Approximation

Assume that the scaling function is such that it reproduces polynomials of order $N-1$.
That means that for each $\alpha = 0, \dots, N-1$,

$$t^\alpha = \sum_k C_k^\alpha \varphi(t-k), \quad \text{for some coefficients } \{C_k^\alpha\}.$$

It is a standard result from interpolation theory that if f is differentiable α times, then it can be well approximated by polynomials, and it is possible to deduce

$$\|f - P_j f\| \le C\, 2^{-j\alpha} \|D^\alpha f\|.$$

Note that if $t^\alpha \in V_j$, then $t^\alpha \perp \tilde{W}_j$, and therefore $\langle t^\alpha, \tilde{\psi}_{j,k} \rangle = 0$, for every wavelet $\tilde{\psi}_{j,k}$, i.e.

$$\int t^\alpha \tilde{\psi}_{j,k}(t)\, dt = 0, \quad \alpha = 0, \dots, N-1.$$

We say that the dual wavelets have N vanishing moments (including $\alpha = 0$). But that means in turn that

$$D^\alpha \mathcal{F}(\tilde{\psi}) = 0, \quad \text{for } \alpha = 0, \dots, N-1.$$

And because

$$\mathcal{F}(\tilde{\psi})(2\omega) = \mathcal{F}(\tilde{\varphi})(\omega)\, \tilde{G}(\omega) \quad \text{and } \mathcal{F}(\tilde{\varphi})(0) = 1,$$

we must have $\tilde{G}(\omega)$ has a zero of order N at $\omega = 0$, which in turn implies that $\left(\tilde{H}(\omega) \overset{!}{=} \frac{G(\omega+\pi)}{\Delta(\omega)} \right)$ $\left(\tilde{G}(\omega) \overset{!}{=} \frac{H(\omega+\pi)}{\Delta(\omega)} \right)$

$$H(\omega) = \left(\frac{e^{-i\omega} + 1}{2} \right)^N Q(\omega), \quad \text{where } Q \text{ is } 2\pi\text{-periodic.}$$

Two dimensional signal processing

$$\mathbb{Z}^2 = \{(k_x, k_y) : \quad k_x = \dots, -1, 0, 1, \dots \qquad k_y = \dots -1, 0, 1, \dots\}$$

A filter in two dimensions is a map that takes a signal (image); and transforms it:

$$H : f \longmapsto g, \quad \text{where}$$
$$f : \mathbb{Z}^2 \longrightarrow \mathbb{R} \ (\text{or } \mathbb{C}); \qquad g : \mathbb{Z}^2 \longrightarrow \mathbb{R} \ (\text{or } \mathbb{C}).$$

The "shift operator", S^n, $n \in \mathbb{Z}^2$, is defined by

$$g = S^n f, \qquad g_k = f_{n-k} \qquad \forall k \in \mathbb{Z}^2.$$

A filter H is "shift invariant" if

$$H(S^n f) = S^n(Hf).$$

The impulse response of a filter is

$$h = H\delta, \qquad \text{where} \quad \delta_k = \begin{cases} 1 & \text{when } k = \binom{0}{0} \\ 0 & \end{cases}$$

If $f = (f_n)_{n \in \mathbb{Z}^2}$, then $g = Hf$ may be written as

$$g = \sum f_n S^n h \overset{\text{df}}{=} h * f$$

The two dimensional (discrete) Fourier transform is

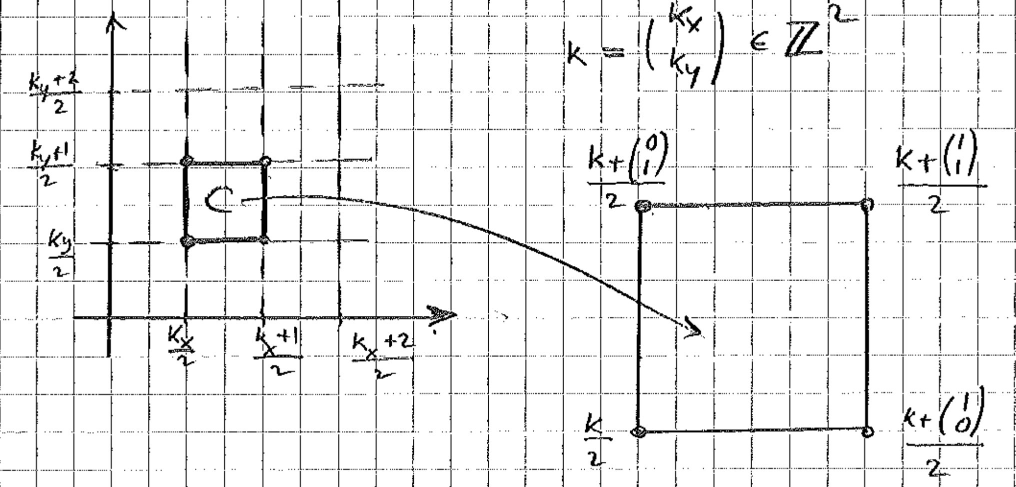
$$F(\xi, \eta) = \sum_{k \in \mathbb{Z}^2} f_k e^{-i(k_x \xi + k_y \eta)},$$

and the convolution theorem is

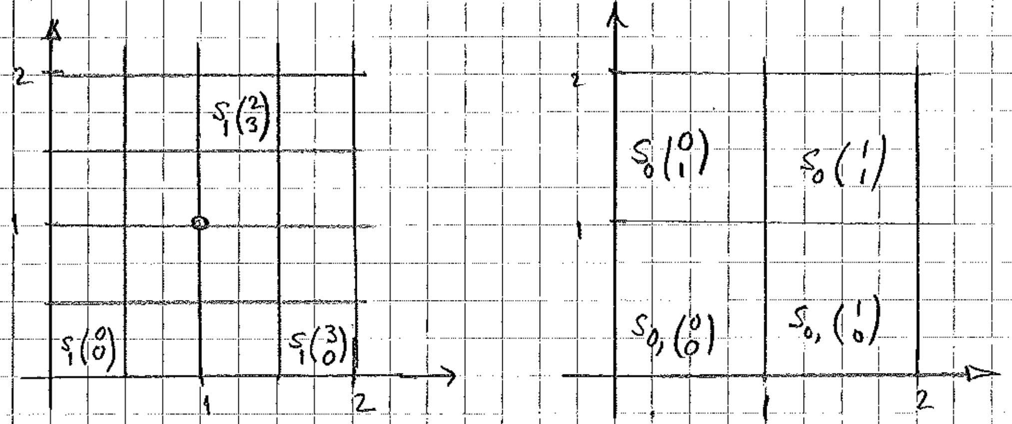
$$g = h * f \iff G(\xi, \eta) = H(\xi, \eta) F(\xi, \eta).$$

Wavelets in higher dimension

The 2-dimensional Haar system



Notation



We construct a function $f_1(x,y)$ that is constant on the identified squares, with values $s_{1,k}$. We can then reduce the resolution by computing the averages:

$$s_{0,k} = \frac{1}{4}\left(s_{1,2k} + s_{1,2k+\binom{1}{0}} + s_{1,2k+\binom{0}{1}} + s_{1,2k+\binom{1}{1}}\right) \qquad \begin{array}{|c|c|}\hline + & + \\ \hline + & + \\ \hline\end{array}$$

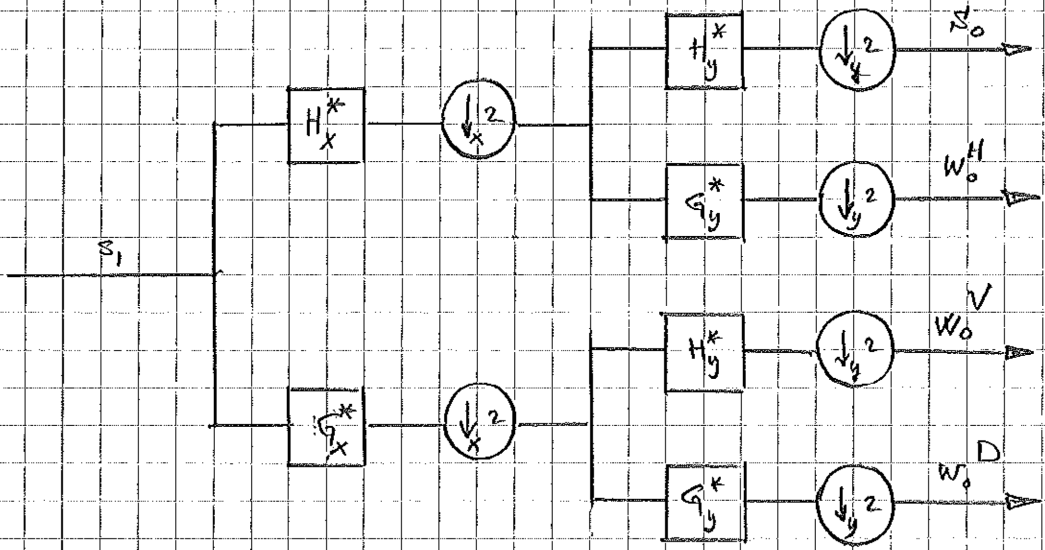
and similarly we may compute differences (but now there are three different ones):

$$W^H_{0,k} = \frac{1}{4}\left(s_{1,2k} + s_{1,2k+\binom{1}{0}} - s_{1,2k+\binom{0}{1}} - s_{1,2k+\binom{1}{1}}\right) \quad \text{H: 'Horizontal'} \quad \begin{array}{|c|c|}\hline - & - \\ \hline + & + \\ \hline\end{array}$$

$$W^V_{0,k} = \frac{1}{4}\left(s_{1,2k} - s_{1,2k+\binom{1}{0}} + s_{1,2k+\binom{0}{1}} - s_{1,2k+\binom{1}{1}}\right) \quad \text{V: "Vertical"} \quad \begin{array}{|c|c|}\hline + & - \\ \hline + & - \\ \hline\end{array}$$

$$W^D_{0,k} = \frac{1}{4}\left(s_{1,2k} - s_{1,2k+\binom{1}{0}} - s_{1,2k+\binom{0}{1}} + s_{1,2k+\binom{1}{1}}\right) \quad \text{D: "Diagonal"} \quad \begin{array}{|c|c|}\hline - & + \\ \hline + & - \\ \hline\end{array}$$

With filter banks:



Separable scaling functions

For the Haar system,

$$\Phi(x,y) = \phi(x)\,\phi(y)$$

$$\Psi^H(x,y) = \phi(x)\,\psi(y)$$

$$\Psi^V(x,y) = \psi(x)\,\phi(y)$$

$$\Psi^D(x,y) = \psi(x)\,\psi(y)$$

Any one dimensional wavelet system may be used to construct a two dimensional system in this way.

The scaling equation in this case is:

$$\begin{cases} \Phi(x,y) = 4 \sum_k h_k \, \Phi(2x-k_x, 2y-k_y) \\ \Psi^\zeta(x,y) = 4 \sum_k g_k \, \Phi(2x-k_x, 2y-k_y), \quad \zeta = D, V, H \end{cases}$$

The MRA is as before:

$$\begin{cases} V_j \subset V_{j+1} \qquad W_j^\sigma \subset V_{j+1}, \qquad \text{and} \\ f_{j+1} = f_j + d_j^H + d_j^V + d_j^D . \end{cases}$$

Multi dimensional Fourier transforms

Notation: $x = (x_1, \dots, x_n)$

$\xi = (\xi_1, \dots, \xi_n)$

If $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ $(k_i \geq 0)$,

let $|k| = k_1 + \dots + k_n$

We define $x^k = x_1^{k_1} \cdot \dots \cdot x_n^{k_n}$

$$\frac{\partial^{|k|}}{\partial x^k} = \frac{\partial^{|k|}}{\partial x_1^{k_1} \dots \partial x_n^{k_n}}$$

Def Let $f : \mathbb{R}^n \longrightarrow \mathbb{C}$; then

$$\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx.$$

Prop The inverse Fourier transform is given by

$$f(x) = \int_{\mathbb{R}^n} e^{2\pi i x \cdot \xi} \hat{f}(\xi)\, d\xi.$$

Almost all rules for the Fourier transform in $\mathbb{R}$ apply:

$f(x,y) \supset \hat{f}(\xi, \eta)$ [Note that here we write f as a function of two arguments, $x, y \in \mathbb{R}$]

$f(ax, by) \supset \frac{1}{|ab|} \hat{f}\left(\frac{\xi}{a}, \frac{\eta}{b}\right)$,

(for $a, b \in \mathbb{R}$)

$f(x-a, y-b) \supset e^{-2\pi i (a\xi + b\eta)} \hat{f}(\xi, \eta)$

$\left(\frac{\partial}{\partial x}\right)^m \left(\frac{\partial}{\partial y}\right)^n f(x,y) \supset (2\pi i \xi)^m (2\pi i \eta)^n \hat{f}(\xi, \eta)$

$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) f(x,y) \supset -4\pi^2 (\xi^2 + \eta^2) \hat{f}(\xi, \eta).$

Distribution in $\mathbb{R}^n$

Def. $S(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ is defined by the family of seminorms

$$\varphi \longmapsto \sup_{x\in\mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \varphi(x) \right|, \quad \alpha \in \mathbb{N}^n, \quad \beta \in \mathbb{N}^n.$$

Def $\varphi_k \longrightarrow 0$ in $S(\mathbb{R}^n)$ if for all $\alpha, \beta \in \mathbb{N}^n$

$$\sup_{x\in\mathbb{R}^n} \left| x^\alpha \frac{\partial^\beta}{\partial x^\beta} \varphi_k(x) \right| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty$$

Def. A tempered distribution in $\mathbb{R}^n$ is a complex valued linear functional T,

$T: S(\mathbb{R}^n) \longrightarrow \mathbb{C}$ such that

$T(\varphi_k) \longrightarrow 0$ when $k \longrightarrow \infty$ for all sequences $\varphi_k \to 0$ in $S(\mathbb{R}^n)$.

Ex $\delta \otimes \delta$ (this could be written as $\delta(x)\delta(y)$)

$$\langle \delta\otimes\delta, \varphi \rangle = \varphi(0,0).$$

Note This is common notation: if $f: \mathbb{R} \to \mathbb{R}$ and $g: \mathbb{R} \to \mathbb{R}$ the function $h(x,y) = f(x)g(y)$ is often denoted by $f \otimes g$.

Observe that $f \otimes g \neq g \otimes f$.

One can define $T_1 \otimes T_2 \in S'(\mathbb{R}^2)$ for any pair $T_1, T_2 \in S'(\mathbb{R})$ by their action on $\varphi = \varphi_1 \otimes \varphi_2 \in S(\mathbb{R}^2)$:

$$\langle T_1 \otimes T_2, \varphi_1 \otimes \varphi_2 \rangle = T_1(\varphi_1)\, T_2(\varphi_2).$$

Problem: Is it enough to define $T_1 \otimes T_2$ on the subset of $S(\mathbb{R}^2)$ given by

$$\varphi = \varphi_1 \otimes \varphi_2 \ ?$$

Solution: One can approximate any $\varphi \in S(\mathbb{R}^2)$ by a sum of tensor products:

$$\varphi(x,y) = \sum_{k=1}^N \varphi_{1k}(x)\, \varphi_{2k}(y) + O(\varepsilon)$$

which means that $\varphi_N(x_1,x_2) = \sum_{k=1}^N \varphi_{1k}(x_1)\varphi_{2k}(x_2) \to \varphi(x_1,x_2)$ in $S(\mathbb{R}^2)$ as $N \to \infty$.

Then $\langle T_1 \otimes T_2, \varphi \rangle = \lim_{N\to\infty} \langle T_1 \otimes T_2, \sum_{k=1}^N \varphi_{1k} \otimes \varphi_{2k} \rangle = \lim_{N\to\infty} \sum_{k=1}^N T_1(\varphi_{1k})\, T_2(\varphi_{2k}).$

The Fourier transform of tensor products

Let $\varphi = \varphi_1 \otimes \varphi_2$, i.e., $\varphi(x,y) = \varphi_1(x)\,\varphi_2(y)$.

Then $\hat{\varphi}(\xi,\eta) = \hat{\varphi}_1(\xi)\,\hat{\varphi}_2(\eta)$ (check that!).

This makes it easy to compute the Fourier transform of many $\varphi : \mathbb{R}^2 \longrightarrow \mathbb{C}$ and tempered distributions $T \in S'(\mathbb{R}^2)$.

Ex: $\dfrac{\partial^{|k|}}{\partial x_1^{k_1}\partial x_2^{k_2}} \varphi_1 \otimes \varphi_2 = \left(\dfrac{\partial^{k_1}}{\partial x_1^{k_1}}\varphi_1\right) \otimes \left(\dfrac{\partial^{k_2}}{\partial x_2^{k_2}}\varphi_2\right) \supset (2\pi i \xi_1)^{k_1}\hat{\varphi}_1(\xi_1) \cdot (2\pi i \xi_2)^{k_2}\hat{\varphi}_2(\xi_2)$.

The Hankel transform and relatives

Many variations of the Fourier transform are derived from the usual Fourier transform in $\mathbb{R}^n$, by using certain symmetries. The Hankel transform is one example.

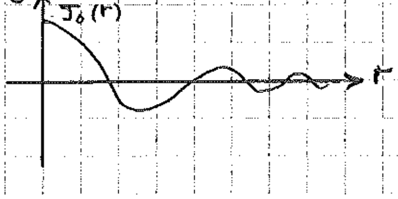
Suppose that $\varphi(x,y) = \psi(r)$ where $r = \sqrt{x^2+y^2}$:
$\varphi(x,y)$ is invariant under rotations in the plane $\mathbb{R}^2$, or in other words, $\varphi(x,y)$ is rotationally symmetric. Then

$$\hat{\varphi}(\xi,\eta) = \iint_{\mathbb{R}^2} e^{-2\pi i(x\xi + y\eta)}\,\varphi(x,y)\,dx\,dy = \{\text{using polar coordinates}\}$$
$$= \int_0^\infty \left(\int_0^{2\pi} e^{-2\pi i r(\xi\cos\theta + \eta\sin\theta)}\,d\theta\right)\psi(r)\,r\,dr.$$

But $\xi\cos\theta + \eta\sin\theta = \sqrt{\xi^2+\eta^2}\,\cos(\theta+\lambda)$ for some λ. Let $\rho = \sqrt{\xi^2+\eta^2}$.

Then
$$\frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r(\xi\cos\theta + \eta\sin\theta)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r\rho\cos(\theta+\lambda)}\,d\theta$$
$$\overset{!!}{=} \frac{1}{2\pi}\int_0^{2\pi} e^{-2\pi i r\rho\cos\theta}\,d\theta \;\vdots = J_0(2\pi r\rho)$$

This is the Bessel function of order 0.
It solves the equation $x^2 f''(x) + x f'(x) + x^2 f(x) = 0$.



Therefore $\hat{\varphi}(\xi,\eta) = \int_0^\infty 2\pi J_0(2\pi r\rho)\,\psi(r)\,r\,dr = \tilde{\psi}(\rho)$.

<u>Def</u> $F(q) = 2\pi \int_0^\infty J_0(2\pi q r) f(r)\, r\, dr$ is the (Zeroth order) Hankel transform of f.

<u>Note</u> The inverse Hankel transform is

$$f(r) = 2\pi \int_0^\infty F(q)\, J_0(2\pi q r)\, q\, dq.$$

Generally, the Fourier transform of a rotationally symmetric function $f: \mathbb{R}^n \to \mathbb{C}$ is computed in a similar way: if

$f(x_1, \dots, x_n) = g\left(\sqrt{x_1^2 + \dots + x_n^2}\right) = g(r)$, then

$$\hat{f}(\xi_1, \dots, \xi_n) = \int_{\mathbb{R}^n} e^{-2\pi i (x_1\xi_1 + \dots + x_n\xi_n)}\, g(r)\, dx_1 \dots dx_n.$$

In polar coordinates,

$(x_1, \dots, x_n) = r(\cos\theta_1, \sin\theta_1 \cos\theta_2, \dots) \equiv r\omega$, where $|\omega| = 1$.

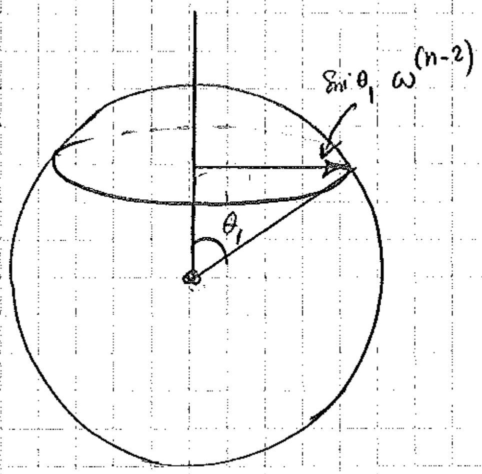
Then $\hat{f}(\xi_1, \dots, \xi_n) = \int_0^\infty \int_{S^{n-1}} e^{-2\pi i \rho r\, \Omega\cdot\omega}\, d\omega^{n-1}\, g(r)\, r^{n-1}\, dr$.

Here $\int_{S^{n-1}} e^{-2\pi i z\, \Omega\cdot\omega}\, d\omega^{n-1}$ only depends on the scalar product between Ω and ω, and therefore we may choose coordinates on S^{n-1} so that $\omega = (\omega_1, \dots, \omega_n)$ & $\Omega\cdot\omega = \omega_1$.

Also $d\omega^{n-1} = d\theta_1 (\sin\theta_1)^{n-2}\, d\omega^{n-2}$

$$\Rightarrow \int_{S^{n-1}} e^{-2\pi i z\, \Omega\cdot\omega}\, d\omega = $$

$$= \int_0^\pi e^{-2\pi i \cos\theta\, z} (\sin\theta)^{n-2}\, d\theta \cdot |S^{n-2}|,$$



where $|S^{n-2}|$ is the "area" of

$S^{n-2} = \{x \in \mathbb{R}^{n-2},\ |x| = 1\}$.

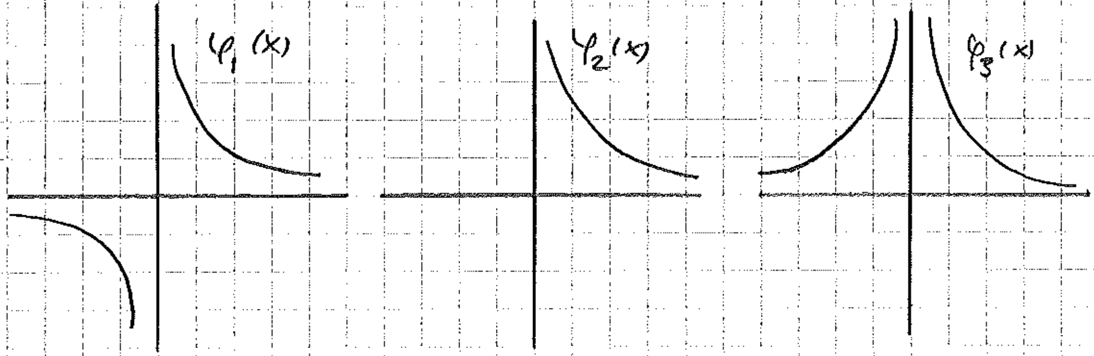
Homogeneous functions and distributions

<u>Def.</u> A function $f : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{C}$ is called homogeneous of degree k if

$$f(tx) = t^k f(x) \quad \text{for all } x \neq 0.$$

<u>Example</u> $\varphi_1(x) = \frac{1}{x}$ is homogeneous of degree -1.

But that is true also for $\varphi_2(x) = \begin{cases} 1/x & \text{if } x>0 \\ 0 & \text{else} \end{cases}$ and $\varphi_3(x) = \frac{1}{|x|}$



Note that if f is homogeneous of degree k,

$$\langle f, \varphi \rangle = \int_{\mathbb{R}^n} f(x)\, \varphi(x)\, dx \underset{\substack{\uparrow \\ \text{(by the homogenity)}}}{=} \int_{\mathbb{R}^n} f(\alpha x)\, \alpha^{-k} \varphi(x)\, dx$$

$$= \int_{\mathbb{R}^n} f(y)\, \alpha^{-k} \varphi\left(\frac{y}{\alpha}\right) \frac{dy}{\alpha^n} = \left\langle f, \frac{1}{\alpha^{n+k}}\, \varphi\left(\frac{\cdot}{\alpha}\right) \right\rangle.$$

A distribution $T \in S'(\mathbb{R})$ is homogeneous of degree k

if $\quad \langle T, \varphi \rangle = \left\langle T, \frac{1}{\alpha^{n+k}}\, \varphi\left(\frac{\cdot}{\alpha}\right) \right\rangle, \quad$ for all $\varphi \in S(\mathbb{R}^n)$.

<u>Ex.</u> $\quad \langle \delta, \varphi \rangle = \varphi(0) = \varphi\left(\frac{0}{\alpha}\right) \frac{1}{\alpha^{n+k}}$

$\Rightarrow$ δ is homogeneous of degree $-n$.

The Fourier transform of homogeneous functions and distributions

Formally, if $f : \mathbb{R} \to \mathbb{R}$ is homogeneous of order k, and

$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\,dx$, then (by the homogeneity of f)

$$\hat{f}(t\xi) = \int_{\mathbb{R}} e^{-2\pi i t x \xi} f(x)\,dx \underset{\uparrow}{=} \int_{\mathbb{R}} e^{-2\pi i y \xi} f\left(\frac{y}{t}\right) \frac{dy}{dt} \overset{\downarrow}{=}$$

(by a change of variables)

$$= \int_{\mathbb{R}} e^{-2\pi i y \xi} f(y) \frac{dy}{t^{k+1}} = \frac{1}{t^{k+1}} \hat{f}(\xi).$$

Thus, if f is homogeneous of degree k, then $\hat{f}$ is homogeneous of degree $-k-1$. But note that this may not be well-defined!

However, if $T \in S'(\mathbb{R})$ is homogeneous of degree λ, then

$\langle \hat{T}, \varphi \rangle = \langle T, \hat{\varphi} \rangle = \langle T, \frac{1}{t^{1+\lambda}} \hat{\varphi}\left(\frac{\cdot}{t}\right) \rangle$, and because

$$\frac{1}{t^{1+\lambda}} \hat{\varphi}\left(\frac{\xi}{t}\right) = \frac{1}{t^{1+\lambda}} \int_{\mathbb{R}} e^{-2\pi i x \frac{\xi}{t}} \varphi(x)\,dx = \int_{\mathbb{R}} e^{-2\pi i \xi y} \varphi(ty) \frac{1}{t^{\lambda}}\,dy$$

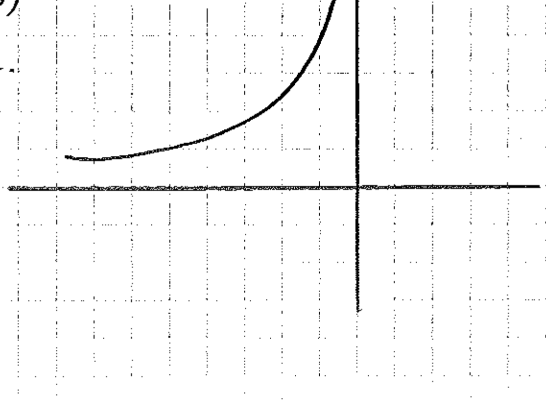
$$\Rightarrow \langle \hat{T}, \varphi \rangle = \langle T, \mathcal{F}(\varphi(t\cdot)) \frac{1}{t^{\lambda}} \rangle = \langle \hat{T}, t^{-\lambda} \varphi(t\cdot) \rangle$$

and therefore $\hat{T}$ is homogeneous of degree $-\lambda-1$.

(recall $\langle \hat{T}, \varphi \rangle = \langle \hat{T}, t^{\alpha+1} \varphi(t\cdot) \rangle$; so we need to set $\alpha = -\lambda - 1$.)

<u>Example</u>: Compute the Fourier transform of

$$f(x) = \begin{cases} (-x)^{-1/2} & (x<0) \\ 0 & \text{else.} \end{cases}$$



<u>Solution:</u> Let $f(x) = \frac{1}{2}\left(\frac{1}{|x|^{1/2}} - \operatorname{sgn} x \, \frac{1}{|x|^{1/2}} \right)$

$f_1(x) = |x|^{-1/2}$ is even, real and homogeneous of degree $-1/2$

$\Rightarrow \hat{f}_1(\xi)$ is even, real and homogeneous of degree $-(-\frac{1}{2})-1 = -\frac{1}{2}$.

Therefore $\hat{f}_1(\xi) = \pm C|\xi|^{-1/2}$.

In the usual way, we have

$\langle \hat{f}, e^{-\pi \cdot^2} \rangle = \langle f_1, e^{-\pi \cdot^2} \rangle > 0 \Rightarrow \hat{f}_1(\xi) = |\xi|^{-1/2}$.

$f_2(x) = \operatorname{Sign}(x) \frac{1}{|x|^{1/2}}$ is odd, real and homogeneous of degree $-\frac{1}{2}$, so

$\hat{f}_2(\xi)$ is odd, imaginary and homogeneous of degree $-\frac{1}{2}$.

Hence $\hat{f}_2(\xi) = i\, c \operatorname{Sign}\xi \, |\xi|^{-1/2}$.

But what is the value of $c \in \mathbb{R}$?

We have $2\pi i \xi \hat{f}_2(\xi) = \mathcal{F}(f_2')$, so

$\langle 2\pi i (\cdot) \hat{f}_2, \varphi \rangle = \langle \mathcal{F}(f_2'), \varphi \rangle = \langle f_2', \hat{\varphi} \rangle = -\langle f_2, \hat{\varphi}' \rangle$.

Then

$\langle -2\pi c |\cdot|^{1/2}, e^{-\pi \cdot^2} \rangle = -\langle \operatorname{Sgn}(\cdot)|\cdot|^{-1/2}, 2\pi(\cdot) e^{-\pi \cdot^2} \rangle$

$= -2\pi \langle |\cdot|^{1/2}, e^{-\pi \cdot^2} \rangle$,

so, we must have $c = 1$.

Finally

$\hat{f}(\xi) = \frac{1}{2}\left(\hat{f}_1(\xi) - \hat{f}_2(\xi)\right) = \frac{1}{2}\left(|\xi|^{-1/2} - i \operatorname{sgn}\xi \, |\xi|^{-1/2}\right)$, for $\xi \in \mathbb{R}$.

For $\xi \in \mathbb{R}$, $\xi > 0$,

$\hat{f}(\xi) = \frac{1-i}{2} \xi^{-1/2}$.

This can be extended to an analytic function

$\hat{f}(\xi) = \frac{1}{(2\pi i \xi)^{1/2}}$ in $\xi \in \mathbb{C} \setminus \{z \in \mathbb{C} : \operatorname{Im} z \le 0, \operatorname{Re} z = 0\}$.

The Abel transform

Let $\xi = x^2$ and $p = r^2$ and define $F_A(x^2) = f_A(x)$, $F(r^2) = f(r)$.

Then

$$F_A(\xi) = \int_0^\infty K(\xi - p) F(p)\, dp = K * F$$

where

$$K(z) = \begin{cases} \frac{1}{(-z)^{1/2}}, & (z < 0) \\ 0 & \text{otherwise} \end{cases}$$

The map $F \longmapsto F_A$ is called the "modified Abel transform".

$$F_A(\xi) = K * F(\xi)$$

implies that

$$\mathcal{F}(F_A) = \mathcal{F}(K)\,\mathcal{F}(F),$$

and recall: $\mathcal{F}(K) = \frac{1}{(-2i\xi)^{1/2}}$. Then we may see that

$$\mathcal{F}(F) = (-2i\xi)^{1/2}\mathcal{F}(F_A)(\xi) = -\frac{1}{\pi}\,\frac{1}{(-2i\xi)^{1/2}}\,2\pi i\xi\,\mathcal{F}(F_A)(\xi)$$

$$= -\frac{1}{\pi}\cdot\frac{1}{(-2i\xi)^{1/2}}\mathcal{F}(F_A') = -\frac{1}{\pi}\mathcal{F}(K)\,\mathcal{F}(F_A').$$

We then see that

$$F(p) = -\frac{1}{\pi} K * F_A'(p).$$

In the original variables

$$f(r) = -\frac{1}{\pi}\int_r^\infty \frac{f_A'(x)}{\sqrt{x^2 - r^2}}\, dx$$

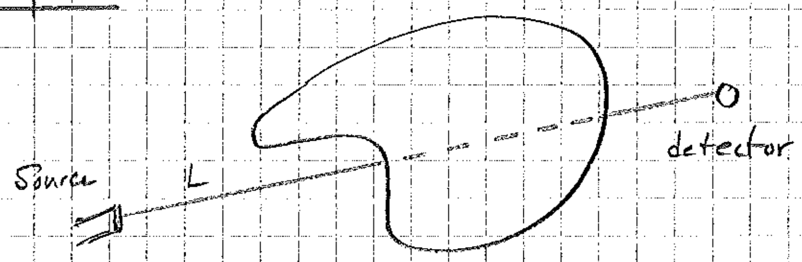
Note that

$$K * K * F' = -\pi F$$

so the operator

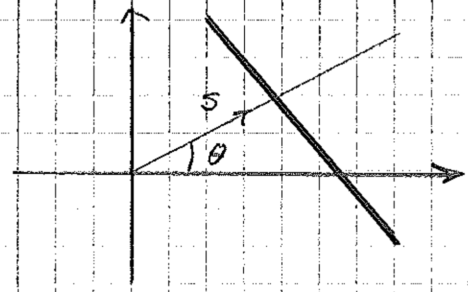
$$F \longmapsto \frac{K}{\sqrt{\pi}} * F$$ corresponds to taking a "half order integral" of F.

The Radon transform



Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ denote the absorbtion per unit length of a ray passing through the object:

$$\int_L f(x(\ell))\, d\ell = \text{absorbtion along the line } L$$



We identify the angle θ and the unit vector θ

Let $x = (x_1, x_2)$.

Def The Radon transform of $f \in S(\mathbb{R}^2)$ is

$$R_\theta f(s) = \int_{x\cdot\theta = s} f(x)\, dx = \int_{\mathbb{R}^2} f(x_1,x_2)\, \delta(x - x\cdot\theta)\, dx_1\, dx_2$$

Exercise Show that if f is a radial function, then

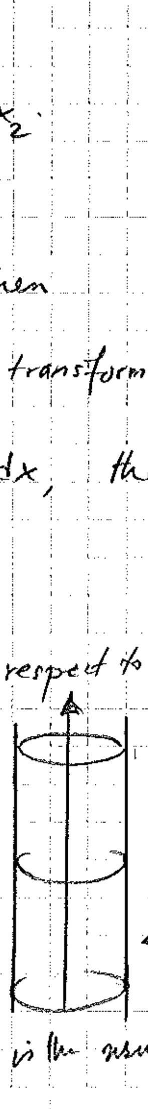
$$R_\theta f(s) = A f(s) \qquad \text{(i.e. the Abel transform.)}$$

Theorem If $f \in S(\mathbb{R}^2)$ and $R_\theta f(s) = \int_{x\cdot\theta = s} f(x)\, dx$, then

$$(R_\theta f)^\wedge(\sigma) = \hat{f}(\theta\sigma),$$

where $(R_\theta f)^\wedge(s)$ is the usual Fourier transform with respect to s.

Pf. Note first that the natural domain of definition for $R_\theta f(s)$ is $\mathbb{R}\times S^1$. But from the definition it is also clear that $R_{-\theta} f(-s) = R_\theta f(s)$. We compute $(R_\theta f)^\wedge(\xi)$:



$$(R_\theta f)^\wedge(\sigma) = \int_{-\infty}^{\infty} e^{-2\pi i \sigma s} R_\theta f(s)\, ds = \int_{-\infty}^{\infty} e^{-2\pi i \sigma s} \int_{\mathbb{R}^2} f(x)\, \delta(x\cdot\theta - s)\, dx\, ds$$

$$= \int_{\mathbb{R}^2} f(x)\, e^{-2\pi i\, \sigma x\cdot\theta}\, dx = \hat{f}(\sigma\theta),$$ where $\hat{f}$ is the usual

(First compute the s-integral)

Fourier transform in $\mathbb{R}^2$.

<u>Corollary</u>: If $f \in \mathcal{S}(\mathbb{R}^2)$ and $R_\theta f(s) \in \mathcal{S}(\mathbb{R}\times S^1)$, then f can be recovered from $R_\theta f(s)$ by first obtaining $\hat{f}(\xi)$ by the theorem above, and then computing the 2-dimensional inverse Fourier Transform.

This is not efficient when discretizing in polar coordinates.

<u>A better inversion formula for the Radon transform</u>

Let $f \in \mathcal{S}(\mathbb{R}^2)$ then

$$f(x) = \int_{\mathbb{R}^2} e^{2\pi i x\cdot\xi}\, \hat{f}(\xi)\, d\xi$$

$$= \int_0^\infty \int_{S^1} e^{2\pi i x\cdot\theta\sigma}\, \hat{f}(\sigma\theta)\, \overbrace{\sigma}^{=|\sigma|}\, d\theta\, d\sigma \qquad \text{(polar coordinate)}$$

$$= \frac{1}{2}\int_{S^1}\int_{\mathbb{R}} e^{2\pi i \sigma\theta\cdot x}\, \hat{f}(\sigma\theta)\, \underbrace{\sigma \operatorname{sgn}\sigma}_{=|\sigma|}\, d\sigma\, d\theta$$

$$= \frac{1}{2}\int_{S^1}\int_{\mathbb{R}} e^{2\pi i \sigma\theta\cdot x} (R_\theta f)^\wedge(\sigma)\, \sigma \operatorname{sgn}\sigma\, d\sigma\, d\theta. \qquad (*)$$

Recall that $\quad 2\pi i\sigma\hat{f}(\sigma) = \mathcal{F}(f')(\sigma)$

$$\operatorname{sign}\sigma = \mathcal{F}\left(\frac{i}{\pi(\cdot)}\right) \longleftarrow \text{Let } \ldots 2\pi$$

Therefore

$$(R_\theta f)^\wedge(\sigma)\,\sigma\operatorname{sgn}\sigma = \frac{1}{2\pi i}\mathcal{F}\left(\frac{d}{ds}R_\theta f(\cdot) * \frac{i}{\pi(\cdot)}\right)(\sigma)$$

$$\Rightarrow (*) = \frac{1}{4\pi^2}\int_{S^1} \underbrace{\left((R_\theta f)' * \frac{1}{(\cdot)}\right)(\theta x)}_{=\int_{\mathbb{R}} \frac{(R_\theta f)'(u)}{u-s}\,du.}\, d\theta$$

<u>Note</u> The map

$$\mathcal{S}(\mathbb{R}\times S^1) \longrightarrow \mathcal{S}(\mathbb{R})$$
$$g(s,\theta) \longmapsto \int_{S^1} g(x\cdot\theta, \theta)\, d\theta$$

is "the dual" of the Radon transform, and it is denoted R^*.

The reason for the name "dual" is the following:

Take $\varphi \in S(\mathbb{R} \times S^1)$. We define

$$\langle g, \varphi \rangle = \int_{S^1} \int_{\mathbb{R}} g(s, \theta)\, \varphi(s, \theta)\, ds\, d\theta.$$

Then

$$\langle Rf, \varphi \rangle = \int_{S^1} \int_{\mathbb{R}} R_\theta f(s)\, \varphi(s, \theta)\, ds\, d\theta = \int_{S^1} \int_{\mathbb{R}} \int_{\mathbb{R}^2} f(x)\, \delta(x \cdot \theta - s)\, \varphi(s, \theta)\, dx\, ds\, d\theta$$

$$= \int_{\mathbb{R}^2} f(x) \int_{S^1} \int_{\mathbb{R}} \delta(x \cdot \theta - s)\, \varphi(s, \theta)\, ds\, d\theta\, dx$$

$$= \int_{\mathbb{R}^2} f(x) \underbrace{\int_{S^1} \varphi(x \cdot \theta, \theta)\, d\theta}_{\in S(\mathbb{R}^2)}\, dx = \langle f, R^* \varphi \rangle$$

This is the usual $\langle \cdot, \cdot \rangle$ in $\mathbb{R}^2$.

The Hilbert transform

Def $\quad F_{H_1}(x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{f(y)}{y - x}\, dx = \left(\frac{1}{\pi(-\cdot)} * f \right)(x)$

Recall that $\quad \mathcal{F}\left(\frac{1}{\pi(-\cdot)} \right)(s) = i \operatorname{sgn} s$,

Hence

$$\left(\frac{1}{\pi(-\cdot)} * \frac{1}{\pi(-\cdot)} * f \right)^{\wedge}(s) = -(\operatorname{sgn} s)^2 \hat{f} = -\hat{f}$$

and then

$$\frac{1}{\pi(-\cdot)} * \frac{1}{\pi(-\cdot)} * f = -f.$$

Therefore

$$f(x) = -\frac{1}{\pi(-\cdot)} * F_{H_1}(x).$$

The analytical signal

Let $f \in S(\mathbb{R})$ be real valued and let

$$g(x) = f(x) - i\, F_{H_1}(x).$$

This is called "the analytical signal". The Fourier transform?

$$\hat{g}(s) = \hat{f}(s) - i\; i\, \operatorname{sgn} s\, \hat{f}(s)$$
$$= \hat{f}(s) + \operatorname{sgn} s\, \hat{f}(s).$$

Then $\hat{g}(s) = 0$ for $s \le 0$.

Why is this called " the analytical signal" ?

Let $\varphi(x,y) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{y}{(x-\xi)^2+y^2} f(\xi)\, d\xi.$

Then
$$\begin{cases} \Delta \varphi = 0 & \text{in} \quad y > 0 \\ \varphi(x,0) = f \end{cases}$$

There is an analytic function $\Phi(z)$ so that

$$\varphi(x,y) = \operatorname{Re} \Phi(z) \qquad (z = x+iy)$$

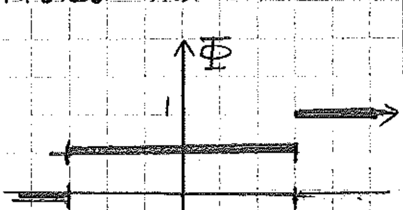
Φ is analytic in $\operatorname{Im} z > 0$ and

$$\lim_{y \to 0^+} \operatorname{Im} \Phi(x+iy) = F_{H_1}(x).$$

Fourier analysis and probability

Let $\xi_1, \xi_2, \dots$ be i.i.d. random variables and let

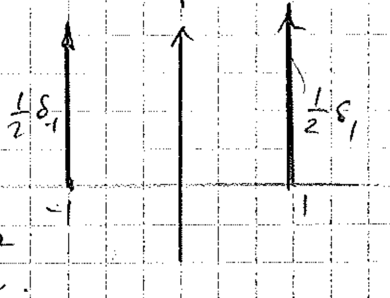
$$\eta_n = \frac{\xi_1+\dots+\xi_n}{\sqrt{n}}.$$



Assume that $P(\xi_i < x) = \Phi(x)$ (all i)

and that $T = \Phi'$ (in $S'(\mathbb{R})$)

Example: $$\Phi = \begin{cases} 0 & \text{when } x<-1 \\ \frac{1}{2} & \text{" } -1<x<1 \\ 1 & x>1 \end{cases}$$



Then $T = \frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$.

This means that $$\xi_i = \begin{cases} 1 & \text{with prob } 1/2 \\ -1 & \text{" " .} \end{cases}$$

If the law of ξ_i is given by T, then the law of $\xi_1+\xi_2$ is given by $T * T$ (when this is well-defined, ...).

If the law of ξ_i is given by T, then the law of $a\xi_i$ is given by $\frac{1}{a}\, T\left(\frac{\cdot}{a}\right)$.

Theorem (The central limit theorem)

Let $\langle T, 1\rangle = 1$, $\langle T, x\rangle = 0$, $\langle T, (\cdot)^2\rangle = \sigma^2 < \infty$.

Then the law of η_n is given by

$$T_n = \underbrace{\sqrt{n}\, T(\sqrt{n}\,\cdot) * \cdots * \sqrt{n}\, T(\sqrt{n}\,\cdot)}_{n \text{ times}},$$

and $$T_n \longrightarrow \frac{1}{\sqrt{4\pi\sigma^2}}\, e^{-\frac{x^2}{4\pi\sigma}}, \quad \text{as } n \longrightarrow \infty.$$

Pf. By the convolution theorem, $\widehat{T_n} = \mathcal{F}\left(\sqrt{n}\, T(\sqrt{n}\,\cdot)\right)^n$, and

$$\mathcal{F}\left(\sqrt{n}\, T(\sqrt{n}\,\cdot)\right) = \hat{T}\left(\frac{\cdot}{\sqrt{n}}\right) = \hat{T}(0) + \hat{T}'(0)\frac{s}{\sqrt{n}} + \frac{1}{2}\hat{T}''(0)\frac{s^2}{n} + \mathcal{O}\left(\frac{s^3}{n^{3/2}}\right).$$

Because $\langle T, 1\rangle = 1$, $\hat{T}(0) = 1$,

and $\langle T, (\cdot)\rangle = 0 \Rightarrow \hat{T}'(0) = 0$.

Also $\langle T, (\cdot)^2\rangle = \sigma^2 \Rightarrow \hat{T}''(0) = -4\pi^2\sigma^2$.

Then
$$\hat{f}_n(s) = \left(1 - \frac{1}{2}\, 4\pi^2\sigma^2 \frac{s^2}{n} + O\left(\frac{1}{n^{3/2}}\right)\right)^n$$
$$= \left(1 - \frac{2\pi^2\sigma^2 s^2}{n}\right)^n \longrightarrow e^{-2\pi^2\sigma^2 s^2}, \quad \text{as } n \to \infty.$$

Note
$$e^{-2\pi^2\sigma^2 s^2} = e^{-\pi(\sqrt{2\pi}\,\sigma s)^2} \subset \frac{1}{\sqrt{2\pi}\,\sigma}\, e^{-\pi\left(\frac{x}{\sqrt{2\pi}\,\sigma}\right)^2} = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-\frac{x^2}{2\sigma^2}}$$

This then means that

$\eta_n \to \eta$, a random variable with density $\frac{1}{\sqrt{2\pi\sigma^2}}\, e^{-x^2/2\sigma^2}$. ☒

The law of large numbers

Here we assume that the ξ_i are i.i.d. with mean $\bar{\xi}$ and that $E\left((\xi - \bar{\xi})^2\right) < \infty$ (this is more than needed). Assume that the law of ξ_i is f. The law of large numbers states that
$$\eta_n = \frac{\xi_1 + \dots + \xi_n}{n} \longrightarrow \delta_{\bar{\xi}} \quad \text{when } n \to \infty.$$

The proof is similar to the proof of the CLT:

$\xi_1 + \dots + \xi_n$ has law $\underbrace{f * f * \dots * f * f}_{n \text{ times}}$

and hence that $\frac{\xi_1 + \dots + \xi_n}{n}$ has law $n \cdot \underbrace{f * \dots * f}_{n\text{-times}}(nx) = g_n(x)$.

Taking the Fourier transform of g_n we see that

$\mathcal{F}(g_n)(s) = \hat{f}(s/n)^n$, and because
$$\hat{f}(s) = 1 - 2\pi i\, \hat{f}'(0)$$
$$= 1 - 2\pi i\, \bar{\xi} s + O(s^2)$$
$$\Rightarrow \hat{f}\left(\frac{s}{n}\right)^n = \left(1 - 2\pi i\, \xi \frac{s}{n} + O\left(\frac{s^2}{n^2}\right)\right)^n = \left\{\left(1 - 2\pi i\, \bar{\xi}\frac{s}{n} = O\left(\frac{s^2}{n^2}\right)\right)^n\right\}$$
$$= \exp\left(n \log\left(1 - 2\pi i\, \bar{\xi}\frac{s}{n} + \dots\right)\right)$$
$$= \exp\left((-2\pi i\, \bar{\xi} s) + O\left(\frac{1}{n}\right)\right) = e^{-2\pi i\, \bar{\xi} s}$$
$$\Rightarrow f \longrightarrow \delta_{\bar{\xi}}.$$

Bochner's theorem

Def. Let $\varphi \in S$ be complex valued. Then

$$\varphi \circledast \varphi(x) = \int \varphi(x+u)\varphi^*(u)\,du$$

Recall $\varphi \circledast \varphi = \varphi * (\varphi^*)^\vee \quad \left(= \varphi * \varphi^*(-\cdot)\right)$,

and $\mathcal{F}(\varphi \circledast \varphi) = |\hat{\varphi}|^2$.

Def. $T \in S'$, $\varphi \in S$. T is called a <u>tempered measure</u> if for all $\varphi \in S$ with compact support (i.e. $\varphi(x) = 0$ for $|x| \geq A$, some A)

$$|T(\varphi)| \leq C \sup_x |\varphi(x)| . \qquad (*)$$

Here C may depend on A. If C does not depend on A, we say that T is a bounded measure.

T is called positive ($T \geq 0$) if for all $\varphi \geq 0$, $T(\varphi) \geq 0$.

T is called positive definite if for all $\varphi \in S$

$$T(\varphi \circledast \varphi) \geq 0 .$$

In $(*)$ the important thing is that the right hand side does not depend on derivatives of φ. So δ is a measure, but δ' is <u>not</u>.

<u>Proposition</u>. Let $T \in S'$. Then $\mathcal{F}(T) \geq 0 \iff T$ is positive definite.

Pf: For all $\varphi \in S$ the following two statements hold:

1) $\hat{T}(\varphi) \geq 0$ when $\varphi \geq 0 \iff \hat{T}(|\varphi|^2) \geq 0$

2) $T(\varphi \circledast \varphi) \geq 0$ for all $\varphi \in S \iff \hat{T}(|\varphi|^2) \geq 0$ for all $\varphi \in S$.

<u>Pf of 1):</u> $\Rightarrow$ is trivial. To prove $\Leftarrow$,

Take $0 \leq \psi \in S$, $\psi(x) = 0$ for $|x| > A$. Let

$\psi_n(x) = \left(\psi(x) + e^{-x^2}/n\right)^{1/2}$. ψ_n is strictly positive, and $\psi_n \in S$,

and also $\psi_n^2(x) = \psi(x) + e^{-x^2}/n \to \psi(x)$, $n \to \infty$. Then

$$0 \leq \hat{T}\left(\psi(x) + \tfrac{1}{n} e^{-x^2}\right) = \hat{T}(\psi) + \tfrac{1}{n}\hat{T}\left(e^{-(\cdot)^2}\right) \longrightarrow \hat{T}(\psi).$$

Therefore $\hat{T}(\psi) \geq 0$.

<u>proof of 2)</u> $\varphi \circledast \varphi$ is even and $\varphi \circledast \varphi = \mathcal{F}^{\circ}\mathcal{F}(\varphi \circledast \varphi)$. Then

$T(\varphi \circledast \varphi) = T(\mathcal{F}\mathcal{F}(\varphi \circledast \varphi)) = \hat{T}(\mathcal{F}(\varphi \circledast \varphi)) = \hat{T}(|\hat{\varphi}|^2)$.

So $T(\varphi \circledast \varphi) \geq 0 \implies \hat{T}(|\hat{\varphi}|^2) \geq 0$. And this completes the proof ∎