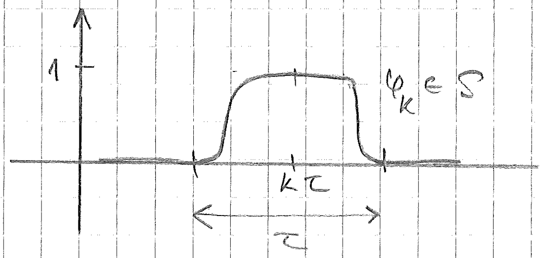
$$= \sum_{k=1}^{N} t_k e^{-2\pi i k(\cdot)} \frac{1}{N} \sum_{l=-N}^{N} \delta_{l/N} = \sum_{k=1}^{N} t_k e^{-2\pi i k\, l/N} \frac{1}{N} \sum_{l=-\infty}^{\infty} \delta_{l/N}$$

$$= \sum_{l=-\infty}^{\infty} \underbrace{\left(\frac{1}{N} \sum_{k=1}^{N} t_k e^{-2\pi i k\, l/N}\right)}_{= C_k} \delta_{l/N}, \qquad \text{Thus } NC_k = \sum_{k=1}^{N} t_k e^{2\pi i k l/N}.$$

<u>Periodic functions</u> Assume that $f \in S$ then $g(x) = \sum_{m=-\infty}^{\infty} f(x - m\tau)$ is periodic with period τ. As a distribution, $g \in S'$, we find that

$$\hat{g} = \sum_{k=-\infty}^{\infty} c_k \delta_{k/\tau} \quad \text{and}$$

$$g = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i \frac{k}{\tau}(\cdot)}.$$



We can compute the c_k by

$$c_k = \hat{g}(\varphi_k) = g(\hat{\varphi}_k) = \int_{-\infty}^{\infty} g(x) \hat{\varphi}_k(x)\, dx = \int_{-\infty} \sum_{m=-\infty}^{\infty} f(x - m\tau) \hat{\varphi}_k(x) dx$$

i.e. $\langle \hat{g}, \varphi_k \rangle$

$$= \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x - m\tau) \hat{\varphi}_k(x) dx = \sum_{m=-\infty}^{\infty} \int_{-\infty}^{\infty} f(x) \hat{\varphi}_k(x + m\tau)\, dx$$

$(x = \tau y)$

$$= \int_{-\infty}^{\infty} f(x) \sum_{m=-\infty}^{\infty} \hat{\varphi}_k(x + m\tau)\, dx = \int_{-\infty}^{\infty} \tau f(\tau y) \sum_{m=-\infty}^{\infty} \hat{\varphi}_k(\tau(y+m))\, dy =$$

$$= \int_{-\infty}^{\infty} \tau f(\tau y) \sum_{m=-\infty}^{\infty} \mathcal{F}\left(\frac{1}{\tau} \varphi_k\left(\frac{\cdot}{\tau}\right)\right)(y+m)\, dy$$

$$= \int_{-\infty}^{\infty} f(\tau y) \sum_{m=-\infty}^{\infty} \underbrace{\varphi_k\left(\frac{m}{\tau}\right)}_{=0 \text{ when } m \neq k} e^{-2\pi i y m}\, dy = \int_{-\infty} f(\tau y) e^{-2\pi i y k} = \frac{1}{\tau} \hat{f}\left(\frac{k}{\tau}\right)$$

So this is method for calculating Fourier Coefficients of certain periodic functions.

Convergence of Fourier Series

Thm. Suppose that f has period 1 and $f \in C^2$. Then,

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}, \quad \text{for all } x \in \mathbb{R}, \quad (*)$$

where
$$c_k = \int_0^1 e^{-2\pi i k x} f(x)\,dx$$

Pf.
$$c_k = \int_0^1 e^{-2\pi i k x} f(x)\,dx = \{\text{twice p I}\} = \int_0^1 \frac{e^{-2\pi i k x}}{(-2\pi i k)^2} f''(x)\,dx$$
$$= O\left(\frac{1}{k^2}\right)$$

So the series $(*)$ is absolutely convergent.

Hence $f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x}$ is continuous, and belongs to S'.

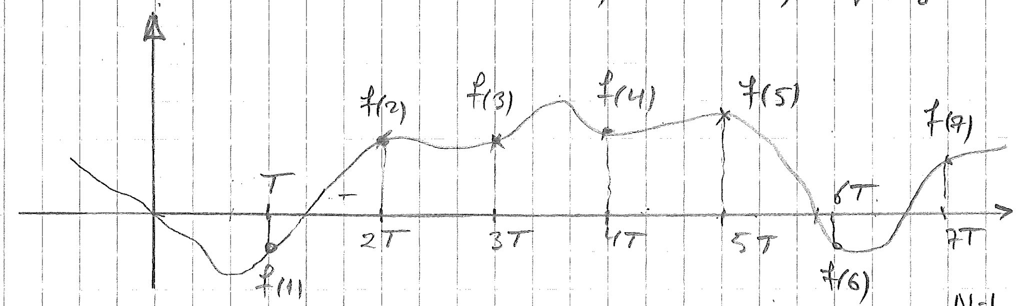
Therefore also $\hat{f} = \mathcal{F}\left(\sum c_k e^{2\pi i k \cdot}\right) \in S'$, and

$$\langle \hat{f}, \varphi \rangle = \langle f, \hat{\varphi} \rangle = \int_{-\infty}^{\infty} f(x)\hat{\varphi}(x)\,dx = \int_0^1 f(x) \sum_{k=-\infty}^{\infty} \hat{\varphi}(x+k)\,dx$$

$$= \int_0^1 f(x) \sum_{k=-\infty}^{\infty} \hat{\varphi}(k) e^{-2\pi i k x}\,dx = \sum_{k=-\infty}^{\infty} \varphi(k) c_k = \left\langle \sum c_k \delta_k, \varphi \right\rangle$$

The DFT and the FFT

[The notation in this part is from Bracewell. Hence T is a "sampling interval", $\tau \in \{0, 1, \dots, N-1\}$, and we write for a function $V(t)$, $f(\tau) = V(t_0 + \tau T)$.]

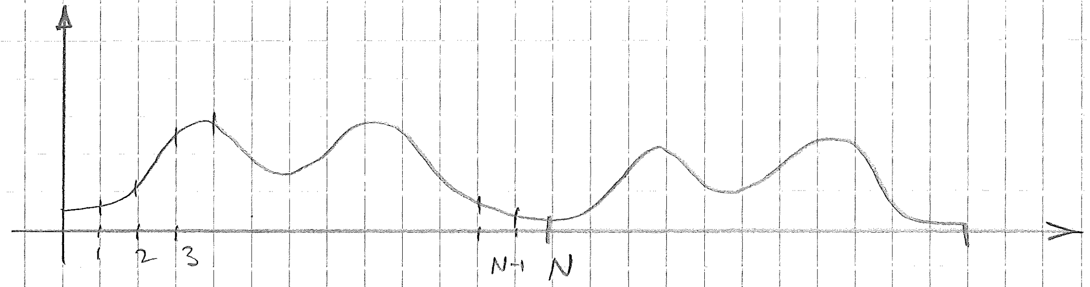


Definition f has discrete Fourier transform $F(\nu) = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau) e^{-2\pi i \nu \tau / N}$

proposition f can be recovered by $f(\tau) = \sum_{\nu=0}^{N-1} F(\nu) e^{2\pi i \nu \tau / N}$.

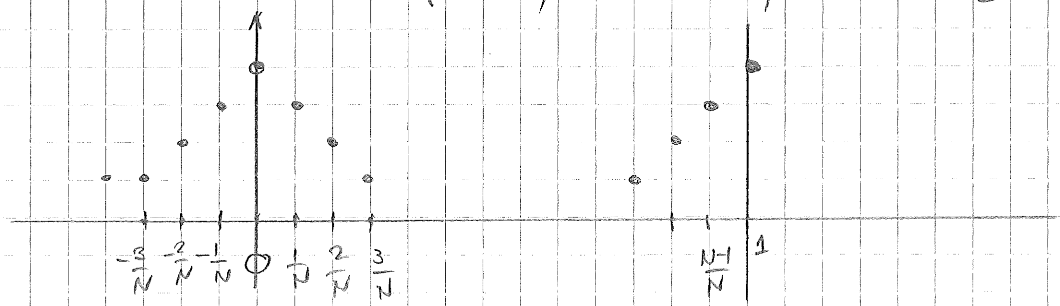
Pf. By direct calculation.

<u>Interpretation</u>



$f(t)$ is periodic with period N, sampling frequency 1

$\hat{f}(\omega)$ has period 1 (and this implies that $\hat{f}(\omega)$ can only be represented up to $|\omega| < \frac{1}{2}$)



It is more natural to consider

$$F_{\frac{N}{2}+1}, \ldots, F_{N-1} \quad \text{as} \quad F_{-\frac{N}{2}+1}, \ldots, F_{-1}.$$

Also if $f(\tau)$ is real, then

$$F(N-\nu) = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau)\, e^{-2\pi i (N-\nu)\tau/N} = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau)\, e^{-2\pi i(-\nu)\tau/N} = F(-\nu) = \overline{F(\nu)}.$$

Hence, there are only N independent <u>real</u> coefficients.

<u>Cyclic convolution</u>: For f and g 2π-periodic, we define

$$f * g(\theta) = \int_0^{2\pi} f(x)\, g(x-\theta)\, dx$$

and for two N-periodic sequences:

$$f * g(\tau) = \sum_{\tau'=0}^{N-1} f(\tau')\, g(\tau-\tau').$$

<u>Note</u>. If f and g are not considered periodic, but only defined on $0, 1, \ldots, N-1$ one should write

$$f * g(\tau) = \sum_{\tau'=0}^{N-1} f(\tau)\, g\big(\tau - \tau' + N H(\tau'-\tau)\big),$$

where H is the Heaviside function.

Computational rules we write $f \supset F$

Note We may write $F(\nu) = \frac{1}{N} \sum_{\tau=-N/2}^{N/2-1} f(\tau)\, e^{-2\pi i \nu\tau/N}$

and $f(\tau) = \sum_{\nu=-N/2}^{\frac{N}{2}-1} F(\nu)\, e^{2\pi i \nu\tau/N}$

provided that N is <u>even</u>.

Rules for Computation

$f(\tau) \supset F(\nu)$ (recall $\tau, \nu \in \{0, 1, \dots, N-1\}$)

$F(\nu) \supset \frac{1}{\nu} f(-\tau)$

$f^*(\nu) \supset F^*(-\nu)$ (for complex sequences)

$f_1 * f_2(\nu) = N F_1(\nu) F_2(\nu)$

[discrete convolution: $f_1 * f_2(\nu) = \sum_{k=0}^{N-1} f_1(\nu-k) f_2(k)$

and here $\nu - k$ must be considered mod $N-1$, i.e. counting

$0 \;\; 1 \;\; 2 \cdots N-1, \; \underset{0}{\overset{\shortparallel}{N}}, \; 1, 2 \cdots$]

$f_1(\tau) f_2(\tau) \supset \sum_{\nu'=1}^{N-1} F_1(\nu') F_2(\nu-\nu')$

$f(0) = \sum_{\nu=0}^{N-1} F(\nu)$

$$\sum_{\tau=0}^{N-1} |f(\tau)|^2 = N \sum_{\nu=0}^{N-1} |F(\nu)|^2 .$$

The packing theorem

The operator $Pack_K$ is defined by

$$f = \{f(0), f(1), \dots, f(N-1)\} \longmapsto$$

$$Pack_K\, f = \{\underbrace{\underbrace{f(0), f(1), \dots, f(N-1)}_{N}, \underbrace{0, 0, \dots, 0}_{N}, \dots, 0, \underbrace{0, 0, 0, \dots, 0}_{N}}_{K \text{ times}}\}.$$

We then have

$$g = Pack_K(f(\tau)) \stackrel{\mathcal{F}}{\supset} G(\nu), \qquad \nu = 0, \dots, NK-1$$

where $G(\nu) = \begin{cases} \frac{1}{K} F\left(\frac{\nu}{K}\right), & \nu = 0, \dots, NK-K \\ 0 & \text{else ?} \end{cases}$

Pf

$$G(\nu) = \frac{1}{KN} \sum_{\tau=0}^{KN-1} g(\nu)\, e^{-2\pi i \tau \nu / KN}$$

$$= \frac{1}{KN} \sum_{\tau=0}^{N-1} f(\nu)\, e^{-2\pi i \tau \nu / KN}$$

$$= \frac{1}{K} F\left(\frac{\nu}{K}\right), \qquad \text{where } \nu = 0, K, 2K, \dots$$

Similarity theorem

$Stretch_K\, f$ (or upsampling) is defined as

$$g(\tau) = \begin{cases} f(\tau/K), & \tau = 0, k, \dots, (N-1)k \\ 0 & \text{else} \end{cases}$$

$$f = \{f(0), f(1), \dots, f(N-1)\}$$

$$g = \{\underbrace{\underbrace{f(0), 0, 0, \dots, 0, 0}_{K}, \underbrace{f(1), 0, \dots, 0}_{K}, 0, \dots, 0, \underbrace{f(N-1), 0, \dots, 0}_{K}}_{N \text{ times}}\}$$

Then $g \supset G$ with

$$G(\nu) = \begin{cases} \frac{1}{K} F(\nu) & \nu = 0, \dots, N-1 \\ \frac{1}{K} F(\nu - N) & \nu = N, \dots, 2N-1 \\ \frac{1}{K} F(\nu - 2N) \dots & \nu = 2N, \dots, 3N-1 \\ \vdots & \end{cases}$$

Examples

$$\{f(0), f(1), f(2), f(3)\} \xrightarrow{\text{Stretch}_2} \{f(0), 0, f(1), 0, f(2), 0, f(3), 0\}$$

$$\{F(0), F(1), F(2), F(3)\} \longrightarrow \frac{1}{2}\{F(0), F(1), F(2), F(3), F(0), F(1), F(2), F(3)\}$$

$$\{f(0), f(1), f(2), f(3)\} \xrightarrow{\text{Repeat}_2} \{f(0), f(1), f(2), f(3), f(0), f(1), f(2), f(3)\}$$

$$\longrightarrow \{F(0), 0, F(1), 0, F(2), 0, F(3), 0\}$$

Note What Bracewell calls stretch is more often called upsampling

The Summation Rule

Ex $\{f_0, f_1, \dots, f_5\} \supset \{F_0, F_1, \dots, F_5\}$

$\{f_0+f_3, f_1+f_4, f_2+f_5\} \supset 2\{F_0, F_2, F_4\}$

$\{f_0+f_2+f_4, f_1+f_3+f_5\} \supset 3\{F_0, F_3\}$

Theorem Let $g(\tau) = \sum_{\ell=0}^{k-1} f(t - \ell \frac{N}{k})$

Then $G(\nu) = k\, F(k\nu)$.

The Decimation Rule (down sampling)

Ex $\{f_0, f_2, f_4\} \supset \{F_0+F_3, F_1+F_4, F_2+F_5\}$

Theorem Let $g(\tau) = f(\tau k)$.

Then $$G(\nu) = \sum_{l=0}^{k-1} F(\nu - \ell \frac{N}{k})$$

Pf. Both theorems are prove by direct (lengthy) computation.

CHALMERS | GÖTEBORGS UNIVERSITET

Repitation # 3

Discrete Fourier Transform (DFT)

$\hat{T} \in S'$, $\hat{T}_1 = \hat{T} \Rightarrow T_N = T$ ($\hat{T}$ periodic with period 1 $\Rightarrow$ T has period N)

$T = \sum_k t_k \delta_k$ & $\hat{T} = \sum_k c_k \delta_{k/N}$

$t_\ell = \sum_{k=1}^{N} c_k e^{2\pi i k(\ell/N)}$, $N c_k = \sum_{\ell=1}^{N} t_\ell e^{-2\pi i \ell (k/N)}$

$g(x) = \sum_{m=-\infty}^{\infty} f(x - m\tau) \in S'$; τ-periode ($f \in S$)

$\hat{g} = \sum_k c_k \delta_{k/\tau}$ $\left(g = \sum_k c_k e^{2\pi i k/\tau (\cdot)}\right)$, $c_k = \frac{1}{\tau}\hat{f}\left(\frac{k}{\tau}\right)$.

Def DFT: $\tau \in \{0, 1, \dots, N-1\}$

$$F(\nu) = \frac{1}{N}\sum_{\tau=0}^{N-1} f(\tau) e^{-2\pi i \nu \tau/N} \quad \overbrace{\leftrightarrow \quad F(\nu) = \frac{1}{N}\sum_{\tau=-N/2}^{N/2-1} f(\tau) e^{-2\pi i \nu \tau/N}}^{\text{"N even"}}$$

$$f(\tau) = \sum_{\nu=0}^{N-1} F(\nu) e^{2\pi i \nu \tau/N} \quad \leftrightarrow \quad f(\tau) = \sum_{\nu=-N/2}^{N/2-1} F(\nu) e^{2\pi i \nu \tau/N}$$

$-\nu$ 0 1 2 ν $N-1$

$F(N-\nu) = \dots = F(-\nu) = F(\nu)^*$

Rules for computation

$f(\tau) \supset F(\nu)$ $(\tau, \nu \in \{0, 1, \dots, N-1\})$

$F(\nu) \supset \frac{1}{N} f(-\tau)$ ...

$f^*(\tau) \supset F^*(-\nu)$ (for complex sequences)

$f_1 * f_2(\tau) \supset N F_1(\nu) F_2(\nu)$

$f_1(\tau) f_2(\tau) \supset \sum_{\nu'=0}^{N-1} F_1(\nu') F_2(\nu - \nu') = F_1 * F_2(\nu)$

$f(0) = \sum_{\nu=0}^{N-1} F(\nu)$

$$\sum_{\tau=0}^{N-1} |f(\tau)|^2 = N \sum_{\nu=0}^{N-1} |F(\nu)|^2.$$

Examples

Let $f = \{f_0, f_1, f_2, f_3\} \supset F = \{F_0, F_1, F_2, F_3\}$

Shift Let $\tau_1 f = \{f_3, f_0, f_1, f_2\}$

Then $\tau_1 f \supset \mathcal{E}_1 f = \{F_0, \omega F_1, \omega^2 F_2, \omega^3 F_3\}$, where $\omega = e^{-2\pi i/N}$

(prove this by direct calculation!)

A diagram

$$\begin{tikzcd}
\{f_0, f_1, f_2, f_3\} = f \arrow[r, "\mathcal{F}"] \arrow[d, "\tau_1"'] \arrow[dr, phantom, "\#"] & F = \{F_0, F_1, F_2, F_3\} \arrow[d, "\mathcal{E}_1"] \\
\{f_3, f_0, f_1, f_2\} \arrow[r, "\mathcal{F}"] & \{F_0, \omega F_1, \omega^2 F_2, \omega^3 F_3\}
\end{tikzcd}$$

Def For a periodic sequence $f = \{f_0, \dots, f_{N-1}\}$ we denote by $\tau_1 f$ the translated sequence $\{f_{N-1}, f_0, \dots, f_{N-2}\}$.

Def. For a periodic sequence $F = \{F_0, F_1, \dots, F_{N-1}\}$ we write $\mathcal{E}_1 F$ for the sequence obtained by multiplication by powers of $\omega = e^{-2\pi i/N}$:

$$\mathcal{E}_1 F = \{F_0, \omega F_1, \omega^2 F_2, \dots, \omega^{N-1} F_{N-1}\}.$$

Stretch (upsampling) and Repeat.

For a sequence $f = \{f_0, \dots, f_{N-1}\}$ we denote by $Z_2 f$, the sequence

$$Z_2 f = \{f_0, 0, f_1, 0, \dots, f_{N-1}, 0\}$$

and similarly we write $Z_k f$ for the sequence

$$\{\underbrace{f_0, 0, 0 \dots, 0}_{k}, \underbrace{f_1, 0, \dots 0}_{k}, \dots, \underbrace{f_{N-1}, 0, 0 \dots, 0}_{k}\}$$

For $F = \{F_0, \dots, F_{N-1}\}$ we write

$$R_k F = \{\underbrace{\underbrace{F_0, \dots, F_{N-1}}_{N}, F_0, \dots, F_{N-1}, \dots, F_0, \dots, F_{N-1}}_{k\text{-times}}\}$$

A diagram:

$$\begin{tikzcd} f \arrow[r, "\mathscr{F}"] \arrow[d, "Z_2"'] \arrow[dr, phantom, "\#"] & F \arrow[d, "\frac{1}{2}R_2"] \\ g \arrow[r, "\mathscr{F}"'] & G \end{tikzcd}$$

obs. per $= N$

$$F = \{F_0, \ldots, F_{N-1}\}$$

$$R_2 F = \{F_0, F_1, \ldots, F_{N-1}, F_0, F_1, \ldots, F_{N-1}\} \leftarrow \text{period } N, \ \# \underline{2N}$$

$$\tfrac{1}{2} R_2 F = \tfrac{1}{2}\{F_0, F_1, F_{N-1}, F_0, F_1, \ldots, F_{N-1}\}$$

$$F = \frac{1}{2N} \sum^{2N-1} \ldots$$

This notation is useful for describing the FFT (Fast Fourier Transform).

Note that the <u>cost</u> of computing the DFT is rather high:

$$F(\nu) = \frac{1}{N} \sum_{\tau=0}^{N-1} \omega^{\tau\nu} f(\nu)$$

$\Rightarrow$ in total $N \times N$ multiplications and $N(N-1)$ additions if all the $\omega^{\tau\nu}$ have been precomputed.

DFT in Matrix form

Let $f = \{f_0, f_1, \ldots, f_{N-1}\} = \begin{bmatrix} f_0 \\ \vdots \\ f_{N-1} \end{bmatrix} \in \mathbb{R}^N$ (or $\mathbb{C}^N$)

i.e. we consider f to be a vector in $\mathbb{R}^N$ or $\mathbb{C}^N$. We also let

$F = \begin{bmatrix} F_0 \\ \vdots \\ F_{N-1} \end{bmatrix} \in \mathbb{C}^N$. Then we have

$$F = \frac{1}{N} \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & & \omega^{N-1} \\ 1 & \omega^2 & \omega^4 & & \omega^{2(N-1)} \\ \vdots & & \vdots & & \vdots \\ 1 & \omega^{N-1} & \omega^{2(N-1)} & & \omega^{(N-1)(N-1)} \end{bmatrix} f$$

where $\omega = e^{-2\pi i/N}$. The matrix can be written

$$\left\{\omega^{(i-1)(j-1)}\right\}_{j=1,\, i=1}^{N,\, N}.$$

Fast Fourier Transform

"decimation in time" (Bracewell's notation)

Recall:

$$\begin{tikzcd} f \arrow[r, "\mathcal{F}"] \arrow[d, "\tau_1"'] & F \arrow[d, "\mathcal{E}_1"] \\ g \arrow[r, "\mathcal{F}"] & G \end{tikzcd} \qquad \begin{tikzcd} f \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{Z}_2"'] & F \arrow[d, "\frac{1}{2} R_2"] \\ g \arrow[r, "\mathcal{F}"] & G \end{tikzcd}$$

One Step in the FFT

(an even number)

(split even and odd)

$$\begin{tikzcd}[column sep=small]
 & \{f_0, f_1, \dots, f_{2N-1}\} \arrow[d] \arrow[dr] & & \\
\{\tilde{F}_0, \dots, \tilde{F}_{2N-2}\} \arrow[dd, "\frac{1}{2} R_2"'] & \{f_0, f_2, \dots, f_{2N-2}\} \arrow[l, "\mathcal{F}"'] \arrow[dd, "\mathcal{Z}_2"] & \{f_1, f_3, \dots, f_{2N-1}\} \arrow[r, "\mathcal{F}"] \arrow[d, "\mathcal{Z}_2"] & \{\tilde{F}_1, \tilde{F}_3, \dots, \tilde{F}_{2N-1}\} \arrow[d, "\frac{1}{2} R_2"] \\
 & & \{f_1, 0, f_3, \dots, f_{2N-1}\} \arrow[d, "\tau_1"] & \{\quad\} \arrow[d, "\mathcal{E}_1"] \\
\{\quad\} \arrow[ddr] & \{f_0, 0, f_2, 0, \dots, f_{2N-2}, 0\} \arrow[l] \arrow[d] & \{0, f_1, 0, f_3, \dots, f_{2N-1}\} \arrow[r] \arrow[dl, "\text{add}"] & \{\quad\} \arrow[ddll] \\
 & \{f_0, f_1, f_2, \dots, f_{2N-1}\} \arrow[d] & & \\
 & \{F_0, F_1, \dots, F_{N-1}\} & &
\end{tikzcd}$$

FFT by decimation in time

1) Split f into $f_O = \{f_1, f_3, \dots, f_{2N-1}\}$ (odds)
   $f_E = \{f_0, f_2, \dots, f_{2N-2}\}$ (evens)
2) Compute DFT of f_O and f_E. $f_O \supset F_O$, $f_E \supset F_E$.
3) Then $f \supset \frac{1}{2} R_2 F_E + \mathcal{E}_1 \frac{1}{2} R_2 F_O$.

Let $\mathcal{X}(N)$ = cost of computing DFT of a sequence of length N.

Then $\mathcal{X}(2N)$ = cost of splitting $\dashrightarrow O(N)$

+ cost of computing $\frac{1}{2}R_2 \dashrightarrow O(N)$

+ $\mathcal{X}(N)$

+ cost of computing $\mathcal{E}_1 \dashrightarrow O(N)$

+ cost of " $\frac{1}{2}R_2 \dashrightarrow O(N)$

+ $\mathcal{X}(N)$

+ Cost of adding $O(N)$.

Hence the total cost is

$$\mathcal{X}(2N) = O(N) + 2\mathcal{X}(N).$$

If $N = 2^m$ then

$$\begin{aligned}
\mathcal{X}(N) &= O(2^m) + 2\mathcal{X}(2^{m-1}) \\
&= O(2^m) + 2\left(O(2^{m-1}) + 2\mathcal{X}(2^{m-2})\right) \\
&= 2\,O(2^m) + 4\mathcal{X}(2^{m-2}) \\
&= 2\,O(2^m) + 4\left(O(2^{m-2}) + 2\mathcal{X}(2^{m-3})\right) \\
&= 2\,O(2^m) + O(2^m) + 2^3\mathcal{X}(2^{m-3}) \\
&= 3\,O(2^m) + 2^3\mathcal{X}(2^{m-2}) = \cdots = m\,O(2^m) + 2^m\mathcal{X}(1) \\
&= m\,O(N) + N \Rightarrow \underline{\underline{\text{total cost} = O(N\log_2 N)}}.
\end{aligned}$$

<u>"Decimation in Frequency"</u>

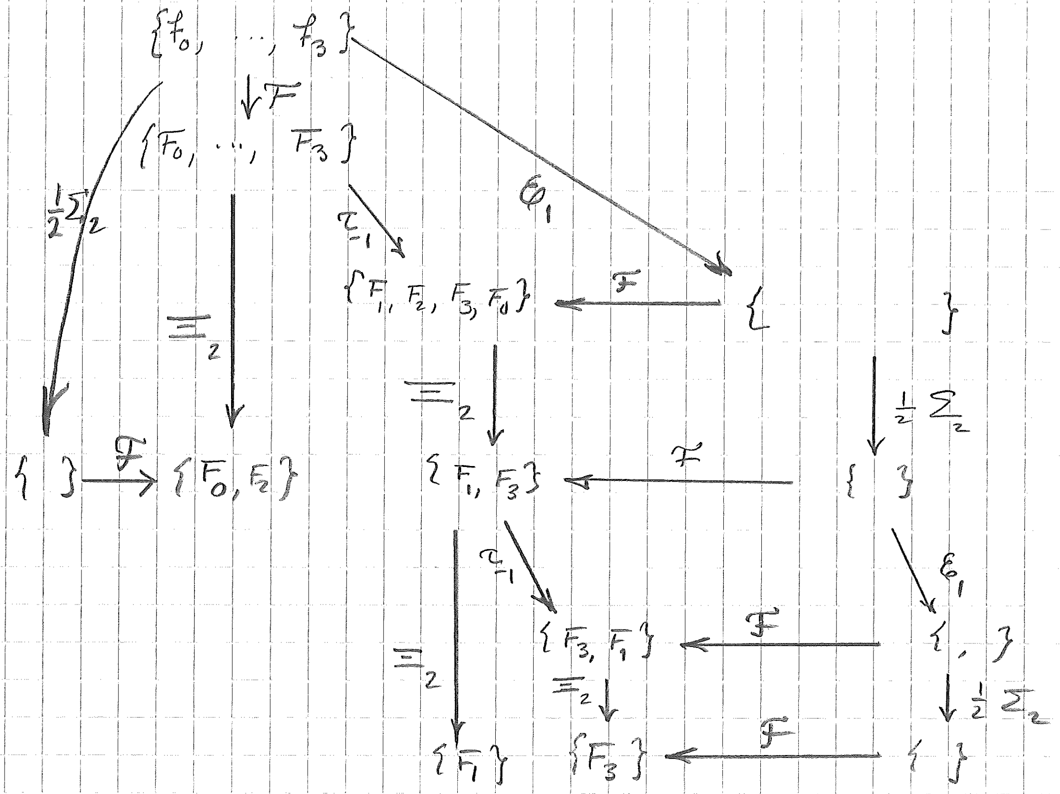
$$\begin{array}{ccc}
\{f_0, f_1, f_2, f_3\} & \xrightarrow{\mathcal{F}} & \{F_0, \dots, F_3\} \\
\mathcal{E}_1 \downarrow & & \downarrow \tau_{-1} \\
\{f_0, \omega f_1, \dots, \omega^3 f_3\} & \xrightarrow{\mathcal{F}} & \{F_1, \dots, F_0\}
\end{array}$$

$$\begin{array}{ccc}
\{f_0, f_1, \dots, f_3\} & \xrightarrow{\mathcal{F}} & \{F_0, \dots, F_3\} \\
\tfrac{1}{2}\Sigma_2 \downarrow & & \downarrow \equiv_2 \\
\tfrac{1}{2}\{f_0+f_2,\ f_1+f_3\} & & \{F_0, F_2\}
\end{array}$$

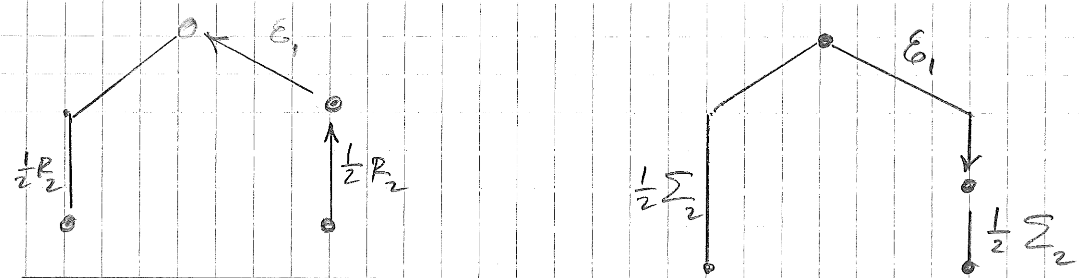
Σ_2: Summation

$(\Sigma_2 f)(\tau) = f(\tau) + f(\tau + \frac{N}{2})$

$\equiv_2$: down sampling



Tree Structures:



The Fourier Transform (again)

$$f \supset \hat{f} \qquad \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx$$

Autocorrelation

Def. $f \star f(x) = \int_{-\infty}^{\infty} f(u) f(u-x)\,du = \int_{-\infty}^{\infty} f(u) f(u+x)\,du$

Normalized $\quad \gamma(x) = \dfrac{f \star f(x)}{\int_{-\infty}^{\infty} |f(u)|^2\,du}$,

Then $\gamma(x) \le 1$, $\gamma(0) = 1$, because $f \star f(x)$ has a maximum at $x = 0$

[and this is because $ab \le \frac{1}{2}(|a|^2 + |b|^2)$, so

$\int_{-\infty}^{\infty} f(u) f(x+u)\,du \le \frac{1}{2}\int_{-\infty}^{\infty} (f(u)^2 + f(x+u)^2)\,du = \int_{-\infty}^{\infty} |f(u)|^2\,du$]

Recall that $f : \mathbb{R} \to \mathbb{C}$, $f \supset \hat{f}$ implies

$$\hat{f}^* = \hat{f}(-\cdot)^* \qquad (z = x+iy \Rightarrow z^* = x-iy)$$

and

$$\mathcal{F}(f^*(-\cdot)) = \hat{f}^*.$$

Therefore

$$\mathcal{F}(f * f^*(-\cdot)) = \hat{f}\,\hat{f}^* = |\hat{f}|^2.$$

And

$$f * f^*(-\cdot)(x) = \int_{-\infty}^{\infty} f(y)\, f(-(x-y))^*\, dy = \int_{-\infty}^{\infty} f(y)\, f(y-x)^*\, dy$$

$$= f \star f^*, \quad \text{and if } f \text{ is real we find}$$

$$f \star f^* \xrightarrow{\mathcal{F}} |\hat{f}|^2.$$

Def $|\hat{f}|^2$ is called "the energy density"

Some relations

1) $\int f(x)\,dx = \hat{f}(0)$

2) $\int x f(x)\,dx = \dfrac{\hat{f}'(0)}{-2\pi i}$

Def $\langle x \rangle = \dfrac{\int x f(x)\,dx}{\int f(x)\,dx} = -\dfrac{1}{2\pi i}\,\dfrac{\hat{f}'(0)}{\hat{f}(0)}$

Def $\int x^2 f(x)\,dx = \dfrac{-1}{4\pi^2}\,\hat{f}''(0)$ (moment of inertia)

$$\langle x^2 \rangle = \int x^2 f(x)\,dx \Big/ \int f(x)\,dx = -\frac{1}{4\pi^2}\,\frac{\hat{f}''(0)}{\hat{f}(0)}$$

Def Variance $\sigma^2 = \langle (x - \langle x \rangle)^2 \rangle$

Note If $f(x) = 0$ for $|x| > a$ and $f \in L^1$, then $\hat{f}(\xi) \in C^\infty$, because

$$|D^k \hat{f}(\xi)| = \left| \int_{-\infty}^{\infty} (-2\pi i x)^k f(x)\,dx \right| \le (2\pi a)^k \int_{-\infty}^{\infty} |f(x)|\,dx < \infty.$$

The uncertainity relation

The Schrödinger equation

$$i\hbar \frac{\partial \psi}{\partial t} = \hat{H}(\psi), \qquad \psi = \psi(r,t), \quad \text{the wave function}$$

For a single particle:

$$i\hbar \frac{\partial \psi}{\partial t}(r,t) = -\frac{\hbar^2}{2m}\Delta\psi(r,t) + V(r)\psi(r,t)$$

$V(r)$: potential energy

$-\frac{\hbar^2}{2m}\Delta\psi$: Kinetic energy ($\nabla\psi$ corresponds to momentum).

$\hat{p} = -i\hbar\frac{\partial}{\partial x}$ (momentum)

x (position)

Heisenberg: you cannot know velocity and position of a particle exactly at one time.

Prel 1) $$|f(x)| = \left|\int_{\mathbb{R}} \hat{f}(\xi)\, d\xi\right| \le \int_{\mathbb{R}} |\hat{f}(\xi)|\, d\xi$$

2) (Cauchy-Schwarz $\le$)

$$\left|\int_{\mathbb{R}} (fg^* + f^*g)\, dx\right|^2 \le 4\int_{\mathbb{R}} |f|^2 dx \int_{\mathbb{R}} |g|^2 dx$$

Recall $|x\cdot y|^2 \le |x|^2|y|^2$ and hence

$$\left|\int fg^* dx\right|^2 \le \int |f(x)|^2 dx \int |g(x)|^2 dx$$

Theorem Define $(\Delta x)^2 = \dfrac{\int x^2 f f^* dx}{\int f f^* dx}$

$$(\Delta \xi)^2 = \frac{\int \xi^2 \hat{f} \hat{f}^* dx}{\int \hat{f} \hat{f}^* dx}$$

Then $\Delta x \, \Delta \xi \geq \dfrac{1}{4\pi}$.

Lemma $\int f' f'^* dx = 4\pi^2 \int \xi^2 |\hat{f}|^2 d\xi$

Pf of Lemma:

use that $\mathcal{F}(Df) = 2\pi i \xi \hat{f}(\xi)$

and that $\displaystyle\int_{\mathbb{R}} |f|^2 dx = \int_{\mathbb{R}} |\hat{f}|^2 dx$ $\square$

Proof of the theorem

$$(\Delta x)^2 (\Delta \xi)^2 = \frac{\int x^2 f f^* dx}{\int f f^* dx} \cdot \frac{\int \xi^2 \hat{f} \hat{f}^* dx}{\int \hat{f} \hat{f}^* dx}$$

$$= \frac{\int xf \, (xf)^* dx \quad \int f' f'^* dx}{4\pi^2 \left(\int_{\mathbb{R}} f f^* dx\right)^2}$$

$$\geq \frac{\left(\int_{\mathbb{R}} (x f f'^* + x f^* f') dx\right)^2}{16\pi^2 \left(\int |f|^2 dx\right)^2} = \frac{\left(\int_{\mathbb{R}} x \frac{d}{dx}(f f^*) dx\right)^2}{16\pi^2 \left(\int |f|^2 dx\right)^2}$$

$$= \frac{\left(\int |f|^2 dx\right)^2}{16\pi^2 \left(\int |f|^2 dx\right)^2} = \frac{1}{16\pi^2}$$

Filters and signals, filter banks

A discrete signal is a (double) sequence

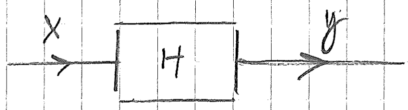
$$x = \{x_k\}_{k=-\infty}^{\infty} = (\cdots, x_{-1}, 0, x_1, \cdots); \qquad x_k \in \mathbb{R} \text{ (or } \mathbb{C})$$

So x is a function $x: \mathbb{Z} \longrightarrow \mathbb{C}$; (or $\mathbb{R}$)

Def. $x \in \ell^2 \iff \sum_{k=-\infty}^{\infty} |x_k|^2 < \infty$ (finite energy)

Def. A filter is an operator; $H: x \longmapsto y$, i.e.

$y = Hx$ is another signal



Def. H is linear if $H(\alpha x + \beta y) = \alpha Hx + \beta Hy$

" " time invariant if $H(Dx) = DH(x)$; for all x

(note Dx is defined by $(Dx)_k = x_{k-1}$)

Def. $\delta = \{\delta_k\}_{k=-\infty}^{\infty}$, $\delta_k = \begin{cases} 1 & \text{if } k=0 \\ 0 & \text{otherwise} \end{cases}$

Def. $h = H\delta$ is called the impulse response of the filter H.

Suppose now that a filter H is Linear Time Invariant (LTI)

We then have $D^n h = H(D^n \delta)$.

Take $x = \{x_k\}_{k=-\infty}^{\infty}$ arbitrary. Then

$$x = \cdots x_{-2} D^{-2}\delta + \underbrace{x_{-1} D^{-1}\delta + x_0 \delta + x_1 D\delta}_{} + x_2 D^2 \delta + \cdots = \cdots$$

$$= \cdots + (0, \ldots, 0, x_{-1}, 0, 0, \ldots) + (0, \ldots, 0, 0, x_0, 0, 0, 0, \ldots) + (0, \ldots, 0, 0, 0, x_1, 0, 0, \cdots)$$

$$= \sum_{n=-\infty}^{\infty} x_n D^n \delta .$$

Hence, if $y = Hx$, we get

$$y = Hx = \sum_{n=-\infty}^{\infty} x_n D^n h = \sum_{n=-\infty}^{\infty} x_n h_{\cdot - n}$$

i.e. $y = \{y_k\}_{k=-\infty}^{\infty}$,

$$y_k = \sum_{n=-\infty}^{\infty} x_n h_{k-n} = x * h = h * x$$

definition of discrete convolution.

So LTI filter is uniquely determined by its impulse response.

Def A Finite Impulse Response filter (FIR) has only finitely many $h_k \neq 0$ (Otherwise one says IIR -filter)

Def An LTI-filter is causal if $h_k = 0$ for $k < 0$.

About the computer assignment

Some matlab commonds (need the manual pages!)

1) Convert a wave file to a vector:

[y, Fs, bits] = wavread('fil.wav')

(auread ('fil.au')

y: data vector

Fs: Sampling rate (HZ)

bits: number of bits / sample

plot (y) (to see the file)

2) To filter the signal:

yf = filter (b, a, y)

y is the input signal (a vector)

b and a are the vectors that define the filter.

3) You can use simulink to create the filters parameters a & b.

- Start simulink

  Creat new model
  signal processing library
  Filter design Library
  Digital Filter Design.

- Double click "filter block"

- choose parameters

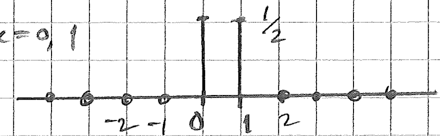
- click "Design Filter"

- File: "export".

Def (Autocorrelation)

$$x \star y = \sum_{n=-\infty}^{\infty} x_{n+\cdot}\, y_n \qquad \left(\text{so } (x \star y)_k = \sum x_{n+k}\, y_k\right)$$

Example Let $h_k = \begin{cases} \frac{1}{2} & \text{when } k=0,1 \\ 0 & \text{else} \end{cases}$



Then $y_k = \frac{1}{2}(x_k + x_{k-1})$.

This is a LTI-filter, that is also causal (a low pass filter)

Example Down sampling by 2, $\downarrow 2$ (note the new symbol: $\downarrow 2$)

If $x = \{x_k\}_{k=1}^{\infty}$

$(\downarrow 2)\, x = (\dots\, x_{-4}, x_{-2}, x_0, x_2, x_4, \dots)$

Upsampling by 2, $\uparrow 2$

$(\uparrow 2)\, x = (\dots\, x_{-2}, 0, x_{-1}, 0, x_0, 0, x_1, 0, x_2, \dots)$

These are linear but not time invariant (why?)

Def The Z-transform of a signal $x = \{x_k\}_{k=-\infty}^{\infty}$ is

$$X(z) = \sum_{k=-\infty}^{\infty} x_k z^{-k}, \qquad z \in \mathbb{C}$$

We write $x \supset X(z)$.

Here the argument must be written out to avoid confusion with the DFT, $X(j) = \dots$

Example $x_k = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases}$

Then $x \supset X(z)$ with

$$X(z) = \sum_{k=0}^{\infty} z^{-k} = \frac{1}{1-\frac{1}{z}} = \frac{z}{z-1}, \quad (z \in \mathbb{C} \setminus \{1\}).$$

The convolution Theorem

If $y = h * x$, then $Y(z) = H(z)\,X(z)$ and vice versa.

Proof
$$Y(z) = \sum_{k=-\infty}^{\infty} z^{-k} \sum_{n=-\infty}^{\infty} h_{k-n} x_n$$
$$= \sum_{k=-\infty}^{\infty} z^{-(k-n)} \sum_{n=-\infty}^{\infty} z^{-n} h_{k-n} x_n$$
$$= \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} z^{-k} z^{-n} h_k x_n = H(z)\,X(z). \quad \square$$

The Discrete Fourier Transform (of infinite sequences)

$$X(\omega) = \sum_{k=-\infty}^{\infty} x_k e^{-i\omega k}.$$

Note, In this definition $e^{-i\omega k}$ NOT $e^{-2\pi i \omega k}$

Other common notation

$$\hat{x}(\omega) = X(\omega)$$

Note that $X(\omega) = X(e^{i\omega}) \equiv X(z)$.

Lemma $X(\omega)$ is 2π-periodic, and

$$x_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{x}(\omega) e^{i\omega k}\, d\omega.$$

Let H be a LTI filter with impulse response h.

Def $H(\omega)$ is called the Frequency response.

Example Let $x_k = e^{i\omega k}$, $|\omega| \leq \pi$. Then $y = Hx$ is given by

$$y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n} = \sum_{n=-\infty}^{\infty} h_n e^{i\omega(k-n)} = e^{i\omega k} \underbrace{\sum_{n=-\infty}^{\infty} h_n e^{-i\omega n}} = e^{i\omega k} \underline{H(\omega)}.$$

Write $H(\omega) = |H(\omega)|\, e^{i\phi(\omega)}$

Amplitude ↑ ; phase function

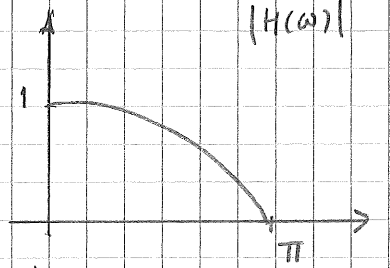
Def If $|H(\omega)| = 1$, H is called an <u>all pass filter</u>.

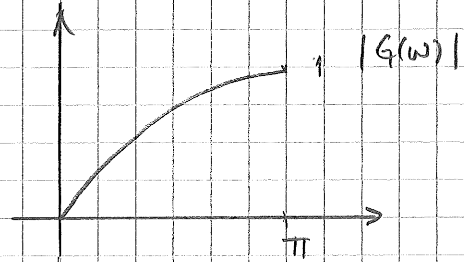
Example a) $h_0 = h_1 = \frac{1}{2}$, $h_k = 0$, for $k \neq 0, 1$.

$$\Rightarrow \quad H(\omega) = \frac{1}{2}\left(1 + e^{-i\omega}\right) = e^{-i\omega/2}\cos(\omega/2)$$

b) $g_k = \begin{cases} \frac{1}{2} & k=0 \\ -\frac{1}{2} & k=1 \\ 0 & \text{else} \end{cases}$. Then

$$G(\omega) = \frac{1}{2}\left(1 - e^{-i\omega}\right) = \cdots = i e^{-i\omega}\sin(\omega/2)$$





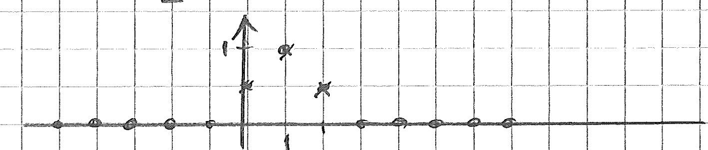
H is a low pass filter (average)

G is a high pass filter. (difference)

Definition: H has linear phase if $\omega \mapsto \phi(\omega)$ is linear

- <u>H is symmetric</u> (if $h_k = h_{-k}$) <u>anti-symmetric</u> (If $h_k = -h_{-k}$)
- The <u>group delay</u> of H is: $\tau(\omega) = -\frac{d\phi}{d\omega}$.

Example Let $h_0 = h_2 = \frac{1}{2}$, $h_1 = 1$ ($h_k = 0$, $k \neq 0,1,2$)



Then $H(\omega) = \frac{1}{2} + 1 \cdot e^{-i\omega} + \frac{1}{2}e^{-i(2\omega)} = e^{-i\omega}\left(1 + \frac{1}{2}e^{i\omega} + \frac{1}{2}e^{-i\omega}\right)$

$$= e^{-i\omega}\underbrace{(1 + \cos\omega)}_{|H(\omega)|}, \quad \phi(\omega) = -\omega \ \ldots \text{ linear.}$$

Example If H is a LTI-filter and if x is a pure frequency, $x_k = e^{i\omega k}$, then

$$y_k = (h * x)_k = \sum_{n=-\infty}^{\infty} h_n e^{i\omega(k-n)} = e^{i\omega k} \sum_{n=-\infty}^{\infty} h_n e^{-i\omega n} = H(\omega) e^{i\omega k}.$$

If $x_k = e^{i\omega_1 k} + e^{i\omega_2 k}$ (sum of two pure frequencies)

$$y_k = H(\omega_1) e^{i\omega_1 k} + H(\omega_2) e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{i\phi(\omega_1)} e^{i\omega_1 k} + |H(\omega_2)| e^{i\phi(\omega_2)} e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{-i\omega_1} e^{i\omega_1 k} + |H(\omega_2)| e^{-i\omega_2} e^{i\omega_2 k}$$

$$= |H(\omega_1)| e^{i\omega_1 (k-1)} + |H(\omega_2)| e^{i\omega_2 (k-1)}.$$

Note the shift $k \longrightarrow k-1$.

Filter banks

Def. A sequence of signals $(\varphi^{(n)})_{n=-\infty}^{\infty}$

[Note: each $\varphi^{(n)}$ is a sequence: $(\ldots \varphi^{(n)}_{-2}, \varphi^{(n)}_{-1}, \varphi^{(n)}_0, \varphi^{(n)}_1, \varphi^{(n)}_2, \ldots)$]

is called a basis if every x can be written $x = \sum c_n \varphi^{(n)}$.

Now we need to restrict x.

Def $\|x\| = \sqrt{\sum |x_k|^2}$. (This is sometimes denoted by $\|x\|_2$ or $\|x\|_{\ell^2}$)

We say that $x^{(n)} \longrightarrow x$ in norm,

if $\|x^{(n)} - x\| \longrightarrow 0$, when $n \longrightarrow \infty$.

Def A Hilbert Space is a "complete inner product space"

Ex. ℓ^2, $\langle x, y \rangle = \sum x_k \bar{y}_k$ (note complex conjugate only)

$$\|x\| = \sqrt{\sum \langle x, x \rangle}.$$

Cauchy-Schwarz : $|\langle x,y\rangle| \le \|x\|\,\|y\|$

A Cauchy sequence is a sequence $(x^{(n)})_{n=1}^{\infty}$ such that

$\forall \varepsilon > 0\ \exists N > 0$ such that $\| x^{(n)} - x^{(m)} \| < \varepsilon$, for $n, m > N$.

Def. A space is complete if every Cauchy sequence is convergent.

An orthogonal basis $(\varphi^{(n)})$ is a basis that satisfies

$$\langle \varphi^{(j)}, \varphi^{(k)} \rangle = \begin{cases} 1 & \text{if } j=k \\ 0 & \text{otherwise.} \end{cases}$$

We can then write

$$x = \sum_n c_n \varphi^{(n)}, \qquad \text{where } c_n = \langle x, \varphi^{(n)} \rangle .$$

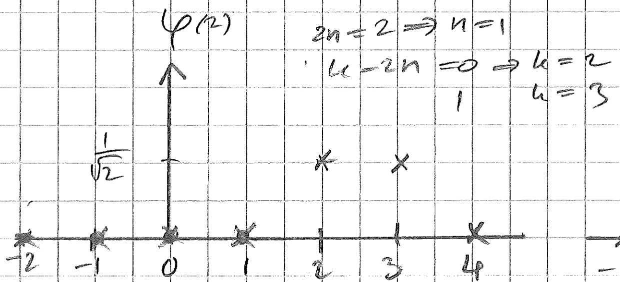
(Compare with the familiar $\mathbb{R}^n$).

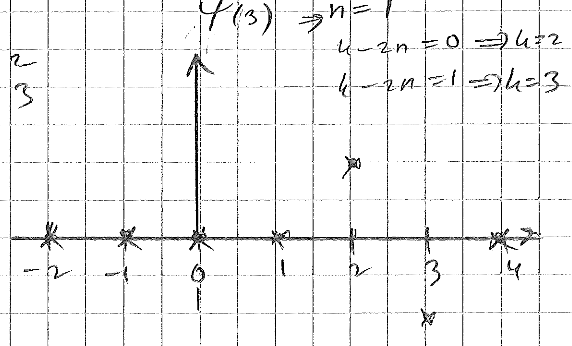
Definition The Haar basis consists of two families of functions,

$$\varphi_k = \begin{cases} \frac{1}{\sqrt{2}} & \text{when } k=0,1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi_k = \begin{cases} \frac{1}{\sqrt{2}} & k=0 \\ -\frac{1}{\sqrt{2}} & k=1 \\ 0 & \text{otherwise} \end{cases}$$

Then let

$$(\varphi^{(2n)})_k = \varphi_{k-2n}, \qquad (\varphi^{(2n+1)})_k = \psi_{k-2n}$$





The coordinates of a sequence $x = (x_k)_{k=-\infty}^{\infty}$

in this basis are

$$y_n^{(0)} := c_{2n} = \langle x, \varphi^{(2n)} \rangle = \frac{1}{\sqrt{2}} \left(x_{2n} + x_{2n+1} \right); \qquad \text{(mean)}$$

$$y_n^{(1)} := c_{2n+1} = \langle x, \varphi^{(2n+1)} \rangle = \frac{1}{\sqrt{2}} \left(x_{2n} - x_{2n+1} \right); \quad \text{(difference)}.$$

These are basis functions in $\ell^2(\mathbb{Z})$ and hence

$$x_k = \sum_n c_n \varphi_k^{(n)}$$

$$= \sum_n y_n^{(0)} (\varphi^{(2n)})_k + \sum_n y_n^{(1)} (\varphi^{(2n+1)})_k$$

$$= \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}.$$

This is almost a convolution, and can be obtained using two filters.

H: $\quad h_k = \varphi_k \qquad h_k^* = \varphi_{-k}$

G: $\quad g_k = \psi_k \qquad g_k^* = \psi_{-k}$

"time reversal".

Hence

$$y_n^{(0)} = \sum_k x_k (\varphi^{(2n)})_k = \sum_k x_k h_{k-2n} = \sum_k x_k h^*_{2n-k}$$

$$= (x * h^*)_{2n}$$

and

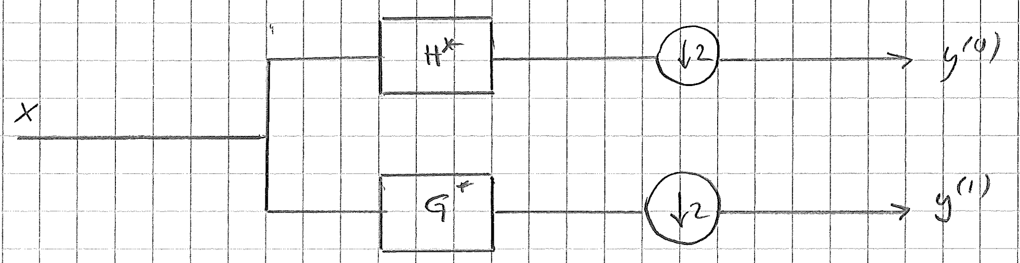
$$y_n^{(1)} = (x * g^*)_{2n}.$$

Recall the up- and downsamplings:

$$(\downarrow 2)\, x = (\ldots, x_{-4}, x_{-2}, x_0, x_2, x_4, \ldots)$$

$$(\uparrow 2)\, x = (\ldots, x_{-2}, 0, x_{-1}, 0, x_0, 0, x_1, 0, x_2, \ldots).$$

We can obtain the coordinates using the two filters:



This procedure is called <u>analysis</u>.

The <u>synthesis</u> is carried out similarly.

$$x_k = \sum_n y_n^{(0)} \varphi_{k-2n} + \sum_n y_n^{(1)} \psi_{k-2n}$$

$$= \sum_n y_n^{(0)} h_{k-2n} + \sum_n y_n^{(1)} g_{k-2n}$$

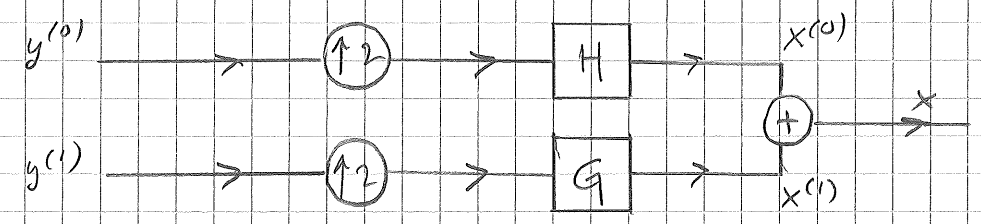
$$= (v^{(0)} * h)_k + (v^{(1)} * g)_k$$

where $v^{(0)} = (\uparrow 2)\, y^{(0)}$ and $v^{(1)} = (\uparrow 2)\, y^{(1)}$

Note that

$$\sum_n y_n h_{k-2n} = \sum_n v_{2n} h_{k-2n} \underset{\text{all odd } v_{2n+1}=0}{=} \sum_n v_n h_{k-n} = (v * h)_k$$

Therefore $x = H\left((\uparrow 2)\, y^{(0)}\right) + G\left((\uparrow 2)\, y^{(1)}\right) = x^{(0)} + x^{(1)}$.



and

$$x = \underbrace{\sum \langle x, \varphi^{(2n)} \rangle \varphi^{(2n)}}_{\text{projection onto even}} + \underbrace{\sum \langle x, \varphi^{(2n+1)} \rangle \varphi^{(2n+1)}}_{\text{projection onto odd.}}$$