DFT and the Z-transform of ⓥ2 and ⓪2

Recall $y = (\downarrow 2)\, x = (\cdots, x_{-2}, x_0, x_2, \cdots)$

$$y_k = x_{2k}$$

$$\Rightarrow \quad Y(z) =: \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} (z^{1/2})^{-2k}$$

$$= \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} x_{2k} (z^{1/2})^{-2k} + \cancel{\sum_{k=-\infty}^{\infty} x_{2k+1} (z^{1/2})^{-(2k+1)}} \right)$$

$$+ \frac{1}{2}\left(\sum_{k=-\infty}^{\infty} x_{2k} (-z^{1/2})^{-2k} + \cancel{\sum_{k=-\infty}^{\infty} x_{2k+1} (-z^{1/2})^{-(2k+1)}} \right)$$

$$= \frac{1}{2}\left(X(z^{1/2}) + X(-z)^{1/2} \right)$$

And with $z = e^{i\omega}$, $z^{1/2} = e^{i\omega/2}$, $-z^{\frac{1}{2}} = e^{i(\frac{\omega}{2}+\pi)}$

we get the DFT

$$Y(\omega) = \frac{1}{2}\left(X(\tfrac{\omega}{2}) + X(\tfrac{\omega}{2}+\pi) \right).$$

The term $X(\frac{\omega}{2}+\pi)$ is an "alias component" which can be removed with a "perfect filter" H^*, G^*.

Upsampling

$$y = (\uparrow 2)\, x = (\cdots, x_{-1}, 0, x_0, 0, x_1, 0, \cdots)$$

$$(\cdots, y_{-2}, y_{-1}, y_0, y_1, y_2, \cdots)$$

Then

$$Y(z) = \sum_{k=-\infty}^{\infty} y_k z^{-k} = \sum_{k=-\infty}^{\infty} y_{2k} z^{-2k} = \sum_{k=-\infty}^{\infty} x_k z^{-2k} = X(z^2)$$

and $Y(\omega) = X(2\omega)$.

It is clear then that

$(\downarrow 2)(\uparrow 2)\, x = x$, but $u = (\uparrow 2)(\downarrow 2)\, x \neq x$ in general.

$$U(z) = \sum_{k=-\infty}^{\infty} u_k z^{-k} = \sum_{k=-\infty}^{\infty} x_{2k} z^{-2k}$$

$$= \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} + \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-(2k+1)} \right]$$

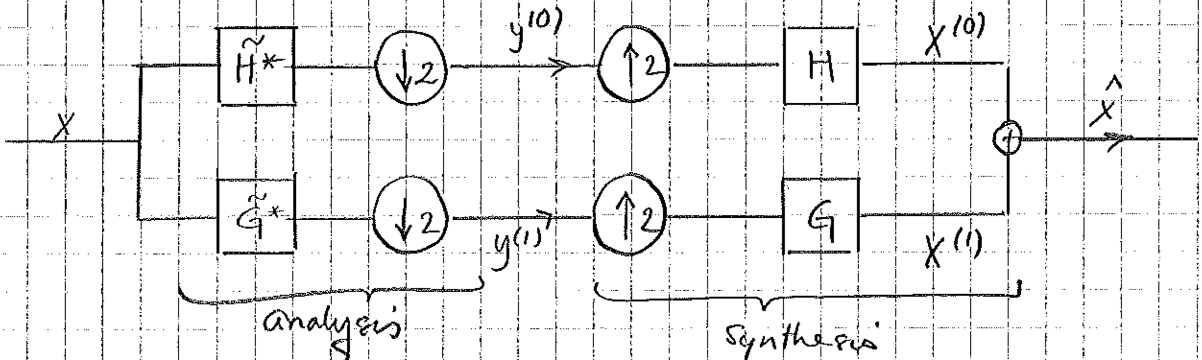
$$+ \frac{1}{2}\left[\sum_{k=-\infty}^{\infty} x_{2k} z^{-2k} - \sum_{k=-\infty}^{\infty} x_{2k+1} z^{-(2k+1)} \right]$$

$$= \frac{1}{2}\Big(X(z) + X(-z) \Big) \qquad (***)$$

which is the same as

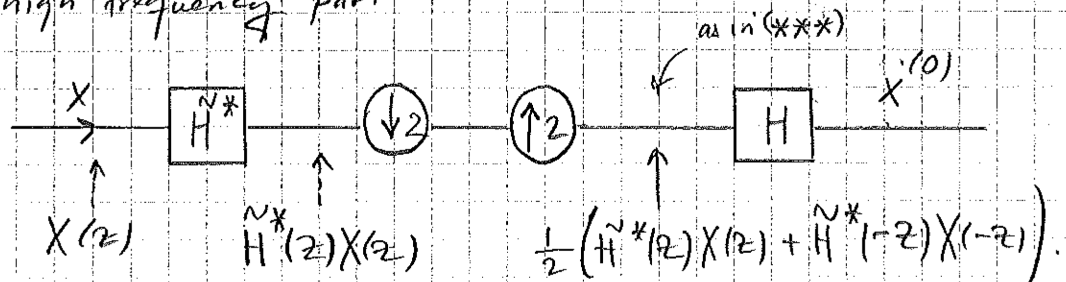
$$U(\omega) = \frac{1}{2}\Big(X(\omega) + X(\omega+\pi) \Big).$$

<u>Perfect reconstruction</u>



A filter bank. In the analysis part, the signal is split into a high frequency part and a low filter part, which gives the signals $y^{(0)}$ and $y^{(1)}$. These then pass the synthesis part to make a reconstruction of the signal. If $\hat{x} = x$, we have achieved "perfect reconstruction".

Expressed in terms of the z-transform, we have, for the high frequency part



So that

$$X^{(0)}(z) = \frac{1}{2} H(z)\left[X(z)\tilde{H}^*(z) + X(-z)\tilde{H}^*(-z)\right]$$

and similarly,

$$X^{(1)}(z) = \frac{1}{2} G(z)\left[X(z)\tilde{G}^*(z) + X(z)\tilde{G}^*(-z)\right]$$

which gives

$$\hat{X}(z) = \frac{1}{2}\left[H(z)\tilde{H}^*(z) + G(z)\tilde{G}^*(z)\right]X(z) + \frac{1}{2}\left[H(z)\tilde{H}^*(-z) + G(z)\tilde{G}^*(-z)\right]X(-z).$$

We therefore achieve perfect reconstruction if and only if the following system is satisfied

$$\begin{cases} H(z)\tilde{H}^*(z) + G(z)\tilde{G}^*(z) = 2 \\ H(z)\tilde{H}^*(-z) + G(z)\tilde{G}^*(-z) = 0. \end{cases}$$

How can one construct such filters?

The "product filters" are obtained by the anstatz,

$$G(z) = -z^{-L}\tilde{H}^*(-z) \qquad L \text{ odd}$$

$$\tilde{G}(z) = -z^{-L} H^*(-z)$$

Recall that the $*$-marked filters are time reversed, so

$$G^*(z) = \sum_{k=-\infty}^{\infty} g_k^* z^{-k} = \sum_{k=-\infty}^{\infty} g_k \left(\frac{1}{z}\right)^{-k} = G\left(\frac{1}{z}\right)$$

and therefore

$$\tilde{G}^*(z) = -\left(-\frac{1}{z}\right)^{-L} H^*\left(\frac{1}{z}\right) = \left(\frac{1}{z}\right)^{-L} H(z)$$

Therefore

$$H(z)\tilde{H}^*(-z) + G(z)\tilde{G}^*(-z) = H(z)\tilde{H}^*(-z) - z^{-L}\tilde{H}^*(-z)\left(\frac{1}{z}\right)^{L} H(z) = 0$$

and

$$H(z)\tilde{H}^*(z) + G(z)\tilde{G}^*(-z) = H(z)\tilde{H}^*(z) - z^{-L}\tilde{H}^*(-z)\cdot\left(\frac{1}{-z}\right)^{-L} H(-z)$$

$$= H(z)\tilde{H}^*(z) + \tilde{H}^*(-z)H(-z)$$

The condition for perfect reconstruction is now given in terms of H and $\tilde{H}$:

$$\boxed{H(z)\,\tilde{H}^*(z) + \tilde{H}^*(-z)\,H(-z) = 2}$$

and a product filter is defined by

$$P(z) = H(z)\,\tilde{H}^*(z).$$

Then, the perfect reconstruction condition becomes

$$P(z) + P(-z) = 2, \qquad \text{i.e.}$$

$$2P_0 + 2\sum_{k=-\infty}^{\infty} P_{2k} z^{-2k} = 2.$$

It follows that $P_0 = 1$, $P_{2n} = 0$ $(n \neq 0)$, and P_{2k+1} can be chose arbitrarily.

Given $P(z)$ we can choose $H(z)$ and $\tilde{H}^*(z)$, but this may be done in many different ways.

Orthogonal Filter banks

Filters that satisfy $H(z) = \tilde{H}(z)$, $G(z) = \tilde{G}(z)$ are called orthogonal. Then

$$P(z) = H(z)\,\hat{H}^*(z) = H(z)\,H\left(\frac{1}{z}\right).$$

In the DFT version

$$P(\omega) = H(\omega)\,\overline{H(\omega)} = |H(\omega)|^2 \geq 0$$

Then $P(\omega)$ is even and real-valued which implies that the coefficients and symmetric, and perfect reconstruction follows from

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 2.$$

The reason for "wording orthogonal" (Why are $H(z)=\tilde{H}(z)$ means orthogonal?)

Recall than in the example of the Haar bases, we had

$$\varphi_k = \begin{cases} \frac{1}{\sqrt{2}}, & k=0,1 \\ 0 & \text{else} \end{cases} \qquad \psi_k = \begin{cases} \frac{1}{\sqrt{2}} & k=0 \\ -\frac{1}{\sqrt{2}} & k=1 \\ 0 & \text{else} \end{cases}$$

and we constructed the basis functions

$$\varphi_k^{(2n)} = \varphi_{k-2n}, \quad \varphi_k^{(2n+1)} = \psi_{k-2n}, \text{ and the}$$

$$y_n^{(0)} = \langle x, \varphi^{(2n)} \rangle, \quad y_n^{(1)} = \langle x, \varphi^{(2n-1)} \rangle; \text{ and in the end}$$

$$x = \sum \langle x, \varphi^{(2n)} \rangle \varphi^{(2n)} + \sum \langle x, \varphi^{(2n+1)} \rangle \varphi^{(2n+1)}.$$

Biorthogonal bases

Often it is advantageous to work with "biorthogonal" bases, which corresponds to expansions of the form

$$x = \sum_n \langle x, \tilde{\varphi}^{(2n)} \rangle \varphi^{(2n)} + \sum_n \langle x, \tilde{\varphi}^{(2n+1)} \rangle \varphi^{(2n+1)}.$$

The basis functions must then satisfy

$$\langle \tilde{\varphi}^{(2k)}, \varphi^{(2n)} \rangle = \delta_{k,n}$$

$$\langle \tilde{\varphi}^{(2k+1)}, \varphi^{(2n+1)} \rangle = \delta_{k,n}$$

$$\langle \tilde{\varphi}^{(2k)}, \varphi^{(2n+1)} \rangle = \langle \tilde{\varphi}^{(2k+1)}, \varphi^{(2n)} \rangle = 0.$$

We shall return to this relations when discussing multi-resolution analysis.

Design of filter banks

This can be done in three steps:

Step 1 Find $p(z)$ that satisfies the condition for perfect reconstruction.

Step 2 Factorize $p(z)$:

$$P(z) = H(z)\,\tilde{H}^*(z).$$

Step 3 Define the high pass filters by

$$G(z) = -z^{-L}\,\tilde{H}^*(-z)$$

$$\tilde{G}(z) = -z^{-L}\,\tilde{H}(-z), \quad \text{where } L \text{ is odd.}$$

Example

The Haar bases. $\quad p(z) = \frac{1}{2}(z+2+z^{-1}) = \frac{1}{\sqrt{2}}(z+1)\cdot\frac{1}{\sqrt{2}}\left(\frac{1}{z}+1\right)$

Example (Ingrid Daubechies)

$$P(z) = \left(\frac{1+z}{2}\right)^N \left(\frac{1+z^{-1}}{2}\right)^N Q_N(z),$$

where $Q_N(z)$ is symmetric:

$$Q_N(z) = a_{N-1} z^{N-1} + \cdots + a_1 z^1 + a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{1-N}$$

$N=1$, then gives the Haar wavelet,

$N=2$: $$p(z) = \left(\frac{1+z}{2}\right)^2 \left(\frac{1+z^{-1}}{2}\right)^2 Q_2(z)$$

$$Q_2(z) = a_1 z^1 + a_0 + a_1 z^{-1}.$$

Wavelets

Recall the Fourier transform (2π-periodic definition)

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t)\, e^{-i\omega t}\, dt = \langle f, e^{i\omega\cdot} \rangle$$

Def. $\langle f, g \rangle = \int f(t)\, \overline{g(t)}\, dt$.

The Fourier transform gives a decomposition in pure frequencies; $e^{i\omega t}$

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \langle f, e^{i\omega\cdot} \rangle\, e^{i\omega t}\, d\omega.$$

The wavelet decomposition is different. Let $\psi(t)$ be a given function, and let

$$\psi_{j,k} = 2^{j/2}\, \psi(\underbrace{2^j t}_{\text{dilation}} - \underbrace{k}_{\text{Translation}}).$$

Under suitable conditions,

$$f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{jk} \rangle\, \psi_{jk}(t).$$

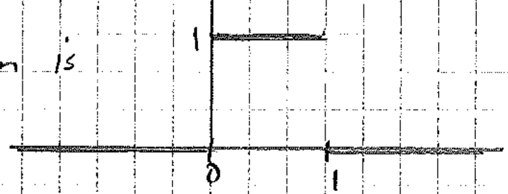
Example The Haar Wavelet.

Let $\varphi(t) = \varphi(2t) + \varphi(2t-1)$ ← (need to check that it is possible)

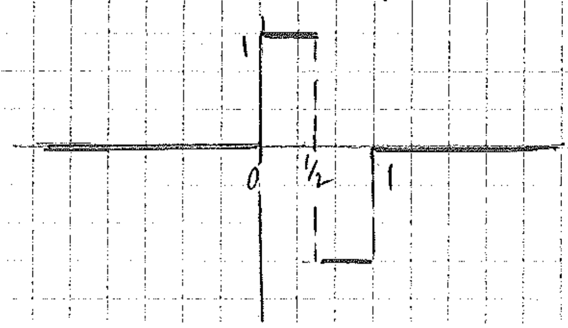
$\psi(t) = \varphi(2t) - \varphi(2t-1)$ ← can always be achieved.

φ is called a scaling function and ψ a wavelet.

The Haar scaling function is



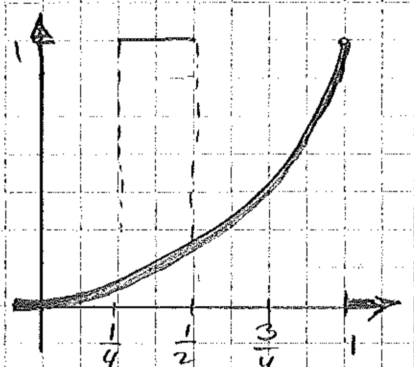
and the wavelet is



Note that $\langle \varphi, \psi \rangle = 0$

$\langle \varphi(\cdot-k), \varphi(\cdot-n)\rangle = 0 \quad (n \neq k)$

$\langle \psi(\cdot-k), \psi(\cdot-n)\rangle = 0 \quad (n \neq k)$



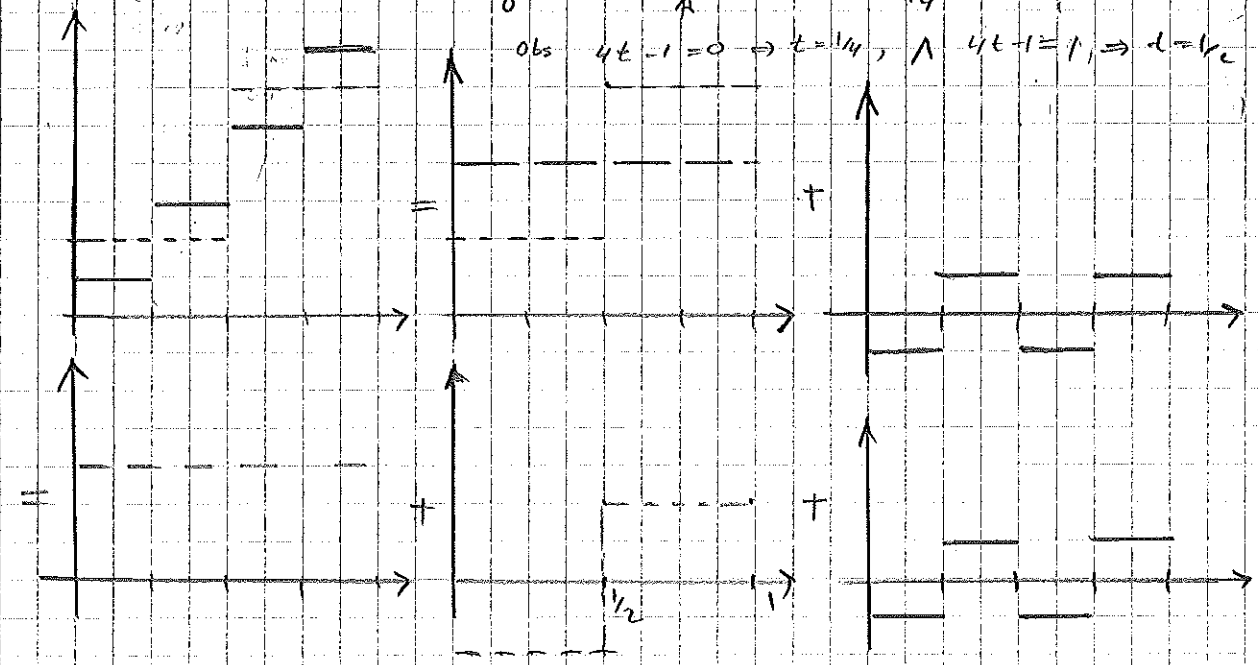
As an example, let $f(t) = \begin{cases} t^2 & 0<t<1 \\ 0 & \text{else} \end{cases}$

The function $f(t)$ can be approximated by step functions, which is the mean of f in each interval.

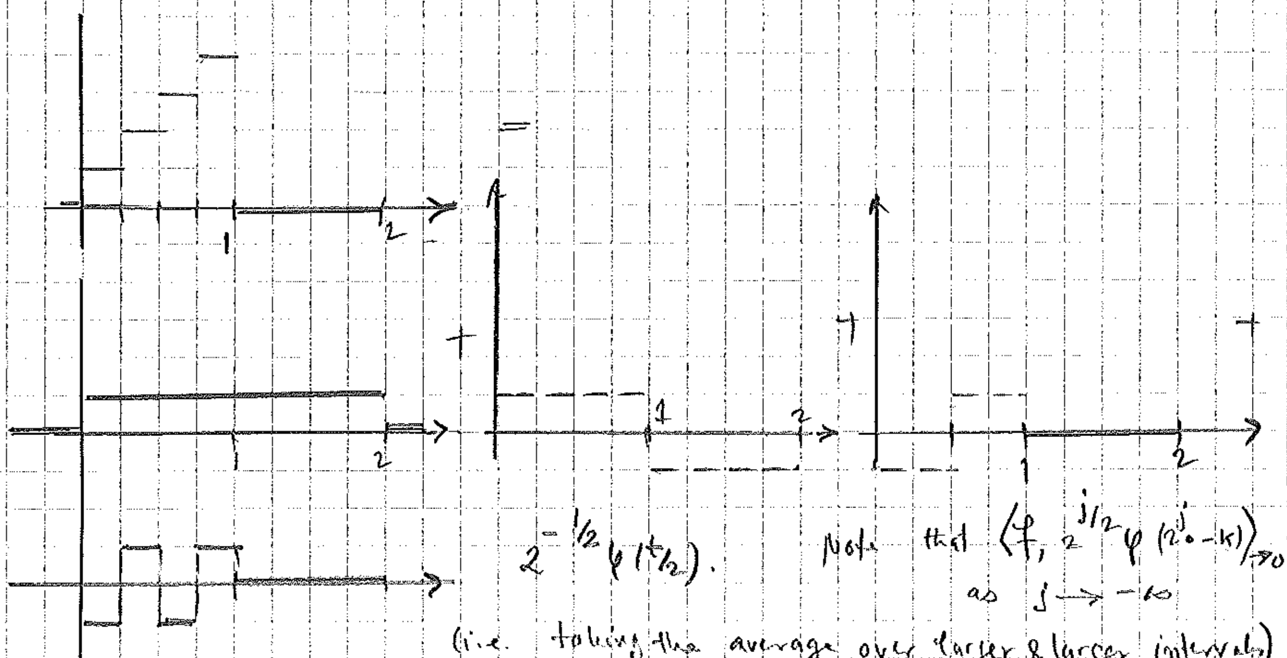
For example, in the interval $[\frac{1}{4}, \frac{1}{2}]$ we have

$$\langle f, 2\varphi(2^2\cdot -1)\rangle = \int_0^1 t^2 \cdot 2\varphi(2^2 t-1)\,dt = \int_{1/4}^{1/2} t^2\, 2\,dt \cong 0.073$$

obs $4t-1=0 \Rightarrow t=1/4$, $\wedge$ $4t-1=1 \Rightarrow t=1/2$



This is the basic idea behind wavelets: We can express the signal as an average plus finer and finer details. Note that we can continue



$2^{-1/2}\varphi(t/2)$. Note that $\langle f, 2^{j/2}\varphi(2^j\cdot -k)\rangle \to 0$ as $j \to -\infty$

(i.e. taking the average over larger & larger intervals)

Multi resolution analysis

The Hilbert space $L^2(\mathbb{R})$.

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \to \mathbb{C} : \int |f(t)|^2 dt < \infty \}$$

(for a proper definition, we need the concept of measurability).

The norm of f is defined as

$$\|f\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2}.$$

properties of a norm: $\|f\| \geq 0$, $\|f\| = 0 \iff f = 0$

$\|cf\| = |c| \, \|f\|$, $c \in \mathbb{C}$

$\|f+g\| \leq \|f\| + \|g\|$.

In a Hilbert space, the norm is defined via a scalar product:

$$\|f\| = \langle f, f \rangle^{1/2}.$$

Recall the Parseval formula: $\int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{\mathbb{R}} |f(t)|^2 dt.$

Closed subspaces and projections

A linear subspace $V \subset L^2(\mathbb{R})$ is closed if $f_n \in V$, $n = 1, \ldots, \infty$ and $f_n \to f$ in $L^2(\mathbb{R})$, implies that $f \in V$.

Example $\{ f \in L^2(\mathbb{R}), \ f(t) = 0 \text{ when } t < 0 \}$

$$V_{BL} = \{ f \in L^2(\mathbb{R}) : |\hat{f}(\omega)| = 0 \text{ when } |\omega| > a \}$$

Def A linear operator $P : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is called a projection if

$$P(Pf) = Pf, \quad \text{for all } f \in L^2(\mathbb{R}).$$

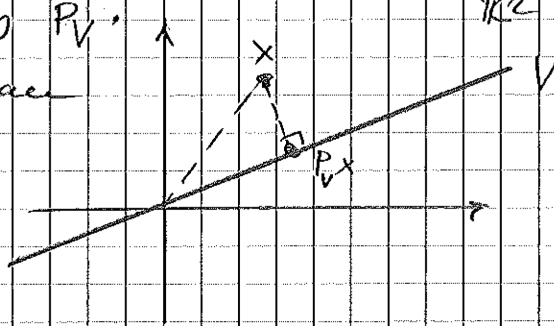
Def The orthogonal projection of f onto the closed subspace V is the unique $w \in V$ such that

$$\|f-w\| \le \|f-v\| \quad \text{for all } v \in V.$$

This projection is denoted by P_V.

Here is an example of a subspace of $\mathbb{R}^2$. It is clear that

$$P_V(P_V x) = P_V x.$$



Note that

$$\langle f - P_V f, w\rangle = 0 \quad \forall w \in V.$$

Bases A family of functions $\{\varphi_k\}_{k\in\mathbb{Z}}$ is a basis for $V \subset L^2(\mathbb{R})$ if any $f \in V$ can be written (uniquely) as

$$f = \sum_k c_k \varphi_k, \qquad \text{where } c_k \in \mathbb{C}.$$

Note that if $\{\varphi_k\}_{k\in\mathbb{Z}}$ is orthonormal, then

$$\|f\|^2 = \langle \sum_k c_k\varphi_k, \sum_k c_k \varphi_k\rangle$$
$$= \sum_{j\neq k} \cancel{\langle \sum c_j\varphi_j, \sum c_k \varphi_k\rangle} + \sum c_k \bar{c}_k \underbrace{\langle \varphi_k, \varphi_k\rangle}_{=1} = \sum |c_k|^2.$$

But for a general basis this is not true.

Def A Riesz basis for a closed subspace $V \subset L^2$ is a basis that satisfies

$$A\,\|f\|^2 \le \sum_k |c_k|^2 \le B\,\|f\|^2,$$

for some constants $A \le 1 \le B$.

<u>Note</u> If $\tilde{f}$ is an approximation of f

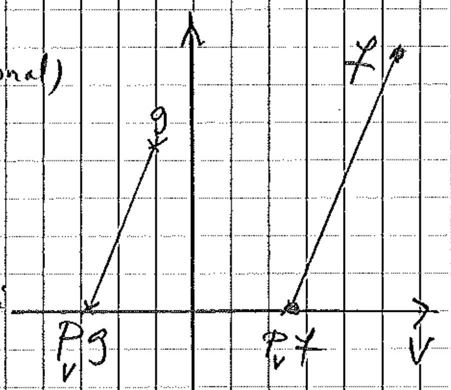
$$A\,\|f-\tilde{f}\| \le \sum_k |c_k - \tilde{c}_k|^2 \le B\,\|f-\tilde{f}\|.$$

A projection onto V: (orthogonal)

$$P_V = \sum \langle f, \varphi_k \rangle \varphi_k$$

An image of a non orthogonal projection:

Here $V = \{(x,0) \in \mathbb{R}^2\}$.

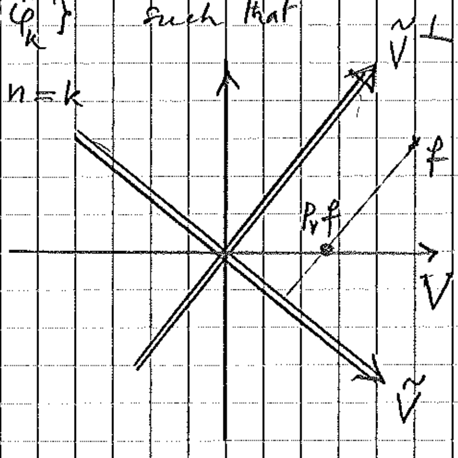


<u>Biorthogonal bases</u>

A biorthogonal basis is a basis $\{\varphi_k\}$ of a subspace V together with a "dual family" $\{\tilde{\varphi}_k\}$ such that

$$\langle \varphi_k, \tilde{\varphi}_n \rangle = \delta_{kn} = \begin{cases} 1 & \text{if } n=k \\ 0 & \text{else} \end{cases}$$

We let $P_V f = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$.

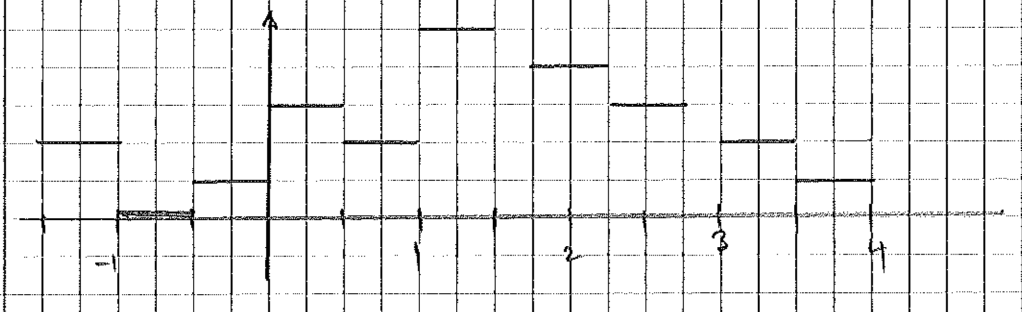


<u>The Haar Sampling function</u>

$$\varphi(t) = \begin{cases} 1 & 0<t<1 \\ 0 & \text{otherwise} \end{cases}$$

This can be used for piecewise constant approximation:

$$f(t) \longmapsto f_1(t) = \sum_k s_{1k}\, \varphi(2t-k)$$



Definition

A multiresolution analysis is a family of closed subspaces $V_j \subset L^2(\mathbb{R})$ such that:

1) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$

2) $f \in V_j \iff f(2\cdot) \in V_{j+1}$, $j \in \mathbb{Z}$

3) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.

4) $\bigcap_j V_j = \{0\}$

5) There is a scaling function $\varphi \in V_0$ such that $\{\varphi(\cdot - k)\}$ is a Riesz basis for V_0.

Def. $\varphi_{j,k} = 2^{j/2} \varphi(2^j t - k)$

Here 4) means that for any $f \in L^2(\mathbb{R})$, there is a sequence of functions $f_k \in V_k$ such that

$$\|f_k - f\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty$$

5) Means that if $f \in V_k$, $k \in \mathbb{Z}$, then

$$f(t) = 0 \quad \text{for all } t.$$

Properties of the scaling function

1) $\int_{-\infty}^{\infty} \varphi(t)\, dt = 1$

2) If $\{\varphi(\cdot - k)\}_{k \in \mathbb{Z}}$ is a basis for V_0, then we must have, that $\{2^{1/2}\varphi(2\cdot - k)\}$ is a basis for V_1. And since $V_0 \subset V_1$, $\varphi \in V_1$ and hence

$$\varphi(t) = 2 \sum_k h_k \varphi(2t - k), \quad \text{for some } \{h_k\}.$$

This is called the scaling equation.

Properties of the Scaling function

3) Let $H(\omega)=\sum_k h_k e^{-i\omega k}$, and let $\hat{\varphi}(\omega)$ be the Fourier transform of φ. Then

$$\hat{\varphi}(\omega)=\sum_k h_k \mathcal{F}\left(2\varphi(2\cdot-k)\right)(\omega)=\sum_{k=-\infty}^{\infty} h_k e^{-ik\omega/2}\hat{\varphi}\left(\tfrac{\omega}{2}\right)=H\left(\tfrac{\omega}{2}\right)\hat{\varphi}\left(\tfrac{\omega}{2}\right), \quad (*)$$

$(*)$ and by induction $\hat{\varphi}(\omega)=\prod_{j\geq 0} H\left(\omega/2^j\right)$

From $\hat{\varphi}(0)=1$, we conclude that

$$\sum_{k=-\infty}^{\infty} h_k = 1.$$

Example: $\varphi(t)=\operatorname{sinc} t=\frac{\sin \pi t}{\pi t} \Longrightarrow \hat{\varphi}(\omega)=\mathbb{1}_{[-\pi,\pi]}$

Wavelets and detail Spaces

In a multi resolution $\{V_j\}$ let f be approximated by f_0 and f_1 in V_0 and V_1 respectively.

Hence, $f_0\in V_0$, $f_1\in V_1$. But also $f_0\in V_1 \Rightarrow f_1-f_0\in V_1$.

For the Haar Wavelet we had $\psi(t)=\begin{cases}1 & 0<t<\frac12\\ -1 & \frac12<t<1\\ 0 & \text{else}\end{cases}$ and

$\psi(t)=\frac12\varphi_{1,0}-\frac12\varphi_{1,1}$.

Def. For a MRA, ψ is called a Wavelet if $W_0\subset V_1$ is spanned by $\{\psi(\cdot-k)\}_{k\in\mathbb{Z}}$ and $V_1=W_0\oplus V_0$, i.e. each $f_1\in V_1$ can be written (uniquely) as

$$f_1=f_0+d_0 \quad \text{with } f_0\in V_0,\ d_0\in W_0.$$

W_0 is called a detail space.

Properties of the Wavelets

1) $\int\psi(t)\,dt=0 \quad (\Leftrightarrow \hat{\psi}(0)=0)$

2) $\psi(t)\in V_1 \Leftrightarrow \psi(t)=2\sum_k g_k\varphi(2t-k)$

hence $\hat{\psi}(t)=G\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right)$, $\quad G(\omega)=\sum_k g_k e^{-ik\omega}$

and because $\hat{\varphi}(0)=1$, $\quad G(0)=\sum_k g_k=0$. $\boxtimes$

MRA and wavelet decomposition

We have $V_1 = V_0 \oplus W_0$, where

V_0 is spanned by $\{\varphi(\cdot - k)\}$, and

W_0 is spanned by $\{\psi(\cdot - k)\}$, both of which are

supposed to be Riesz bases.

Let $\psi_{j,k}(t) = 2^{j/2}\psi(2^{j/2}t - k)$ and define

W_j = linear span of $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$, i.e.

$W_j = \{ d_j(t) = \sum_k w_{jk}\, \psi_{jk}(t),\ \ w_{jk} \in \mathbb{C} \}$.

Let J be an integer, corresponding to highest resolution, or in other words, the finest detail of interest.

Take f_J be an approximation of $f \in L^2(\mathbb{R})$. Then

$$
\begin{aligned}
f_J \in V_J &= W_{J-1} \oplus V_{J-1} \\
&= W_{J-1} \oplus W_{J-2} + V_{J-2} \\
&= \cdots \\
&= W_{J-1} \oplus W_{J-2} \oplus W_{J-3} \oplus \cdots \oplus W_{j_0} \oplus V_{j_0}
\end{aligned}
$$

Then we can write

$$
\begin{aligned}
f_J(t) &= d_{J-1}(t) + d_{J-2}(t) + \cdots + d_{j_0}(t) + f_{j_0}(t) \\
&= \sum_{j=j_0}^{J-1} \sum_k w_{j,k}\, \psi_{jk}(t) + \sum_k s_{j_0 k}\, \varphi_{j_0 k}(t).
\end{aligned}
$$

The last sum $\sum_k s_{j_0k}\varphi_{j_0k}(t)$ converges to zero as $j \to \infty$, because $\bigcap V_j = \{0\}$.

Also, because $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ one can choose

$f_J \to f$ in L^2, $f_J \in V_J$. So letting $J \to \infty$ we find

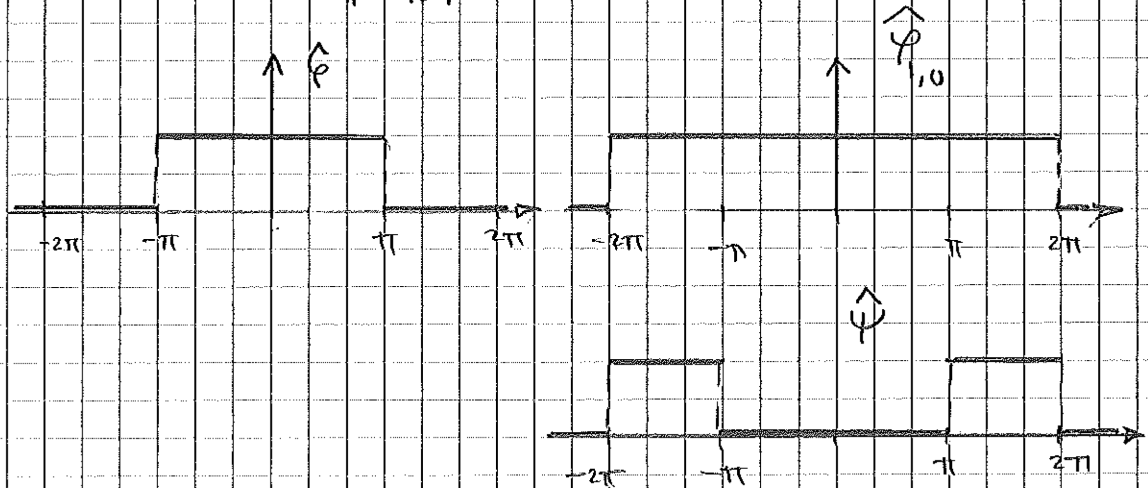
$$\underline{f(t) = \sum_{j,k} w_{j,k}\, \psi_{j,k}(t).} \longleftarrow \quad \underline{\text{This is the wavelet decomposition of } f.}$$

Example $\varphi(t) = \operatorname{sinc} t = \frac{\sin \pi t}{\pi t}$

Corresponding to $\hat{\varphi}(\omega) = \mathbb{1}_{-\pi < \omega < \pi}$.

Then V_0 is the set of bandlimited functions, with cutoff π, and V_j the set of bandlimited functions with cutoff $2^j \pi$.

Here $\hat{\psi}(\omega) = \mathbb{1}_{\pi < |\omega| < 2\pi}$



Orthogonal Wavelet decomposition

General properties of scaling function φ and wavelet ψ:

$$\begin{cases} \varphi(t) = \sum_k h_k \, 2\varphi(2t-k); & \int \varphi(t)\,dt = 1 \\ \hat{\varphi}(\omega) = H(\tfrac{\omega}{2})\, \varphi(\tfrac{\omega}{2}) \quad \text{where} \quad H(\omega) = \sum h_k e^{-i\omega k}, & \hat{\varphi}(0) = 1 \\ \hat{H}(0) = 1 \iff \sum h_k = 1 \end{cases}$$

$$\psi(t) = \sum_k g_k \, 2\varphi(2t-k); \qquad \int \psi(t)\,dt = 0$$

$$\hat{\psi}(\omega) = G(\tfrac{\omega}{2})\, \hat{\varphi}(\tfrac{\omega}{2}); \qquad G(0) = 0, \qquad G(\omega) = \sum g_k e^{-ik\omega}$$

NB The decomposition $V_{j+1} = V_j \oplus W_j$.

It is defined by the filter G. However, there is only one way of writing $V_{j+1} = V_j \oplus W_j$ so that

$\langle v, w \rangle = 0$ for all $v \in V_j$, $w \in W_j$.

For an orthonormal system we require

$$\int_{-\infty}^{\infty} \varphi(t-k)\overline{\varphi(t-n)}\,dt = \begin{cases} 0 & \text{when } n \neq k \\ 1 & \text{when } n = k. \end{cases}$$

In the scaling equation:

$$\int \varphi(t-n)\overline{\varphi(t-m)}\,dt = 4\sum_{k,k'} h_k \overline{h_{k'}} \int \varphi(2(t-n)-k)\overline{\varphi(2(t-m)-k')}\,dt$$

$$= 2\sum_{k,k'} h_k \overline{h_{k'}} \int \varphi(t-2n-k)\overline{\varphi(t-2m-k')}\,dt = \begin{cases} 1 & \text{if } 2n+k = 2m+k' \\ 0 & \text{else} \end{cases}$$

$$= 2 \sum_{2n+k=2m+k'} h_k \overline{h_{k'}} .$$

Without loss of generality, we may take $n = 0$

$$\Rightarrow \int \varphi(t)\overline{\varphi(t-m)}\,dt = 2\sum_{k=2m+k'} h_k \overline{h_{k'}} = 2\sum_{k'} h_{k'+2m} \overline{h_{k'}}$$

so $$2\sum_k \overline{h_k}\, h_{k+2m} = \begin{cases} 1 & \text{if } m=0 \\ 0 & m \neq 0 \end{cases} \qquad (a)$$

Next we demand that $\{\psi(t-k)\}_{k=-\infty}^{\infty}$ is an ON-basis for W_0.

$$\int_{-\infty}^{\infty} \psi(t-k)\overline{\psi(t-m)}\,dt = \begin{cases} 1 & \text{if } k=m \\ 0 & \text{else} \end{cases} \Leftrightarrow$$

$$2\sum_k g_k\, \overline{g_{k+2m}} = \begin{cases} 1 & \text{if } m=0 \\ 0 & m\neq 0. \end{cases} \qquad (b)$$

And finally we demand $V_0 \perp W_0$, which means that

$$\int_{-\infty}^{\infty} \varphi(t-k)\overline{\psi(t-n)}\,dt = 0 \quad \text{for all } k, n$$

Which means that

$$\sum_m h_{m+2k}\, \overline{g_{m+2n}} = 0. \qquad (c)$$

The Haar Wavelet is an example showing that such systems exist.

Once constructed such a system can be used to approximate $f \in L^2(\mathbb{R})$:

$$f \approx f_j + d_j \quad \text{where} \quad f_j = P_j f = \sum_k \langle f, \varphi_{jk} \rangle \varphi_{jk}$$
$$d_j = Q_j f = \sum_k \langle f, \psi_{jk} \rangle \psi_{jk} .$$

Proposition. The ON-condition implies that

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$
$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1$$
$$H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega+\pi)} = 0$$

Proof (of the 1st statement)

$$H(\omega)\overline{H(\omega)} + H(\omega+\pi)\overline{H(\omega+\pi)} = \{\text{assuming } h_k \text{ real}\}$$
$$= \Big(\sum_k h_k e^{-i\omega k}\Big)\Big(\sum_{k'} h_{k'} e^{i\omega k'}\Big) + \Big(\sum_k h_k (-1)^k e^{-i\omega k}\Big)\Big(\sum_{k'} h_{k'} (-1)^{k} e^{i\omega k'}\Big)$$
$$= \sum_{kk'} h_k h_{k'} \left(1 + (-1)^{k+k'}\right) e^{-i\omega(k-k')} = 2 \sum_{k+k'=2m} h_k h_{k'} e^{-i\omega(k-k')}$$
$$= \left\{ \begin{matrix} u = k-k' \\ k' = k-u \end{matrix} \right\} = 2 \sum_{2k-u=2m} h_k h_{k-u} e^{-i\omega u} = \left\{ \begin{matrix} \text{need} \\ u = 2u' \end{matrix} \right\}$$
$$= 2 \sum_{k,u'} h_k h_{k-2u'} e^{-i\omega 2u'} = \sum_{k'} e^{-i\omega 2u'} \underbrace{2\sum_k h_k h_{k-2u'}}_{= \begin{cases} 1 & \text{if } u'=0 \\ 0 & \text{else} \end{cases}} = 1 .$$

Proposition (conditions on the scaling function and the wavelet)

1) $\sum |\hat{\varphi}(\omega + 2k\pi)|^2 = 1$

2) $\lim_{j \to -\infty} \hat{\varphi}(2^{-j}\omega) = 1$ (obvious if $\hat{\varphi}$ continuous at $\omega = 0$).

3) $\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega)$, H 2π-periodic

4) $\sum_j |\hat{\psi}(2^j\omega)|^2 = 1$

5) $\sum_{j \geq 0} \hat{\psi}(2^j\omega)\overline{\hat{\psi}(2^j(\omega + 2k\pi))} = 0$, when k is odd

6) $\sum_{j \geq 0} \sum_k |\hat{\psi}(2^j(\omega + 2k\pi))|^2 = 1$.

Proof of 1):

$$\int_{\mathbb{R}} \varphi(t)\,\overline{\varphi(t-k)}\,dt = \begin{Bmatrix} 0 & k\neq 0 \\ 1 & k=0 \end{Bmatrix} = \delta_k$$

$$\Leftrightarrow \quad \delta_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\,\overline{e^{-ik\omega}\,\hat{\varphi}(\xi)}\,d\xi$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\xi)\,e^{i\omega k}\,\overline{\hat{\varphi}(\xi)}\,d\xi$$

$$= \sum_n \frac{1}{2\pi}\int_0^{2\pi} |\hat{\varphi}(\xi+2\pi n)|^2 e^{i(\xi+2\pi n)k}\,d\xi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{ik\xi}\sum_n |\hat{\varphi}(\xi+2n\pi)|^2\,d\xi.$$

So δ_k is the Fourier series of the periodic function

$$r(\omega) = \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega+2\pi n)|^2,$$ and this implies that $r(\omega)=1$.

The continuous Wavelet transform

We have seen how to write $f\in L^2(\mathbb{R})$ as a "Wavelet series"

$$f = \sum_{j,k} c_{jk}\,\psi_{jk}.$$

Here is a different kind of decomposition: Take $\psi\in L^2(\mathbb{R})$, and assume that

$$C_\psi = \frac{1}{2\pi}\int_{\mathbb{R}} \frac{1}{|\xi|}\,|\hat{\psi}(\xi)|^2\,d\xi < \infty.$$

If $\psi\in L^1(\mathbb{R})$, then $\hat{\psi}(\xi)$ is continuous, and then we must have $\hat{\psi}(0) = \int_{\mathbb{R}}\psi(x)\,dx = 0$.

The continuity of $\hat{\psi}$ follows from

$$|\hat{\psi}(\xi+h)-\hat{\psi}(\xi)| = \left|\int_{-\infty}^{\infty}\left(e^{-i(\xi+h)x}-e^{-i\xi x}\right)f(x)\,dx\right|$$

$$\le \int_{-\infty}^{\infty}|e^{-ihx}-1|\,|f(x)|\,dx \le \int_{|x|<M}|e^{-ihx}-1|\,|f(x)|\,dx + \int_{|x|>M}|f(x)|\,dx$$

Given $\varepsilon>0$, take M so large that $\int_{|x|>M}|f(x)|\,dx < \frac{\varepsilon}{2}$, and

and then δ so small that

$$|1-e^{-ihx}| < \varepsilon\left(2\int|f(x)|dx\right)^{-1}, \quad \text{for } |x|\le M, \ |h|<\delta;$$

We then find that $|\hat{\psi}(\xi+h)-\hat{\psi}(\xi)| \le \varepsilon$, for $|h|<\delta$.

<u>Define</u> $\psi^{ab}(x) = |a|^{-1/2}\psi\left(\frac{x-b}{a}\right)$

[This is a <u>doubly indexed family of wavelets</u>]

<u>Definition</u>

$$\left(T^{wav}f\right)(a,b) = \langle f, \psi^{ab}\rangle = \int f(x)\,\overline{\psi\left(\frac{x-b}{a}\right)}\,|a|^{-1/2}\,dx$$

<u>Note</u> Because $\|\psi^{ab}\|_{L_2}=1$, $\left|\left(T^{wav}f\right)(a,b)\right| \le \|f\|_{L^2}$.

<u>Proposition</u> (The resolution of the identity)

For all $f, g \in L^2(\mathbb{R})$, the following equality holds:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(T^{wav}f\right)(a,b)\,\overline{\left(T^{wav}g\right)(a,b)}\,\frac{1}{|a|^2}\,da\,db = C_\psi\langle f,g\rangle$$

[This result should be compared with

$$\int f(x)\,\overline{g(x)}\,dx = \frac{1}{2\pi}\int \hat{f}(\xi)\,\overline{\hat{g}(\xi)}\,d\xi \ ]$$

We can rewrite the equality as

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(T^{wav}f\right)(a,b)\,\overline{\int_{-\infty}^{\infty} g(x)\,|a|^{-1/2}\,\psi\left(\frac{x-b}{a}\right)dx}\,\frac{1}{|a|^2}\,da\,db$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(T^{wav}f\right)(a,b)\,\psi\left(\frac{x-b}{a}\right)\frac{1}{|a|^{2+\frac{1}{2}}}\,da\,db\right)\overline{g(x)}\,dx$$

$$=\int_{-\infty}^{\infty}\underbrace{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(T^{wav}f\right)(a,b)\,\psi^{ab}(x)\,\frac{1}{|a|^2}\,da\,db\right)}_{\text{so, "in a weak sense" this must be equal to } f.}\overline{g(x)}\,dx$$

Proof $\iint \frac{da\,db}{|a|^2} \left(T^{wav} f\right)(a,b) \overline{\left(T^{wav} g\right)(a,b)}$

$$= \iint \frac{da\,db}{|a|^2} \left[\frac{1}{2\pi}\int d\xi\, \hat{f}(\xi)\, |a|^{1/2} e^{-ib\xi}\, \overline{\hat{\psi}(a\xi)}\right] \times$$

$$\times \left[\frac{1}{2\pi}\int d\xi'\, \overline{\hat{g}(\xi')}\, |a|^{1/2} e^{ib\xi}\, \hat{\psi}(a\xi')\right] \qquad (*)$$

i.e. We have expressed $\left(T^{wav} f\right)(a,b)$ using Parseval's formula on

$$T^{wav} f(a,b) = \int_{-\infty}^{\infty} f(x)\, |a|^{-1/2} \psi\left(\frac{x-b}{a}\right) dx$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, \mathcal{F}\left(a^{-1/2} \psi\left(\frac{\cdot - b}{a}\right)\right)(\xi)\, d\xi = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\xi)\, |a|^{1/2} e^{ib\xi} \hat{\psi}(a\xi)\, d\xi .$$

Next we note that

$$\int_{-\infty}^{\infty} \underbrace{\hat{f}(\xi)\, |a|^{1/2}\, \overline{\hat{\psi}(a\xi)}}_{\equiv F_a(\xi)}\, e^{-ib\xi}\, d\xi = \hat{F}_a(b)$$

where ξ and b are treated as dual variables in a Fourier transform.

Similarly, $\int e^{ib\xi'} \underbrace{|a|^{1/2}\, \overline{\hat{g}(\xi')}\, \hat{\psi}(a\xi')}_{\equiv \overline{G_a(\xi)}} d\xi' = \overline{\hat{G}_a(b)}$

Therefore we can write

$$(*) = \frac{1}{4\pi^2}\int \frac{da}{a^2}\int db\, \hat{F}_a(b)\, \overline{\hat{G}_a(b)} = \frac{1}{2\pi}\int \frac{da}{a^2}\int F_a(\xi)\, \overline{G_a(\xi)}\, d\xi$$

$$= \frac{1}{2\pi}\int \frac{da}{a}\int \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, |\hat{\psi}(a\xi)|^2 d\xi$$

$$= \frac{1}{2\pi}\int \hat{f}(\xi)\, \overline{\hat{g}(\xi)} \underbrace{\int \frac{|\hat{\psi}(a\xi)|^2}{|a|} da}_{= \int |\hat{\psi}(a)|^2 \frac{da}{a} = C_\psi} d\xi$$

$$= C_\psi\, \frac{1}{2\pi}\int \hat{f}(\xi)\, \overline{\hat{g}(\xi)}\, d\xi = C_\psi \langle f, g \rangle .$$

Note that all integrals of the form $\int \frac{F(xy)}{y} dy$ are independent of x, if the integral is convergent). ☐

The family $\{\psi^{ab}\}_{\substack{a\in\mathbb{R}\\ b\in\mathbb{R}}}$ is uncountable, but in the discrete wavelet case, we have only countably many $\psi_{j,k}$.

How many ψ_{jk}'s do we really need?

<u>Definition</u> A family of functions $(\varphi_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a <u>frame</u> if there exists $0 < A \leq B$ such that for all $f \in \mathcal{H}$,

$$A\,\|f\|^2 \leq \sum_j |\langle f, \varphi_j\rangle|^2 \leq B\|f\|^2.$$

A and B are called the <u>frame bounds</u>.

A frame is called tight if $A = B$.

<u>Proposition</u> If $(\varphi_j)_{j\in J}$ is a tight frame with $A=1$ and $\|\varphi_j\| = 1$ for all $j\in J$, then (φ_j) is an ON-basis.

<u>Proof</u> If $\langle f, \varphi_j\rangle = 0$ for all $j\in J$, then $\|f\| = 0$ and therefore $(\varphi_j)_{j\in J}$ span all of $\mathcal{H}$.

Otherwise let $\mathcal{H}_0$ be a span of $(\varphi_j)_{j\in J}$ and let $\mathcal{H}_1$ be the orthogonal complement to $\mathcal{H}_0$ in $\mathcal{H}$, i.e. $\mathcal{H}_0 \perp \mathcal{H}_1$ and $\mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}$.
Take $g \in \mathcal{H}_1$, $\|g\| = 1$. But then $\langle g, \varphi_j\rangle = 0$ for all $j\in J$, which contradicts the tightness of the frame).

Also $\|\varphi_j\|^2 = \sum_{j'} |\langle \varphi_j, \varphi_{j'}\rangle|^2 = \|\varphi_j\|^4 + \sum_{j'\neq j} |\langle\varphi_j, \varphi_{j'}\rangle|^2$

But $\|\varphi_j\|^2 = \|\varphi_j\|^4 = 1 \Rightarrow \sum_{j'\neq j} |\langle \varphi_j, \varphi_{j'}\rangle| = 0$

so $\langle \varphi_j, \varphi_{j'}\rangle = 0$ if $j \neq j'$.

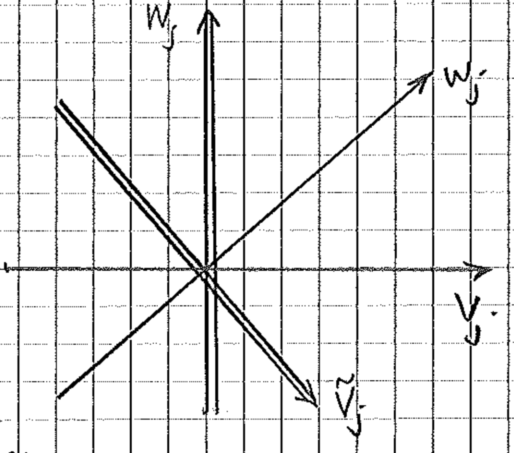
If a frame is tight $\sum_{j \in J} |\langle f, \varphi_j \rangle|^2 = A \|f\|^2 \Rightarrow$

$$A \langle f, g \rangle = \sum_{j \in J} \langle f, \varphi_j \rangle \langle \varphi_j, g \rangle \Rightarrow f = \frac{1}{A} \sum \langle f, \varphi_j \rangle \varphi_j.$$

For a frame that is not an ON-basis, there may be many ways of writing $f = \sum c_j \varphi_j$.

The most "economical" way is by aid of the "dual frame" $\tilde{\varphi}_j$.

Biorthogonal systems



The figure indicates how V_j and W_j together span $\mathbb{R}^2$, which serves as a image of V_{j+1}.

But they are not orthogonal.

Instead there is a space $\tilde{W}_j$ that is the orthogonal complement to V_j. $\tilde{W}_j$ is the detail space of a "dual MRA", $\{\tilde{V}_j\}_{j=-\infty}^{\infty}$; $\tilde{V}_j \subset \tilde{V}_{j+1}$ ---

There is a dual scaling function $\tilde{\varphi}$ that satisfies a scaling equation

$$\tilde{\varphi}(t) = 2 \sum \tilde{h}_k \tilde{\varphi}(2t-k)$$

and dual mother wavelet $\tilde{\psi}$ that satisfies $\tilde{\psi}(t) = 2 \sum \tilde{g}_k \tilde{\varphi}(2t-k)$.

The conditions for biorthogonality are

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,n} \rangle = \delta_{kn}; \qquad \langle \psi_{j,k}, \tilde{\psi}_{j,n} \rangle = \delta_{kn}$$

$$\langle \varphi_{j,k}, \tilde{\psi}_{j,n} \rangle = 0, \qquad \langle \tilde{\varphi}_{j,k}, \psi_{j,n} \rangle = 0.$$

One may then derive orthogonality conditions for $H, G, \tilde{H}$ and $\tilde{G}$:

$$\begin{cases} \tilde{H}(\omega) \overline{H(\omega)} + \tilde{H}(\omega+\pi) \overline{H(\omega+\pi)} = 1 \\ \tilde{G}(\omega) \overline{G(\omega)} + \tilde{G}(\omega+\pi) \overline{G(\omega+\pi)} = 1 \\ \tilde{G}(\omega) \overline{H(\omega)} + \tilde{G}(\omega+\pi) \overline{H(\omega+\pi)} = 0 \\ \tilde{H}(\omega) \overline{G(\omega)} + \tilde{H}(\omega+\pi) \overline{G(\omega+\pi)} = 0. \end{cases}$$

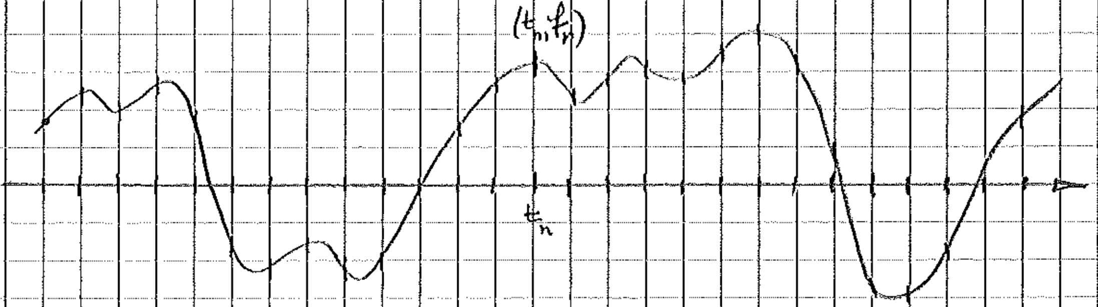
The discrete wavelet transform

Question: How to compute the scaling and wavelet coefficients?

$$f_J(t) = \sum_{j'=J_0}^{J-1} \sum_k w_{jk} \psi_{j,k}(t) + \sum_k s_{J_0} \varphi_{J_0,k}(t).$$

First: We have to decide a finest scale, J_0 and determine

$$s_{J,k} = \langle f, \varphi_{J,k} \rangle.$$



Usually the signal is sampled to give a discrete signal $(f_n)_{n=-\infty}^{\infty}$.
In the case of the Haar wavelet the $s_{J,k}$ are simply the f_n, but for higher order wavelets the calculation is more involved.
Note that the signal must not contain too high frequencies.

Once the $s_{J,k}$ are known, we may proceed recursively:

Assume that $s_{j+1,k} = \langle f, \varphi_{j+1,k} \rangle$ are known.
we may write $f_{j+1} = \sum \langle f, \varphi_{j+1,k} \rangle \varphi_{j+1,k}$ (obs! notation)
and we know that $f_{j+1} = f_j + d_j$.
Hence

$$f_{j+1}(t) = \sum_k s_{j+1,k} \varphi_{j+1,k}(t) = \sum_k s_{j,k} \varphi_{j,k}(t) + \sum_k w_{j,k} \psi_{j,k} = f_j(t) + d_j(t).$$

We may compute the scalar product with $\varphi_{j,\ell}$, to get

$$\sum_k s_{j+1,k} \langle \varphi_{j+1,k}, \varphi_{j,\ell} \rangle = \sum_k s_{j,k} \underbrace{\langle \varphi_{j,k}, \varphi_{j,\ell} \rangle}_{\delta_{k,\ell}} + \sum_k w_{j,k} \underbrace{\langle \psi_{j,k}, \varphi_{j,\ell} \rangle}_{=0}$$

Because $\varphi_{j,k} = \sqrt{2} \sum_m h_n \varphi_{j+1, m+2k}$.

We check this:

$$\varphi_0 = \varphi(t) = \sum_m h_m \, 2\varphi(2t-m).$$

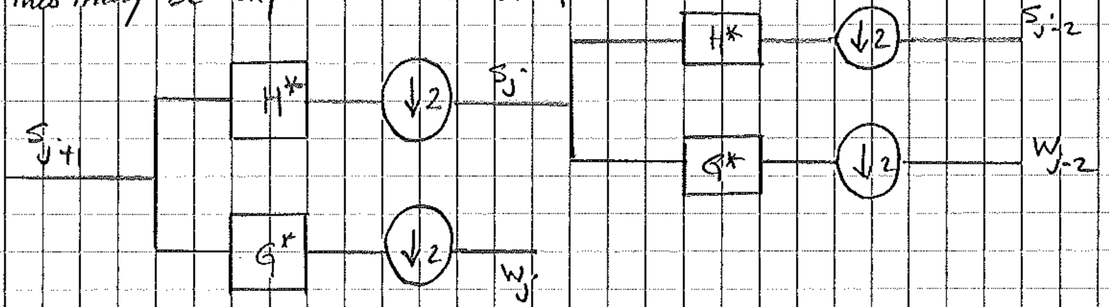
Then
$$\varphi_{0,k} = \varphi(t-k) = \sum_m h_m \sqrt{2}\sqrt{2}\, \varphi(2t-m-2k)$$
$$= \sum_m \sqrt{2}\, h_m \, \varphi_{1,2k+m}(t)$$

So
$$\langle \varphi_{j+1,k}, \varphi_{l,k} \rangle = \sqrt{2} \sum_m h_m \underbrace{\langle \varphi_{j+1,k}, \varphi_{j+1,m+2l} \rangle}_{\delta_{k,m+2l}} = \sqrt{2}\, h_{k-2l}.$$

A similar calculation holds for $\langle f_{j+1}, \psi_{j,k} \rangle$, and in summary

$$s_{j,k} = \sqrt{2} \sum_l h_{l-2k}\, s_{j+1,l}, \qquad w_{j,k} = \sqrt{2} \sum_l g_{l-2k}\, s_{j+1,l}.$$

This may be expressed with a filter bank:



The inverse calculation is carried out similarly.

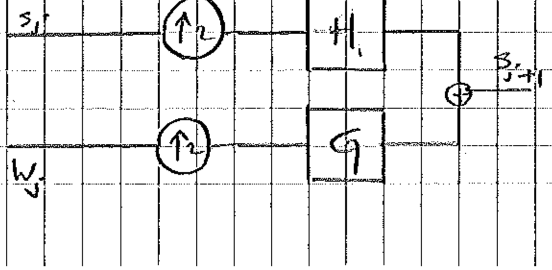
Write

$$f_{j+1} = \sum_l s_{j,l}\varphi_{j,l} + \sum_l w_{j,l}\psi_{j,l} =$$
$$= \sqrt{2} \sum_l \sum_m s_{j,l} h_m \varphi_{j+1,m+2l} + \sqrt{2} \sum_l \sum_m w_{j,l}\, g_m \varphi_{j+1,m+2l}$$

and

$$s_{j+1,k} = \langle f_{j+1}, \varphi_{j+1,k} \rangle = \sqrt{2} \sum_l \sum_m \left(s_{j,l} h_m + w_{j,l} g_m \right) \langle \varphi_{j+1,m+2l}, \varphi_{j+1,k} \rangle$$
$$= \sqrt{2} \sum \left(s_{j,l} h_{k-2l} + w_{j,l}\, g_{k-2l} \right)$$

This corresponds to the following filter:
(The factors $\sqrt{2}$ may be included in the filters).



Biorthogonal systems

Let $\{V_j\}$ be a MRA, and let

$\{\tilde{V}_j\}$ be a dual MRA.

The scaling function and mother wavelets for $\tilde{V}_j$ satisfies

$$\tilde{\varphi}(t) = 2\sum \tilde{h}_k \tilde{\varphi}(2t-k)$$

$$\tilde{\psi}(t) = 2\sum \tilde{g}_k \tilde{\varphi}(2t-k)$$

The biorthogonality conditions are,

$$\langle \varphi_{j,k}, \tilde{\varphi}_{j,l} \rangle = \delta_{k,l}$$

$$\langle \psi_{j,k}, \tilde{\psi}_{j,l} \rangle = \delta_{k,l}$$

$$\langle \varphi_{j,k}, \tilde{\psi}_{j,l} \rangle = 0$$

$$\langle \tilde{\varphi}_{j,k}, \psi_{j,l} \rangle = 0$$

At the finest resolution we may then write

$$f_J(t) = \sum_k \langle f, \tilde{\varphi}_{J,k} \rangle \varphi_{J,k}(t) = \sum_k s_{J-1,k}\varphi_{J-1,k} + \sum_k w_{J-1,k}\psi_{J-1,k}$$

and just as in the orthogonal case,

$$f(t) = \sum_{j,k} \langle f, \tilde{\psi}_{j,k} \rangle \psi_{j,k}.$$

Note that in this formula we have both $\tilde{\psi}_{j,k}$ and $\psi_{j,k}$.

Approximation

Assume that the scaling function is such that it reproduces polynomials of order $N-1$.
That means that for each $\alpha = 0, \ldots, N-1$,

$$t^\alpha = \sum_k C_k^\alpha \varphi(t-k), \quad \text{for some coefficients } \{C_k^\alpha\}.$$

It is a standard result from interpolation theory that if f is differentiable α times, then it can be well approximated by polynomials, and it is possible to deduce

$$\|f - P_j f\| \leq C\, 2^{-j\alpha} \|D^\alpha f\|.$$

Note that if $t^\alpha \in V_j$, then $t^\alpha \perp \tilde{W}_j$, and therefore $\langle t^\alpha, \tilde{\psi}_{j,k} \rangle = 0$, for every wavelet $\tilde{\psi}_{j,k}$, i.e.

$$\int t^\alpha \tilde{\psi}_{j,k}(t)\, dt = 0, \qquad \alpha = 0, \ldots, N-1.$$

We say that the dual wavelets have N vanishing moments (including $\alpha = 0$). But that means in turn that

$$D^\alpha \mathcal{F}(\tilde{\psi}) = 0, \text{ for } \alpha = 0, \ldots, N-1.$$

And because

$$\mathcal{F}(\tilde{\psi})(2\omega) = \mathcal{F}(\tilde{\varphi})(\omega)\, \tilde{G}(\omega) \quad \text{and } \mathcal{F}(\tilde{\varphi})(0) = 1,$$

we must have $\tilde{G}(\omega)$ has a zero of order N at $\omega = 0$, which in turn implies that

$$H(\omega) = \left(\frac{e^{-i\omega}+1}{2}\right)^N Q(\omega), \qquad \text{where } Q \text{ is } 2\pi\text{-periodic.}$$