Fourier and wavelet analysis

Fourier transform;

$$\begin{cases} \text{in } \mathbb{R}: & \hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\, dx \\ \text{in } \mathbb{R}^n: & " = \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(x)\, dx, \qquad x = (x_1, \dots, x_n) \in \mathbb{R}^n \end{cases}$$

The course:

- The Fourier transform
- Distribution theory
- Transforms related to Fourier transform (Radon-, Hankel-, Z-transforms, ...)
- The wavelet transforms
- Multiresolution analysis
- Discrete transforms, Sampling
- Applications (three Lab assignments).

Notation

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\,dx, \qquad \longleftrightarrow \qquad \hat{f}(\xi) = (\mathcal{F} f)(\xi) \longleftrightarrow f(x) \overset{\mathcal{F}}{\supset} \hat{f}(\xi).$$

Basic properties

- Linearity: $f+g \supset \hat{f}+\hat{g}$

  $\alpha f \supset \alpha \hat{f}$ ($\alpha \in \mathbb{R}$ or $\mathbb{C}$)

- Scaling: $f \supset \hat{f} \iff \frac{1}{a} f(\frac{x}{a}) \supset \hat{f}(a\xi)$, $(a>0)$

  (notation: $\frac{1}{a} f(\frac{\cdot}{a}) \supset \hat{f}(a\xi)$)

- "If f is reasonably regular/smooth so is $\hat{f}$." (Will be made precise later)

Fourier transform in L_2:

$L_2(\mathbb{R}) := \{f : f \text{ measurable}, \ \& \ \int_{\mathbb{R}} |f(x)|^2 dx < \infty\}$

Scalar product: $\langle f, g \rangle = \int_{\mathbb{R}} f(x) \overline{g(x)}\,dx$

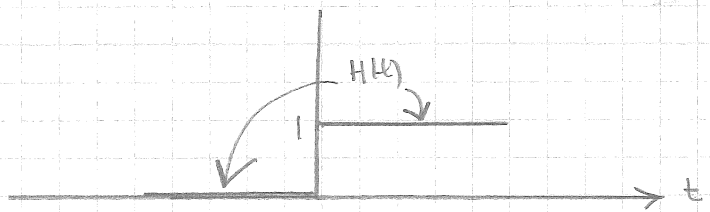
Parseval: $\int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\hat{f}(\xi)|^2 d\xi$

By "regular/smooth" we will mean Schwartz class $\mathcal{S}$.

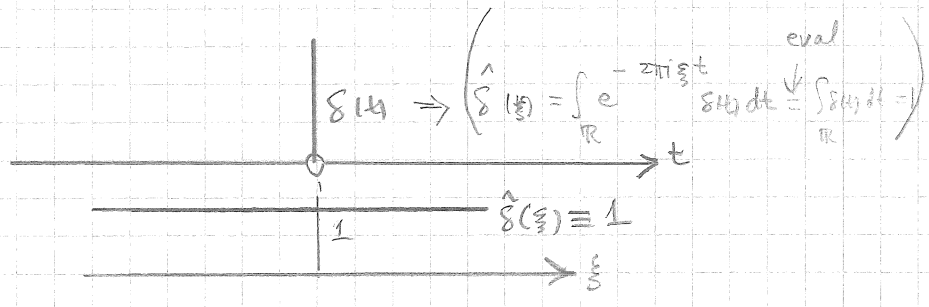
Often: $f(t)$ is a signal:

$f(t) = \sin(\omega t)$

$f(t) = H(t) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$



"$f(t) = \delta(t)$" : The dirac "δ-function"



These are **not** in L_2, nor in $\mathcal{S}$.

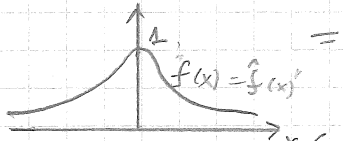
Fundamental The Fourier inversion formula

$$\hat{f}(\xi) = \int_{-\infty}^{\infty} e^{-2\pi i x \xi} f(x)\,dx \quad \Longleftrightarrow \quad f(x) = \int_{-\infty}^{\infty} e^{2\pi i x \xi} \hat{f}(\xi)\,d\xi.$$

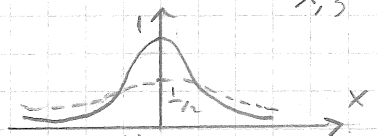
[Which means that: $f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \left(\int_{\mathbb{R}} e^{-2\pi i y \xi} f(y)\,dy \right) dx = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} e^{2\pi i (x-y)\xi} f(y)\,dy \right) dx$

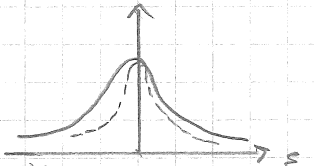
$= \int_{\mathbb{R}} (f * e^{2\pi i \cdot \xi})(x)\,dx$]

Example (Important) $\mathcal{F}(e^{-\pi x^2}) = e^{-\pi \xi^2}$.



(Scaling $a = 2$: $\frac{1}{2} e^{-\pi (\frac{x}{2})^2} \supset e^{-\pi (2\xi)^2}$.

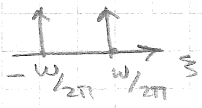




Example $f(t) = \sin(\omega t) = \frac{1}{2i}\left(e^{i\omega t} - e^{-i\omega t}\right) \supset \frac{1}{2i} \int_{\mathbb{R}} e^{-2\pi i t \xi} \left(e^{i\omega t} - e^{-i\omega t}\right) dt = \frac{1}{2i} \int_{\mathbb{R}} \left\{ e^{-2\pi i t (\xi - \frac{\omega}{2\pi})} - e^{-2\pi i t (\xi + \omega/2\pi)} \right\} e^{-}$

$\Rightarrow \hat{f}(\xi) = \mathcal{F}(\sin(\omega t))(\xi) = \{s = -t\} = \frac{1}{2i} \int_{\mathbb{R}} \left(e^{2\pi i s (\xi - \frac{\omega}{2\pi})} - e^{2\pi i s (\xi + \frac{\omega}{2\pi})} \right) ds \cdots$

$f(t) = \cos(\omega t) = \frac{1}{2}(e^{i\omega t} + e^{-i\omega t}) \supset \frac{1}{2} \int_{\mathbb{R}} \left(e^{-2\pi i t (\xi - \frac{\omega}{2\pi})} + e^{-2\pi i t (\xi + \frac{\omega}{2\pi})} \right) dt = \{s = -t\} = \cdots \frac{1}{2}\left(\delta_{\frac{\omega}{2\pi}} + \delta_{-\frac{\omega}{2\pi}}\right)$



Other transforms

Hankel:

$$\hat{f}(\xi_1, \xi_2) = \iint_{\mathbb{R}^2} e^{-2\pi i(\xi_1 x_1 + \xi_2 x_2)} f(x_1, x_2)\, dx_1\, dx_2$$

(2D $\mathcal{F}$-transf)

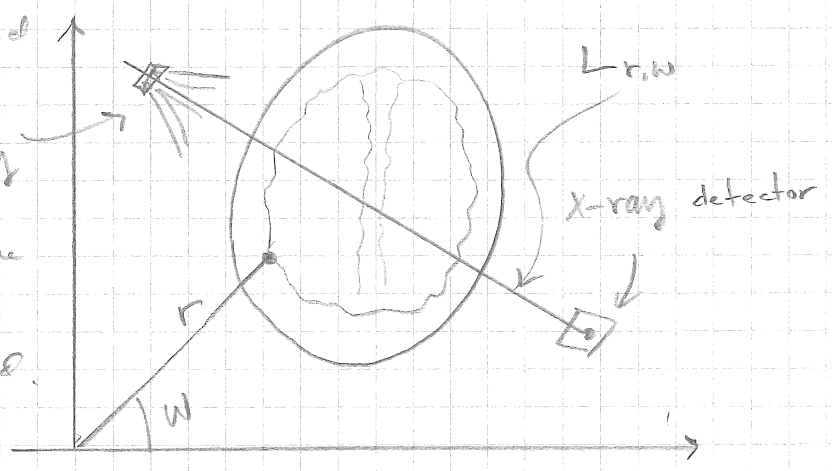
Let $f(x_1, x_2) = F\left(\sqrt{x_1^2 + x_2^2}\right) = F(r) \rightarrow x_1 = r\cos\theta, \quad x_2 = r\sin\theta$

Then $\hat{f}(\xi_1, \xi_2) = \widetilde{\mathcal{F}}\left(\sqrt{\xi_1^2 + \xi_2^2}\right) = \widetilde{F}(\rho) \rightarrow \xi_1 = \rho\cos\alpha, \quad \xi_2 = \rho\sin\alpha$

$F(r) \longmapsto \widetilde{F}(\rho)$ is the Hankel transform.

Radon: If $f(x_1, x_2)$ is the density of the head at point (x_1, x_2), then

$\int_{L_{r,w}} f(x_1, x_2)\, dl$ gives a measure of the damping along $L_{r,w}$. This can be measured.



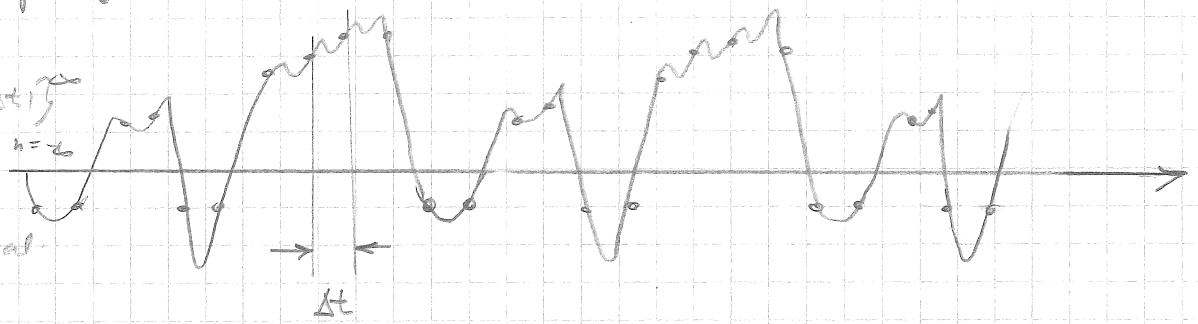
Notation: $R_w f(r) := \int_{L_{r,w}} f(x_1, x_2)\, dl$

is the Radon transform.

Important (Computer tomography) Because there is a formula for recovering f from $R_w f(r)$.

Discrete transforms, Sampling, Filters

$f(t) \longrightarrow \{f_n\}_{n=-\infty}^{\infty} = \{f(n\Delta t)\}_{n=-\infty}^{\infty}$

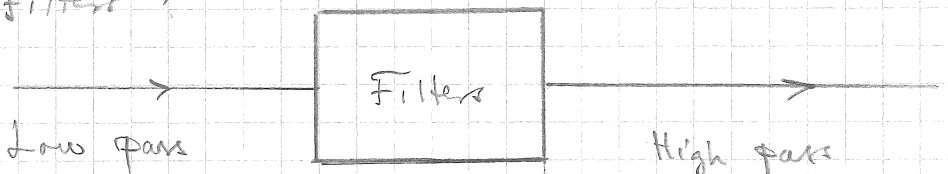


$\{f_n\}$ is a discrete signal.

Questions:

How often do we need to sample?

How we construct digital filters?



Answer:

We:

Fast Fourier transforms

What functions do have a Fourier transforms?

$L^2(\mathbb{R})$

S (Smooth functions with rapid decay)

$L^1(\mathbb{R}) = \{f : \int_{\mathbb{R}} |f(x)|\,dx < \infty\}$

Nok! $f(x) \in L^1(\mathbb{R}) \Rightarrow$

$$|\hat{f}(\xi)| = \left|\int_{\mathbb{R}} e^{-2\pi i x\xi} f(x)\,dx\right|$$

$$\leq \int_{\mathbb{R}} |e^{-2\pi i x\xi} f(x)|\,dx$$

$$= \int_{\mathbb{R}} |f(x)|\,dx < \infty$$

$\therefore$ $(f \in L^1 \Rightarrow \hat{f} \in L_\infty)$.

Hausdorff Young's inequality.

$$f \in L^p, \quad (1 \le p \le 2) \Rightarrow \hat{f} \in L^q, \quad \frac{1}{p} + \frac{1}{q} = 1.$$

Convolution theorem.

Suppose that $f(x) \supset \hat{f}(\xi)$ and $g(x) \supset \hat{g}(\xi)$, then

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y)\, g(y)\, dy \supset \hat{f}(\xi)\, \hat{g}(\xi).$$

Other properties of convolution:

Commutative: $f * g = g * f$

associative: $f * (g * h) = (f * g) * h$

distributive: $f * (g + h) = f * g + f * h$.

The Fourier transform (Further properties)

1) $f \supset \hat{f} \Rightarrow {}^{1a)} Df \supset 2\pi i \xi \hat{f}(\xi) \wedge {}^{1b)} -2\pi i (\cdot) f \supset D\hat{f}$.

Pf $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx \Rightarrow$

$$D\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} \left(-2\pi i x f(x)\right) dx = \mathcal{F}\left(-2\pi i (\cdot) f\right) \Rightarrow 1b)$$

$$\mathcal{F}(Df) = \int e^{-2\pi i x \xi} (Df)(x)\, dx \overset{PI}{=} -\int D_x\left(e^{-2\pi i x \xi}\right) f(x)\, dx = 2\pi i \xi \int e^{-2\pi i x \xi} f(x)\, dx = 2\pi i \xi\, \hat{f}(\xi) \Rightarrow 1a)$$

2) Translation: $\tau_a : f(\cdot) \longmapsto f(\cdot - a)$

$$\mathcal{F}(\tau_a f)(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(\underbrace{x-a}_{:=y})\, dx = \int_{\mathbb{R}} e^{-2\pi i (y+a)\xi} f(y)\, dy$$

$$= e^{-2\pi i a \xi}\, \hat{f}(\xi).$$

$$\tau_a \hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x(\xi - a)} f(x)\, dx$$

$$= \int_{\mathbb{R}} e^{-2\pi i x \xi} \left[e^{2\pi i a x} f(x) \right] dx = \mathcal{F}\left(e^{2\pi i a(\cdot)} f \right)$$

$f(x)$	$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$
$Df(x)$	$(2\pi i \xi)\, \hat{f}(\xi)$
$-2\pi i (\cdot) f$	$(D\hat{f})(\xi)$
$\frac{1}{a} f(\frac{x}{a})$	$\hat{f}(a\xi)$
$\tau_a f = f(x-a)$	$e^{-2\pi i a \xi} \hat{f}(\xi)$
$e^{2\pi i a x} f(x)$	$\tau_a \hat{f}(\xi)$
$(f * g)(x)$	$\hat{f}(\xi)\, \hat{g}(\xi)$

The class $\mathcal{S}$ and $\mathcal{S}'$

Distributions (generalized functions) were introduced by Laurent Schwartz (~1940) and S Sobolev (~1935), to give a mathematically rigorous theory of mathematical objects like Dirac's "δ-function".

[Note: H.A. Schwarz (no "t") in Cauchy–Schwarz was German]

Definition The function class $\mathcal{S}$ are complex valued functions f of a real variable, such that

$f : \mathbb{R} \longrightarrow \mathbb{C}$ satisfies

$$\sup_{x \in \mathbb{R}} \left| |x|^{\alpha} D^{\beta} f(x) \right| < \infty, \quad \text{for any choice of } \alpha \geq 0 \text{ and } \beta \geq 0.$$

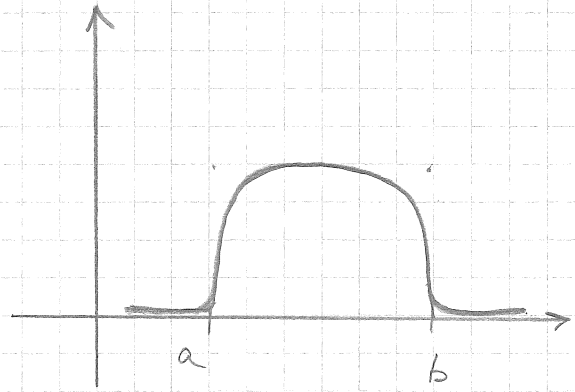
(obs! $\beta = 0 \Rightarrow |f(x)| \leq |x|^{-\alpha}$ for any choice of $\alpha \geq 0$)

Example $f(x) = e^{-ax^2}$ $(a > 0)$

(question: what if $a = i$? $a < 0$?)

Example

$$f(x) = \begin{cases} 0 & x \leq a \\ e^{-\frac{1}{(x-a)^2} - \frac{1}{(x-b)^2}} & a < x < b \\ 0 & x \geq b \end{cases}$$



the function has "compact support" and is C^{∞}, but not (real analytic (i.e. its power series is not convergent everywhere)!

Example Let $g \in L^1(\mathbb{R})$. Then $g_{\varepsilon}(x) := \int_{\mathbb{R}} g(y) \frac{1}{\varepsilon} f\left(\frac{x-y}{\varepsilon}\right) dy \longrightarrow g(x)$ as $\varepsilon \to 0$

In other words: $g * \frac{1}{\varepsilon} f\left(\frac{\cdot}{\varepsilon}\right) \longrightarrow g(x)$ almost everywhere as $\varepsilon \to 0$.

Note that the Fourier transform is well-defined for $f \in \mathcal{S}$.

Properties of $\mathcal{S}$

Lemma I) If $f \in \mathcal{S}$ and if $g(x) = x^\alpha D^\beta f(x)$, $(\alpha, \beta \in \mathbb{Z}^+)$ Then $g \in \mathcal{S}$.

II) $f \in \mathcal{S} \Rightarrow \hat{f} \in \mathcal{S}$.

Pf I): Calculus

II): Let $\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx$, then

$$D^\beta \hat{f}(\xi) = \int_{\mathbb{R}} f(x) \left(\frac{d}{d\xi}\right)^\beta \left(e^{-2\pi i x \xi}\right) dx = \int_{\mathbb{R}} f(x) \underbrace{(-2\pi i x)^\beta}_{\text{decays rapidly}} e^{-2\pi i x \xi} dx.$$

(Why is this o.k.?)

$$\Rightarrow |D^\beta \hat{f}(\xi)| \le \int_{\mathbb{R}} |f(x)| |2\pi x|^\beta dx < \infty. \quad (*)$$

Also $(2\pi i \xi)\hat{f}(\xi) = \int_{\mathbb{R}} f(x) \underbrace{(2\pi i \xi) e^{-2\pi i x \xi}}_{= -\frac{d}{dx}\left(e^{-2\pi i x \xi}\right)} dx \overset{PI}{=} \int_{-\infty}^{\infty} \left(\frac{d}{dx}(f(x))\right) e^{-2\pi i x \xi} dx$

$$= \hat{\mathcal{F}}(Df)$$

$$\Rightarrow |\xi \hat{f}(\xi)| \le \frac{1}{2\pi} \int_{\mathbb{R}} |Df(x)| dx < \infty.$$

This can be repeated for any power ξ^α which together with $(*)$ concludes the result.

Example $f(x) = e^{-\pi x^2} \Rightarrow \hat{f} = f$

[Note if $\mathcal{F}f = f$, then f is a 'fix point' for $\mathcal{F}$. Are there any other fix points?]

$f(x) = e^{-\pi x^2}$

Pf. $f \in \mathcal{S}$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2} e^{-2\pi i x \xi}\, dx \implies$

$$D\hat{f}(\xi) = \int_{\mathbb{R}} e^{-\pi x^2}(-2\pi i x)\, e^{-2\pi i x \xi}\, dx = i \int_{\mathbb{R}} D\left(e^{-\pi x^2}\right) e^{-2\pi i x \xi}\, dx$$

$$= i\, \mathcal{F}(Df) = -2\pi \xi \hat{f}(\xi).$$

$\mathcal{F}(Df) = 2\pi i (\cdot) \hat{f}$

Further $\hat{f}(0) = \int_{\mathbb{R}} e^{-\pi x^2}\, dx = 1$ (see any calculus book).

Hence $\begin{cases} \hat{f}'(\xi) + (2\pi\xi)\hat{f}(\xi) = 0 \\ \hat{f}(0) = 1 \end{cases} \implies \hat{f}(\xi) = e^{-\pi \xi^2}$ ☒

The Fourier inversion formula: Suppose that $f \in \mathcal{S}$ and $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi i x \xi} f(x)\, dx$. Then

$$f(x) = \int_{\mathbb{R}} e^{2\pi i x \xi} \hat{f}(\xi)\, d\xi.$$

Pf Suppose we can show that $\boxed{f(0) = \int_{\mathbb{R}} \hat{f}(\xi)\, d\xi \text{ for all } f \in \mathcal{S},}$ then we are done;

Since then $f(a) = \underbrace{\tau_{-a} f}_{\text{new } f}(0) = \int_{\mathbb{R}} \mathcal{F}(\tau_{-a} f)\, d\xi = \int_{\mathbb{R}} e^{2\pi i \overset{x}{a} \xi} \hat{f}(\xi)\, d\xi.$

$\tau_{-a} f \Leftrightarrow -x$

I) Assume that $f(0) = 0$. Then $g(x) = f(x)/x \in \mathcal{S}$, and $f(x) = x\, g(x)$.

If $g(x) \supset \hat{g}(\xi)$ then $-2\pi i x\, g(x) \supset D\hat{g} \implies \hat{f}(\xi) = \hat{g}'(\xi)/(-2\pi i) \implies$

$\int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \hat{g}'(\xi)\, d\xi = 0$ because $\hat{g}(\xi) \in \mathcal{S}$.

II) If $f(0) \neq 0$, Consider $\tilde{f}(x) = f(x) - f(0) e^{-\pi x^2} \supset \hat{f}(\xi) - f(0) e^{-\pi \xi^2}$,

Then $\tilde{f}(0) = 0 = \int_{-\infty}^{\infty} \hat{\tilde{f}}(\xi)\, d\xi = \int_{-\infty}^{\infty} \hat{f}(\xi) - \int_{-\infty}^{\infty} f(0) e^{-\pi \xi^2}\, d\xi = \hat{\tilde{f}}(\xi)$

$= \int_{-\infty}^{\infty} \hat{f}(\xi)\, d\xi - f(0)$. ☒

The class $\mathcal{S}'(\mathbb{R})$

Notation $\langle \varphi_1, \varphi_2 \rangle = \int_{\mathbb{R}} \varphi_1(x)\, \varphi_2(x)\, dx$ **Note!** This is a scalar product only if φ_1 and φ_2 are real-valued.

Definition A linear mapping $T: \mathcal{S} \longrightarrow \mathbb{C}$ must satisfy

(I) $T(\varphi_1 + \varphi_2) = T\varphi_1 + T\varphi_2$, $\varphi_1, \varphi_2 \in \mathcal{S}$ (II) $T(\alpha\varphi_1) = \alpha\, T\varphi_1$.

Definition A linear mapping $T: \mathcal{S} \longrightarrow \mathbb{C}$ ($\varphi \longmapsto T\varphi$) is called a tempered distribution if for any sequence $\{\varphi_n\}_{n=1}^{\infty}$, $\varphi_n \in \mathcal{S}$, such that for all $\alpha, \beta \in \mathbb{Z}^+$

$$\lim_{n \to \infty} \sup_{x \in \mathbb{R}} |x|^{\alpha} D^{\beta} \varphi_n(x)| = 0$$

we have

$$\lim_{n \to \infty} T(\varphi_n) = 0.$$

Example Take f such that $f(x)/(1+x^2)^{\alpha}$ is integrable for some $\alpha \geq 0$, and let

$$T(\varphi) = \langle f, \varphi \rangle = \int_{-\infty}^{\infty} f(x)\, \varphi(x)\, dx,$$

Then T is a tempered distribution.

Note We may identify f with T (write $f(\varphi)$ for $T(\varphi)$). This must not be confused with $f(x)$, where we think of $f: \mathbb{R} \longrightarrow \mathbb{C}$.

Note S is a linear space: $\varphi_1, \varphi_2 \in S$, $\alpha_1, \alpha_2 \in \mathbb{C} \Longrightarrow \alpha_1\varphi_1 + \alpha_2\varphi_2 \in S$, $(\varphi_1 \equiv 0 \in S)$

To say that $\varphi_n \longrightarrow \varphi$ in S is equivalent to saying that

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left| |x|^{\alpha} D^{\beta}(\varphi_n(x) - \varphi(x)) \right| = 0. \qquad (*)$$

- The family of limits $(*)$, (indexed by α and β) define a <u>topology</u> in S.

Def. If X and Y are topological spaces, and $f: X \longrightarrow Y$, we say that f is continuous if $f(x_n) \longrightarrow f(x)$ in Y, whenever $x_n \longrightarrow x$ in X.

We have $T: S \longrightarrow \mathbb{C}$, so a tempered distribution is a continuous linear map $S \longrightarrow \mathbb{C}$.

<u>Exercise</u> If $f \in C(\mathbb{R})$, $g \in C(\mathbb{R})$ and $f = g$ in S', then $f(x) = g(x)$ for all $x \in \mathbb{R}$.

<u>Proof</u>. $f = g$ in S' means that for all $\varphi \in S$,

$$\langle f, \varphi \rangle = \langle g, \varphi \rangle \iff \int_{\mathbb{R}} (f(x) - g(x)) \varphi(x)\, dx = 0.$$

Let $h(x) = f(x) - g(x)$, and assume that there is an $x_0 \in \mathbb{R}$ such that $h(x_0) > 0$.

Because $h \in C(\mathbb{R})$, there is an interval $I_\varepsilon = (x_0 - \varepsilon, x_0 + \varepsilon)$ such that $h(x) > \frac{1}{2} h(x_0)$, $x \in I_\varepsilon$.

Take $\varphi \in S$, such that $\varphi(x) = 0$ for $x \in \mathbb{R} \setminus I_\varepsilon$ and $\varphi(x) > 0$, when $x \in I_\varepsilon$. Then

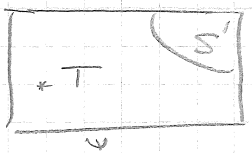
$$\langle h, \varphi \rangle = \int_{-\infty}^{\infty} h(x)\varphi(x)\, dx = \int_{I_\varepsilon} h(x)\varphi(x)\, dx \geq \frac{1}{2} h(x_0) \int_{I_\varepsilon} \varphi(x)\, dx > 0. \quad \lightning$$

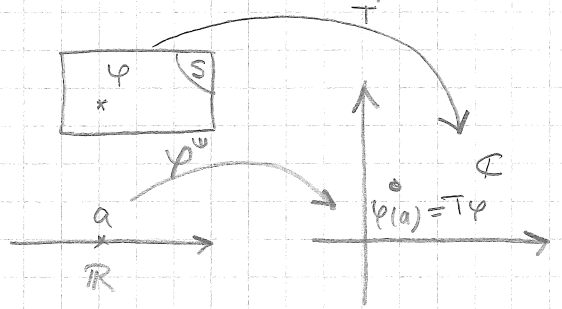
But this contradicts the statement $\langle f, \varphi \rangle = \langle g, \varphi \rangle$ for all φ.

Example $T: \mathcal{S} \longrightarrow \mathbb{C}$

$\varphi \longmapsto \varphi(a)$ ($a \in \mathbb{R}$ i.e. φ is evaluated at the point a)

this is the Dirac "δ-function" at a: $\delta_a(\varphi) \stackrel{?}{=} \varphi(a)$





1) T is linear

2) Let $\varphi_n \in \mathcal{S}$, $\varphi_n \longrightarrow 0$ in $\mathcal{S}$

$\sup_{x \in \mathbb{R}} \left| x^\alpha D^\beta \varphi_n(x) \right| \longrightarrow 0, \quad (n \to \infty)$

In particular $\varphi_n(a) \longrightarrow 0$, as $n \to \infty$, $T(\varphi_n) = \varphi_n(a) \to 0$.

Note. In this case only φ and no derivatives need to be evaluated.

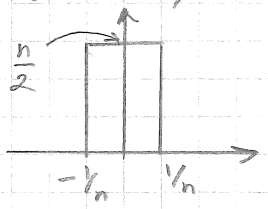
Example Let $f_n(x) = \sqrt{n}\, e^{-n\pi x^2}$, $n = 1, 2, \ldots$, Then $f_n \in \mathcal{S}'$ and for all $\varphi \in \mathcal{S}$,

$f_n(\varphi) = \langle f_n, \varphi \rangle \longrightarrow \varphi(0) = \delta_0(\varphi)$ when $n \to \infty$. We say that $f_n \to \delta_0$ in $\mathcal{S}'$.

Proof

$$\left[\langle f_n, \varphi \rangle = \int_{\mathbb{R}} f_n(x)\varphi(x)\,dx = \sqrt{n} \int e^{-\pi n x^2} \varphi(x)\,dx = \{\sqrt{n}\,x = y\} = \int e^{-\pi y^2} \varphi\left(\frac{y}{\sqrt{n}}\right) dy \to \varphi(0) \int e^{-\pi y^2} dy = \varphi(0), \; n \to \infty \right]$$

(by uniform or dominated conv)



(Other choice of f_n with similar properties.)

Example $f(x) = e^x$ is not a tempered distribution,

because it is growing too fast as $x \longrightarrow \infty$.

Take $\varphi(x) = e^{-\sqrt{1+x^2}} \in \mathcal{S}$, then $\langle e^{(\cdot)}, \varphi \rangle = \int_{-\infty}^{\infty} e^x\, e^{-\sqrt{1+x^2}}\, dx$ which is divergent.

(Prove this)

Example Let $\delta_n : \varphi \longmapsto \varphi(n)$, and set $\amalg = \sum_{n=-\infty}^{\infty} \delta_n$ so $\langle \amalg, \varphi \rangle = \sum_{n=-\infty}^{\infty} \varphi(n)$.

This is a tempered distribution. (Prove it!)

Next we wish to prove that tempered distributions share many properties with ordinary functions. They can be differentiated, Fourier transformed etc.

Differentiation of distributions: Let $f \in C^1(\mathbb{R})$, and assume that it is not growing very fast (it may be bounded for example). Then

$$\langle f', \varphi \rangle = \int_{-\infty}^{\infty} f'(x)\varphi(x)\,dx = -\int_{-\infty}^{\infty} f(x)\varphi'(x)\,dx$$

which is well-defined for all $\varphi \in \mathcal{S}$.

Definition Let T be a tempered distribution, Then we define DT by

$$\langle DT, \varphi \rangle = -\langle T, D\varphi \rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Multiplication by a function: Let $f(x) \in C(\mathbb{R})$ and $g(x) \in C^\infty(\mathbb{R})$ and assume that there is an $\alpha \in \mathbb{Z}^+$ such that $|g(x)|/(1+x^2)^\alpha$ is bounded. Then we have

$$\langle fg, \varphi \rangle = \int_{\mathbb{R}} f(x)g(x)\varphi(x)\,dx = \langle f, g\varphi \rangle.$$

Definition Let $T \in \mathcal{S}'$ and let g be as above. Then we define gT by

$$\langle gT, \varphi \rangle = \langle T, g\varphi \rangle.$$

This is o.k. because $g\varphi \in \mathcal{S}$ if $\varphi \in \mathcal{S}$.

Translation Let $f \in C(\mathbb{R})$ and write $f_\tau(x) = f(x-\tau)$. Then

$$\int_{\mathbb{R}} f_\tau(x)\,\varphi(x)\,dx = \int_{\mathbb{R}} f(x-\tau)\,\varphi(x)\,dx = \{y = x-\tau\} = \int_{\mathbb{R}} f(x)\,\varphi(x+\tau)\,dx = \langle f, \varphi_{-\tau}\rangle.$$

Definition For $T \in \mathcal{S}'$ we define T_τ by $\underline{\underline{\langle T_\tau, \varphi\rangle = \langle T, \varphi_{-\tau}\rangle}}$.

But these definitions are only useful if DT, gT and T_τ satisfy some good properties.

Proposition If $T \in \mathcal{S}'$ then also DT, gT and T_τ belong to $\mathcal{S}'$.

Proof (for DT) We have that $DT : \varphi \longmapsto -\langle T, D\varphi\rangle$. Then DT is obviously linear (why?).

Take $\{\varphi_n\}$ with $\varphi_n \in \mathcal{S}$ and let $\varphi_n \to 0$ in $\mathcal{S}$. Then

$$\sup_x \left|x^\alpha D^\beta D\varphi_n(x)\right| = \sup_x \left|x^\alpha D^{\beta+1}\varphi_n(x)\right| \longrightarrow 0, \text{ as } n \to \infty, \text{ and so,}$$

because $T \in \mathcal{S}'$. $\langle T, D\varphi_n\rangle \longrightarrow 0$ as $n \to \infty$.

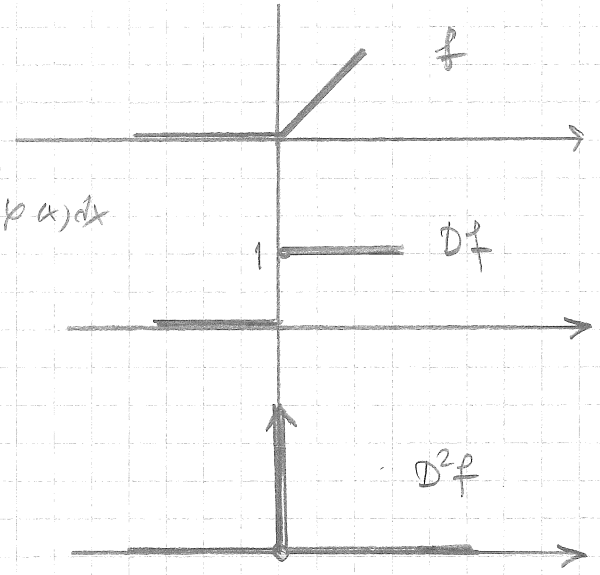
The structure theorem Let $T \in \mathcal{S}'$. Then there exist functions $f_j \in C(\mathbb{R})$ such that

$$T = \sum_j D^{\beta_j} f_j.$$

that means that any tempered distribution can be written as linear combination of (distributional) derivatives of continuous functions.

☒ The pf is rather difficult.

Example Let $f(x) = \begin{cases} x, & \text{when } x > 0 \\ 0 & \text{when } x \le 0 \end{cases}$

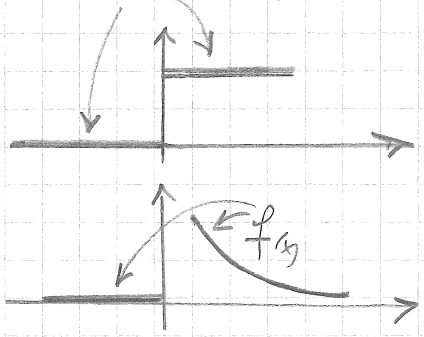


Then $\langle D^2 f, \varphi \rangle = \int_{\mathbb{R}} f(x)\, D^2\varphi(x)\, dx = \int_0^\infty x\, D^2\varphi(x)\, dx = -\int_0^\infty D\varphi(x)\, dx$

$= \varphi(0)$, so $D^2 f = \delta_0$

Example Let $f(x) = \frac{1}{2} e^{-3/2} H(x)$, with

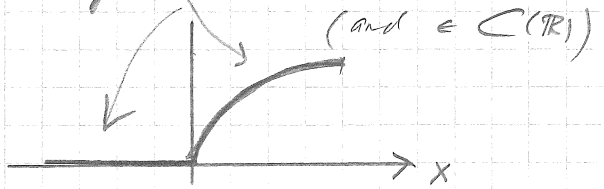
$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & x > 0 \end{cases}$$



[Note. Usually it does not matter if $H(0)$ is defined differently.]

Define a distribution T by $\langle T, \varphi \rangle = \lim_{\varepsilon \to 0+} \left(-\frac{1}{2} \int_\varepsilon^\infty x^{-3/2} \varphi(x)\, dx + \frac{1}{\varepsilon^{1/2}} \varphi(0) \right)$.

Then T is a tempered distribution. In fact $T = D^2 g$, where $g(x) = 2x^{1/2} H(x) \in \mathcal{S}'$



(But as a function $g(x)$ is not in $\mathcal{S}'$, because

$\int_{-\infty}^{\infty} g(x)\varphi(x)\, dx$ is divergent, in general.)

Let $\varphi_n \longrightarrow 0$ in S.

Then $\langle g, \varphi_n \rangle = \int_0^\infty x^{1/2} \varphi_n(x)\,dx = \int_0^\infty \frac{x^{1/2}}{1+x^2}(1+x^2)\varphi_n(x)\,dx \longrightarrow 0$, when $n \to \infty$,

because if $\varphi_n \longrightarrow 0$ in S, so does $(1+x^2)\varphi_n$ and $\int_0^\infty \frac{2x^{1/2}}{1+x^2}\,dx$ is convergent.

Next, $\langle D^2 g, \varphi \rangle = -\langle Dg, D\varphi \rangle = \langle g, D^2\varphi \rangle = \int_0^\infty 2x^{1/2}\varphi''(x)\,dx = \underbrace{\int_\varepsilon^\infty 2x^{1/2}\varphi''(x)\,dx}_{:= F_\varepsilon} + \underbrace{2\int_0^\varepsilon x^{1/2}\varphi''(x)\,dx}_{\to 0,\ \text{as } \varepsilon \to 0}$

$$F_\varepsilon = -\int_\varepsilon^\infty x^{-1/2}\varphi'(x)\,dx + \left[2x^{1/2}\varphi'(x)\right]_\varepsilon^\infty$$

$$= -\left[x^{-1/2}\varphi(x)\right]_\varepsilon^\infty + \int_\varepsilon^\infty -\tfrac{1}{2}x^{-3/2}\varphi(x)\,dx + \underbrace{\left[2x^{1/2}\varphi'(x)\right]_\varepsilon^\infty}_{\to 0 \text{ as } \varepsilon \to 0}$$

$$= -\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}\varphi(0) + \underbrace{\varepsilon^{-1/2}(\varphi(\varepsilon) - \varphi(0))}_{\to 0 \text{ as } \varepsilon \to 0}.$$

In conclusion

$$\langle g, D^2\varphi \rangle = -\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}(\varphi(\varepsilon) - \varphi(0)) + 2\int_0^\varepsilon x^{1/2}\varphi''(x)\,dx + \varepsilon^{-1/2}\varphi(0)$$

$$= \lim_{\varepsilon \to 0}\left(-\tfrac{1}{2}\int_\varepsilon^\infty x^{-3/2}\varphi(x)\,dx + \varepsilon^{-1/2}\varphi(0)\right)$$

T is called the finite part of f.

Proposition [The Plancherel formula] Let $f, \varphi \in \mathcal{S}$. Then

$$\int_{\mathbb{R}} \hat{f}(x)\varphi(x)\,dx = \int_{\mathbb{R}} f(x)\hat{\varphi}(x)\,dx$$

Proof.

$$\int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2\pi i x y} f(y)\,dy\right)\varphi(x)\,dx = \int_{\mathbb{R}}\left(\int_{\mathbb{R}} e^{-2\pi i x y}\varphi(x)\,dx\right) f(y)\,dy.$$

The change of order of integration is justified because $f, \varphi \in \mathcal{S}$.

The Fourier transform of distributions. Because $\langle \hat{f}, \varphi\rangle = \langle f, \hat{\varphi}\rangle$ for $f \in \mathcal{S}$, we define the Fourier transform of $T \in \mathcal{S}'$ by

$$\langle \hat{T}, \varphi\rangle = \langle T, \hat{\varphi}\rangle \quad \text{for all } \varphi \in \mathcal{S}.$$

Proposition $\hat{T} \in \mathcal{S}'$ (this we have to prove)

1) Linear: clear because $\varphi \longmapsto \hat{\varphi}$ is linear.
2) Take $\varphi_n \longrightarrow 0$ in $\mathcal{S}$. Then $\hat{\varphi}_n \longrightarrow 0$ in $\mathcal{S}$, so $\langle \hat{T}, \varphi_n\rangle = \langle T, \hat{\varphi}_n\rangle \to 0$ as $n \to \infty$.

[Note! That $\hat{\varphi}_n \to 0$ in $\mathcal{S}$ follows from the calculations needed to prove that $\hat{\varphi}_n \in \mathcal{S}$.]

Example. Let $\beta \in \mathbb{Z}^+$. Compute $\mathcal{F}(D^\beta \delta_0)$:

$$\langle \mathcal{F}(D^\beta\delta_0), \varphi\rangle = \langle D^\beta\delta_0, \hat{\varphi}\rangle = (-1)^\beta\langle \delta_0, D^\beta\hat{\varphi}\rangle = \langle \delta_0, \mathcal{F}\left((2\pi i\,\cdot)^\beta\varphi\right)\rangle = \int e^{-i2\pi\xi x}(2\pi i x)^\beta\varphi(x)\,dx\Big|_{\xi=0} = \int_{\mathbb{R}}(2\pi i x)^\beta\varphi(x)\,dx$$

Conclusion: $\mathcal{F}(D^\beta\delta_0) = (2\pi i\,\cdot)^\beta \Rightarrow \mathcal{F}(\delta_0) = 1$ for $\beta = 0$

Theorem (The Fourier inversion formula for tempered distributions)

Let $\check{\varphi}(-x) = \varphi(-x)$, where $\varphi \in \mathcal{S}$, and let $\check{T}$ be defined by

$$\langle \check{T}, \varphi \rangle = \langle T, \check{\varphi} \rangle \quad \text{for } T \in \mathcal{S}'.$$

Then, for all $T \in \mathcal{S}'$, $\underline{\underline{\mathcal{F}\mathcal{F}T = \check{T}}}$.

Pf. $\check{\varphi}(x) = \varphi(-x) = \int_{\mathbb{R}} e^{2\pi i \xi(-x)} \hat{\varphi}(\xi)\, d\xi = \mathcal{F}\hat{\varphi}(x) = \mathcal{F}\mathcal{F}\varphi(x)$. Hence

$$\langle \mathcal{F}\mathcal{F}T, \varphi \rangle = \langle \mathcal{F}T, \mathcal{F}\varphi \rangle = \langle T, \mathcal{F}\mathcal{F}\varphi \rangle = \langle T, \check{\varphi} \rangle = \langle \check{T}, \varphi \rangle.$$

Further properties of the Fourier transform of Tempered distribution:

Proposition 1) The Fourier transform is linear:

$$\mathcal{F}(T_1 + T_2) = \mathcal{F}(T_1) + \mathcal{F}(T_2)$$
$$\mathcal{F}(\alpha T_1) = \alpha \mathcal{F}(T_1).$$

2) Let $T \in \mathcal{S}'$ and $f \in C^\infty$ such that for all $\beta > 0$ there is α such that

$$\sup_{x \in \mathbb{R}} \left((1+x^2)^{-\alpha} |D^\beta f(x)| \right) < \infty.$$

Then

$$\mathcal{F}(DT) = 2\pi i(\cdot)\, \mathcal{F}(T) \longleftrightarrow \mathcal{F}(-2\pi i(\cdot)T) = D\hat{T}$$
$$\mathcal{F}(fT) = \hat{f} * \hat{T} \longleftrightarrow \mathcal{F}(\hat{\varphi} * T) = \check{f}\, \mathcal{F}(T)$$
$$\mathcal{F}(\tau_s T) = e^{-2\pi i s(\cdot)} \hat{T} \longleftrightarrow \mathcal{F}(e^{2\pi i(\cdot)s} T) = \tau_s \hat{T}$$

Moreover: $D(\hat{f} * T) = (D\hat{f}) * T + \hat{f} * DT$

Pf All follows directly from the definitions except that we have not yet defined the convolution for the distributions

Recall that if $\varphi_1, \varphi_2 \in \mathcal{S}$. Then $\underline{\mathcal{F}(\varphi_1 * \varphi_2) = \hat{\varphi}_1 \hat{\varphi}_2}$.

If $\hat{T} \in \mathcal{S}'$ and $\hat{f} \in \mathcal{S}'$ and $f \in C^\infty$ has moderate growth. Then, $fT \in \mathcal{S}'$ (Because we can multiply distributions and functions). Hence we can define $\hat{f} * \hat{T}$ by

$$\hat{f} * \hat{T} = \mathcal{F}(fT), \text{ i.e.}$$

$$\langle \hat{f} * \hat{T}, \varphi \rangle = \langle \mathcal{F}(fT), \varphi \rangle = \langle fT, \hat{\varphi} \rangle$$

Taking Fourier transforms, we find

$$\check{f}\check{T} = \mathcal{F}\mathcal{F}(fT) = \mathcal{F}(\hat{f} * \hat{T}) \iff \check{f}\hat{T} = \mathcal{F}(\hat{f} * T)$$

Also

$$\mathcal{F}\left(D(\hat{f} * T)\right) = (2\pi i(\cdot))\mathcal{F}(\hat{f} * T) = \underbrace{(2\pi i(\cdot))\check{f}}_{= \mathcal{F}(Df)}\hat{T} = \mathcal{F}(D\hat{f} * T) = \mathcal{F}(\hat{f} * DT).$$

Differential Equations

Recall that if $f \in C^1(\mathbb{R})$ and $f' = 0$, then f is a constant.

What can we say if $T \in \mathcal{S}'(\mathbb{R})$ and $DT = 0$ in $\mathcal{S}'(\mathbb{R})$?

Lemma Let $T \in \mathcal{S}'$ and assume that $\underline{(\cdot)T = 0}$ in $\mathcal{S}'$. Then there is a constant $a \in \mathbb{C}$ such that $\underline{T = a\delta}$.

Proof Let $\psi \in \mathcal{S}$ with $\psi(0) = 0$, and let $\varphi(x) = \frac{\psi(x)}{x}$. Then $\varphi \in \mathcal{S}(\mathbb{R})$.

For $T \in \mathcal{S}'(\mathbb{R})$,

$$T(\psi) = \underbrace{T((\cdot)\varphi)} = (\cdot)T\varphi = 0. \qquad (*)$$

{note: This should be interpreted as $[(\cdot)T]\varphi = \langle (\cdot)T, \varphi \rangle$.}

Next we take $\varphi \in \mathcal{S}$ arbitrary, and fix $\varphi_1 \in \mathcal{S}$ with $\varphi_1(0) = 1$. We can write

$$\varphi(x) = \underbrace{\varphi(x) - \varphi(0)\varphi_1(x)}_{\equiv \psi(x),\ \psi(0)=0} + \varphi(0)\varphi_1(x),$$ Then

$$T(\varphi) = T(\psi) + T(\varphi(0)\varphi_1) \overset{(*)}{=} T(\varphi_1)\varphi(0) \quad \text{so} \quad \underline{T = T(\varphi_1)\delta}.$$

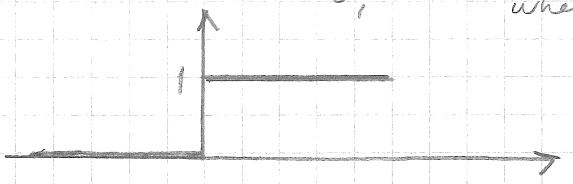
Note that the result depends on the choice of φ_1, but in the C^1-case, $f' \equiv 0$ implies that f is a constant but it does not say which constant.

Corollary Let $T \in \mathcal{S}'(\mathbb{R})$ and assume that $DT = 0$. Then $\hat{T} = a\delta$ for some $a \in \mathbb{C}$.

Proof $0 = \mathcal{F}(DT) = (2\pi i(\cdot))\hat{T} \overset{\text{"recall } \delta\text{"}}{\Longrightarrow} \hat{T} = a\delta$ for some $a \in \mathbb{C}$ $\Rightarrow T = a$, a constant.

$\langle T, \hat{\varphi} \rangle = \langle \hat{T}, \varphi \rangle = \langle a\delta, \varphi \rangle = a\varphi(0) = a\int \varphi = \langle a, \hat{\varphi} \rangle$. ∎

Example Let H be the "Heaviside" step function $H(x) = \begin{cases} 1, & \text{when } x > 0 \\ 0, & \text{when } x < 0 \end{cases}$

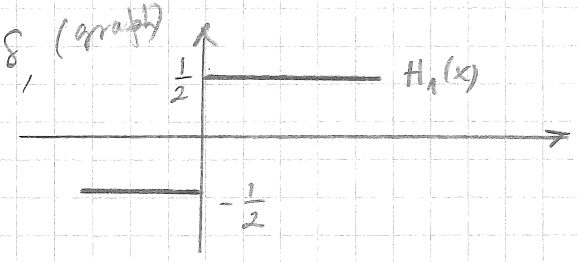


then $\mathcal{F}H = \frac{1}{2\pi i(\cdot)} + \frac{1}{2}\delta$, where

$$\left\langle \frac{1}{\cdot}, \varphi \right\rangle = \lim_{\varepsilon \to 0} \int_{|x|>\varepsilon} \frac{1}{2\pi i x} \varphi(x)\, dx$$

(the Cauchy principal value).

Proof Let $H_1 = H - \frac{1}{2}\cdot$ Then $DH_1 = DH = \delta$, (graph)



and hence $2\pi i(\cdot)\,\hat{H}_1 = 1$. $\xleftarrow{\mathcal{F}}$

Consequently $\hat{H}_1 = \frac{1}{2\pi i(\cdot)} + a\delta$ ← (Why do you have to add a δ? because $DT = 0 \Rightarrow \hat{T} = a\delta$).

But H_1 is odd $\Rightarrow \mathcal{F}H_1$ is odd.

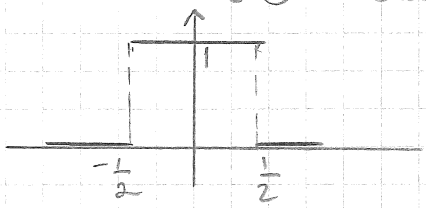
And $\frac{1}{(\cdot)}$ is odd, δ is even, and hence $a = 0$.

Then $\hat{H} = \mathcal{F}\left(H_1 + \frac{1}{2}\right) = \frac{1}{2\pi i(\cdot)} + \frac{1}{2}\delta$. □

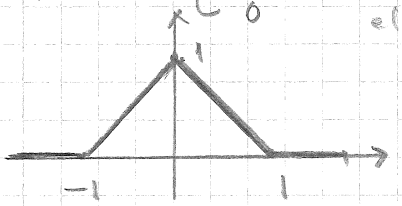
$$\boxed{\operatorname{sig}(x) = 2H(x) - 1}$$

Common notation (Generalized functions)

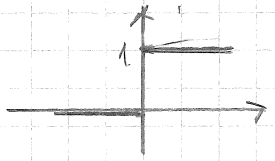
$\Pi(x) = \begin{cases} 1 & \text{when } |x| < \frac{1}{2} \\ 0 & \text{otherwise} \end{cases}$



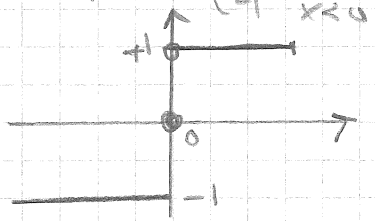
$\Lambda(x) = \begin{cases} 1-x, & 0 \le x < 1 \\ x+1, & -1 < x < 0 \\ 0 & \text{else} \end{cases}$



$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$



$\operatorname{Sign}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$



(Generalized functions) $\rightsquigarrow$

$\operatorname{Sinc}(x) = \dfrac{\sin \pi x}{\pi x}$; Check that $\displaystyle\int_{-\infty}^{\infty} \operatorname{Sinc}(x)\,dx = 1$, (if the divergent integral is properly defined)

$$\mathcal{F}(\Pi) = \int_{-1/2}^{1/2} e^{-2\pi i x \xi}\,dx = \frac{e^{\pi i \xi} - e^{-\pi i \xi}}{2\pi i \xi} = \frac{\sin \pi \xi}{\pi \xi} = \operatorname{Sinc} \xi.$$

$$\boxed{\mathcal{F}(\Pi) = \operatorname{Sinc}}$$

Both Π and Sinc are even, so $\mathcal{F}\mathcal{F}(\Pi) = \check{\Pi} = \Pi = \mathcal{F}(\sin(\cdot)) \Rightarrow$

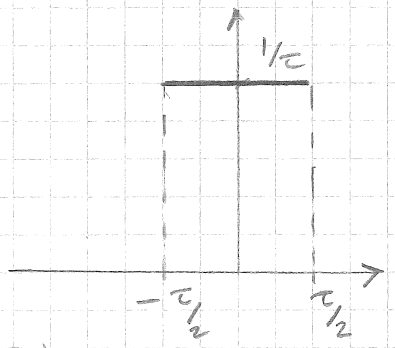
$$\boxed{\Pi = \mathcal{F}(\operatorname{Sinc}(\cdot))}$$

and $\underbrace{\Pi(0)}_{=1} = \displaystyle\int_{-\infty}^{\infty} \mathcal{F}(\Pi)\,d\xi = \int_{-\infty}^{\infty} \frac{\sin \pi \xi}{\pi \xi}\,d\xi \overset{(*)}{=} 1.$

(*)

Distributions and Generalized functions

Let $\Pi(x) = \begin{cases} 1, & \text{when } |x| < 1/2 \\ 0, & \text{else} \end{cases}$, Then $\frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right) = \begin{cases} \frac{1}{\tau}, & |x| < \tau/2 \\ 0 & \text{elsewhere} \end{cases}$



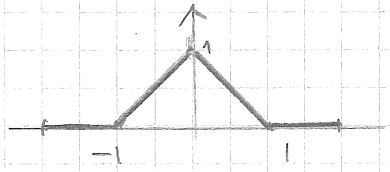
Then, $\displaystyle\int_{-\infty}^{\infty} \frac{1}{\tau}\Pi\left(\frac{x}{\tau}\right)dx = \int_{-\infty}^{\infty} \Pi(x)\,dx = 1$, and for $\varphi \in C(\mathbb{R})$

$$\int_{-\infty}^{\infty} \frac{1}{\tau} H\left(\frac{x}{\tau}\right)\varphi(x)\,dx = \int_{-\infty}^{\infty} \Pi(x)\varphi(\tau x)\,dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} \varphi(\tau x)\,dx \longrightarrow \varphi(0) \quad \text{when } \tau \to 0.$$

because $\varphi(\tau x) \longrightarrow \varphi(0)$ uniformly in $-\frac{1}{2} \le x \le \frac{1}{2}$.

Hence we can see δ as the limit of $\frac{1}{\tau} H\left(\frac{\cdot}{\tau}\right)$ when $\tau \longrightarrow 0$.

Definition Let T_n be a family of distributions, $T_n \in S'(\mathbb{R})$, we say that $T_n \to T$ in S, if for every $\varphi \in S(\mathbb{R})$ $\lim_{n\to\infty} \langle T_n, \varphi \rangle = \langle T, \varphi \rangle$.

Let $\Lambda(x) =$ 

, Then as before

$$\int_{-\infty}^{\infty} \frac{1}{\tau} \Lambda\left(\frac{x}{\tau}\right) \varphi(x)\, dx = \int_{-\infty}^{\infty} \Lambda(x)\, \varphi(\tau x)\, dx \longrightarrow \varphi(0) \quad \text{as} \quad \tau \to 0.$$

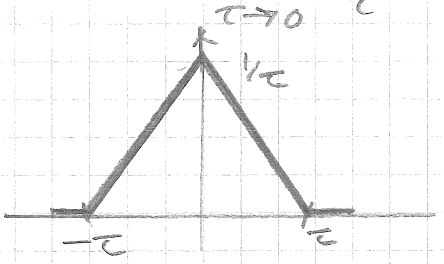
There are many other examples: $\int_{-\infty}^{\infty} \frac{1}{\tau} e^{-\pi (x/\tau)^2} \varphi(x)\, dx \longrightarrow \varphi(0)$

$$\int_{-\infty}^{\infty} \frac{1}{\tau} \operatorname{sinc}\left(\frac{x}{\tau}\right) \varphi(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{\tau} \frac{\sin(\pi x/\tau)}{\pi x/\tau} \varphi(x)\, dx = \int_{-\infty}^{\infty} \frac{\sin \pi x}{\pi x} \varphi(\tau x)\, dx.$$

The last one is more difficult to prove, because $\operatorname{sinc} x$ is not absolutely integrable, but rather it is necessary to rely on cancellations due to the oscillatory behavior of $\sin(\pi x)$.

What about derivatives of generalized functions?

With $\delta = \lim_{\tau\to 0} \frac{1}{\tau}\Pi\left(\frac{\cdot}{\tau}\right)$ in $\mathcal{S}'$ you can't do much. But for $\delta = \lim_{\tau\to 0} \frac{1}{\tau}\Lambda\left(\frac{\cdot}{\tau}\right)$



you have

$$D\left(\frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\right) = \frac{1}{\tau^2}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right) - \Pi\left(\frac{x-\tau/2}{\tau}\right)\right).$$

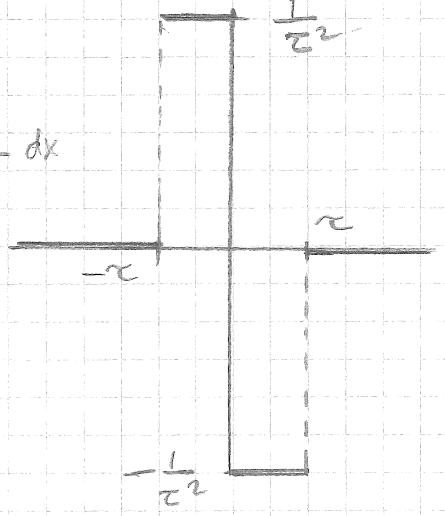
Hence

$$\int_{-\infty}^{\infty} D\left(\frac{1}{\tau}\Lambda\left(\frac{x}{\tau}\right)\right)\varphi(x)\,dx = \int_{-\infty}^{\infty}\left(\Pi\left(\frac{x+\tau/2}{\tau}\right) - \Pi\left(\frac{x-\tau/2}{\tau}\right)\right)\frac{1}{\tau^2}\varphi(x)\,dx$$

$$= \int_{-\infty}^{\infty}\left(\Pi(x+1/2) - \Pi(x-1/2)\right)\frac{1}{\tau}\varphi(\tau x)\,dx = \int_{-1/2}^{1/2}\frac{\varphi(\tau(x-1/2)) - \varphi(\tau(x+1/2))}{\tau}\,dx$$

$$= \{\text{Taylor}\} = \int_{-1/2}^{1/2}\frac{1}{\tau}\left(\varphi'(0)\left(\tau(x-\tfrac{1}{2}) - \tau(x+1/2)\right) + \Theta(\tau^2)\right)d\tau$$

$$= \varphi'(0)\int_{-1/2}^{1/2}\left(-1 + \Theta(\tau)\right)d\tau \longrightarrow -\varphi'(0), \quad \text{when } \tau\to 0.$$



But we also have $\langle D\delta, \varphi\rangle = -\varphi'(0)$. So it seems that $D\left(\frac{1}{\tau}\Lambda\left(\frac{\cdot}{\tau}\right)\right) \longrightarrow \delta'$ in $\mathcal{S}'$.

Is this always true? Could you compute $\lim_{\tau\to 0} D\left(\frac{1}{\tau}\operatorname{sinc}\left(\frac{x}{\tau}\right)\right)$?

The great advantage with studying $\mathcal{S}$ and $\mathcal{S}'$ is that:

You move all difficult operations to $\varphi \in \mathcal{S}$.