Wavelets

Recall the Fourier transform of 2π-periodic functions:

$$\hat{f}(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt = \langle f, e^{i\omega \cdot} \rangle$$

Def. $\quad \langle f, g \rangle = \int_{\mathbb{R}} f(t)\, \overline{g(t)}\, dt.$

The Fourier transform gives a decomposition in pure frequencies $e^{i\omega t}$:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \underbrace{\langle f, e^{i\omega \cdot} \rangle}_{\hat{f}(\omega)} e^{i\omega t}\, d\omega.$$

The <u>Wavelet decomposition</u> is different:

Let $\varphi(t)$ be a given function and let

$$\psi_{j,k} = 2^{j/2} \varphi(2^j t - k)$$

dilation, translation.

Under suitable conditions

$$f(t) = \sum_{j,k=-\infty}^{\infty} \langle f, \psi_{jk} \rangle \psi_{jk}(t).$$

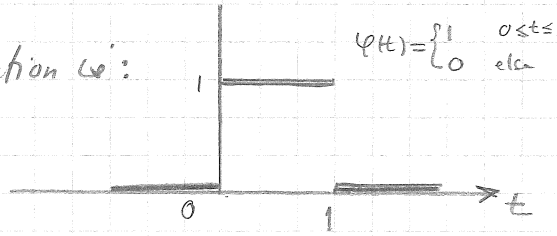
Example: The Haar Wavelet.

Let $\varphi(t) = \varphi(2t) + \varphi(2t-1)$ ← [need to check that it is possible]

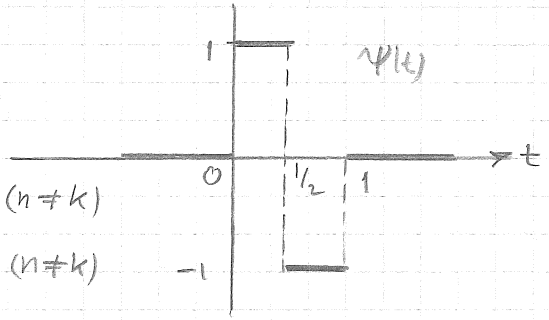
$\psi(t) = \varphi(2t) - \varphi(2t-1)$ ← [can always be achieved]

φ is called a <u>scaling function</u> and ψ a <u>wavelet</u>.

The Haar scaling function is:



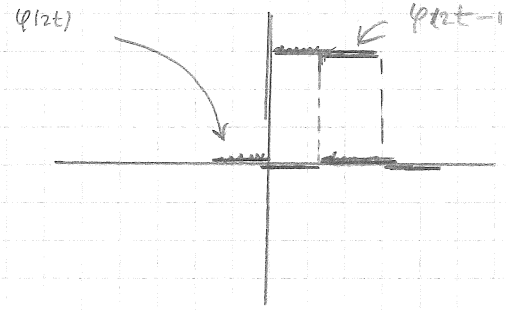
and the wavelet is



Note that $\langle \varphi, \psi \rangle = 0$

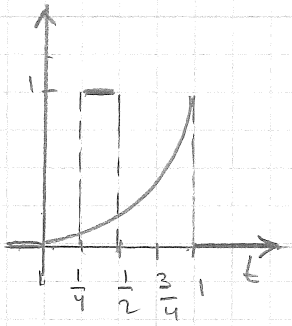
$\langle \varphi(\cdot - k), \varphi(\cdot - n) \rangle = 0, \quad (n \neq k)$

$\langle \psi(\cdot - k), \psi(\cdot - n) \rangle = 0, \quad (n \neq k)$



Example

Let $f(t) = \begin{cases} t^2 & 0<t<1 \\ 0 & \text{else} \end{cases}$



The function $f(t)$ can be approximated by step functions, which is the mean of f in each interval.

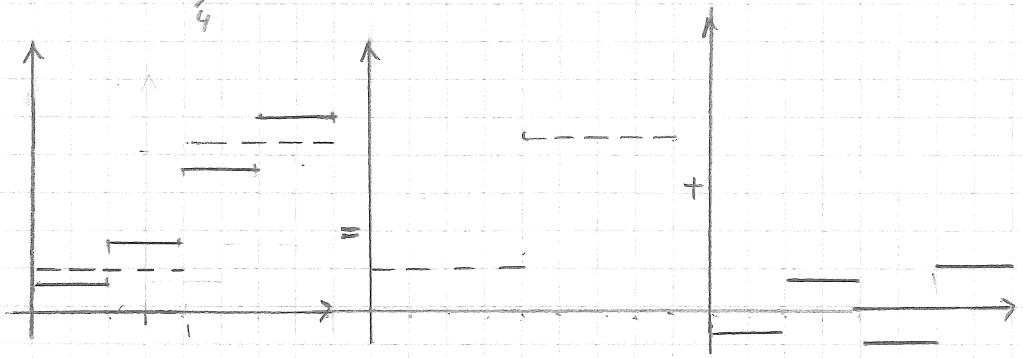
For example in the interval $[\frac{1}{4}, \frac{1}{2}]$

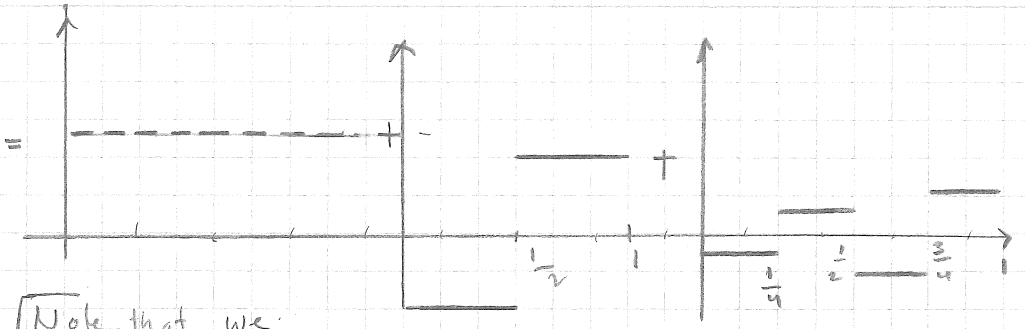
(Second interval: $j=2$)

$$\langle f, 2\varphi(2^2 \cdot -1) \rangle = \int_0^1 t^2 \, 2\varphi(4t-1)\, dt$$

(Obs! $0<4t-1<1 \Leftrightarrow \frac{1}{4} \leq t \leq \frac{1}{2}$)

$$= \int_{1/4}^{1/2} t^2 \cdot 2 \, dt \cong 0.073.$$





[Note that we can continue.]

This is the basic idea behind wavelets: We can expand a signal as an average plus finer and finer details.

Note that $\langle f, 2^{j/2} \varphi(2^j \cdot - k) \rangle \to 0$ as $j \to -\infty$

(i.e. taking the average over larger and larger intervals)

Multiresolution analysis

The Hilbert Space $L^2(\mathbb{R})$.

$$L^2(\mathbb{R}) = \{ f : \mathbb{R} \longrightarrow \mathbb{C} : \int |f(t)|^2 dt < \infty \}$$

(for a proper definition we need the concept of measurability).

The norm of f is defined as:

$$\|f\| = \left(\int_{\mathbb{R}} |f(t)|^2 dt \right)^{1/2}.$$

properties of a norm: $\|f\| \geq 0$, $\|f\| = 0 \Leftrightarrow f = 0$,

$$\|cf\| = |c| \, \|f\|, \quad c \in \mathbb{C}.$$

$$\|f+g\| \leq \|f\| + \|g\|$$

In a Hilbert Space, the norm is defined via a scalar product

$$\|f\| = \langle f, f \rangle^{1/2}.$$

Recall the Parseval's formula: $\int_{\mathbb{R}} |\hat{f}(\omega)|^2 d\omega = 2\pi \int_{\mathbb{R}} |f(x)|^2 dx$

Closed subspaces and projections

A linear subspace $V \subset L^2(\mathbb{R})$ is <u>closed</u> if $f_n \in V$, $n = 1, \ldots, \infty$ and $f_n \longrightarrow f$ in $L^2(\mathbb{R})$ implies that $f \in V$.

<u>Example</u> $\{ f \in L^2(\mathbb{R}) : f(t) = 0 \text{ when } t < 0 \}$

$$V_{BL} = \{ f \in L^2(\mathbb{R}) : |\hat{f}(\omega)| = 0 \text{ when } |\omega| > a \}$$

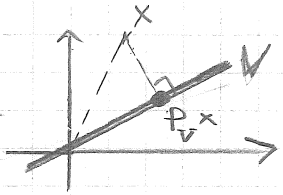
<u>Def</u> A linear operator $P : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is called a <u>projection</u> if

$$P(Pf) = Pf \quad \text{for all } f \in L^2(\mathbb{R}).$$

<u>Def</u> The orthogonal projection of f onto the closed subspace V is the unique $w \in V$ such that

$$\|f - w\| \leq \|f - v\| \quad \text{for all } v \in V.$$

This projection is denoted by P_V.



Here is an example of a subspace of $\mathbb{R}^2$

It is clear that $P_V(P_V x) = P_V x$.

$\langle x - P_V x, w \rangle = 0, \quad \forall w \in V.$

Bases

A family of functions $\{\varphi_k\}_{k\in\mathbb{Z}}$ is a <u>basis</u> for $V \subset L^2(\mathbb{R})$, if any $f \in V$ can be written (uniquely) as

$$f = \sum_k c_k \varphi_k, \qquad c_k \in \mathbb{C}.$$

Note that if $\{\varphi_k\}_k$ is <u>orthogonal</u>, then

$$\|f\|^2 = \left\langle \sum_k c_k \varphi_k, \sum_k c_k \varphi_k \right\rangle$$

$$= \sum_{j\neq k} \left\langle \sum_j c_j \varphi_j, \sum_k c_k \varphi_k \right\rangle + \sum c_k \bar{c}_k \underbrace{\langle \varphi_k, \varphi_k \rangle}_{=1}$$

$$= \sum_k |c_k|^2.$$

But for a general basis this is not true.

Def A <u>Riesz basis</u> for a closed subspace $U \subset L^2$ is a basis that satisfies

$$A\|f\|^2 \leq \sum_k |c_k|^2 \leq B\|f\|^2$$

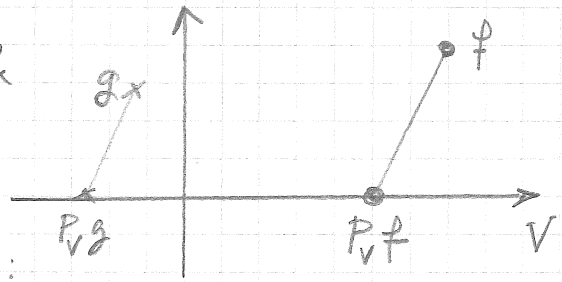
for some constants $A \leq 1 \leq B$.

Note If $\tilde{f}$ is an approximation of f

$$A\|f-\tilde{f}\|^2 \leq \sum_k |c_k - \tilde{c}_k|^2 \leq B\|f-\tilde{f}\|^2.$$

A projection onto V: (orthogonal)

$$P_V f = \sum \langle f, \varphi_k \rangle \varphi_k$$



An image of a non-orthogonal projection:

Here $V = \{(x,0) \in \mathbb{R}^2\}$.

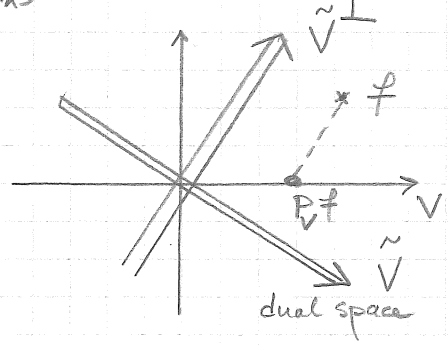
Biorthogonal bases

A biorthogonal basis is a basis $\{\varphi_k\}$ of a subspace V together with a "dual family" $\{\tilde{\varphi}_k\}$ such that

$$\langle \varphi_k, \tilde{\varphi}_n \rangle = \delta_{kn} = \begin{cases} 1 & \text{if } k=n \\ 0 & \text{else.} \end{cases}$$

We let $P_V f = \sum_k \langle f, \tilde{\varphi}_k \rangle \varphi_k$.

Note: Not OG-projection.

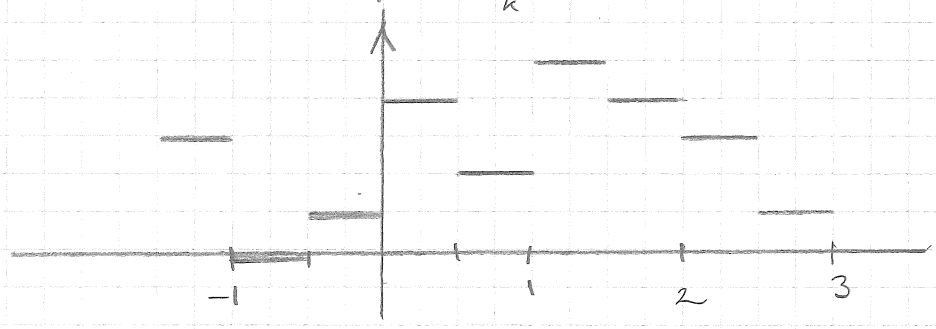


The Haar sampling function

$$\varphi(t) = \begin{cases} 1 & 0 < t < 1 \\ 0 & \text{otherwise} \end{cases}$$

This can be used for piecewise constant approximation:

$$f(t) \longmapsto f_1(t) = \sum_k s_{1k}\, \varphi(2t-k)$$



Definition A multiresolution analysis is a family of closed subspaces $V_j \subset L^2(\mathbb{R})$ such that:

1) $V_j \subset V_{j+1}$, $j \in \mathbb{Z}$
2) $f \in V_j \iff f(2\cdot) \in V_{j+1}$, $j \in \mathbb{Z}$
3) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$.
4) $\bigcap_j V_j = \{0\}$
5) there is a scaling function $\varphi \in V_0$ ...

... such that $\{\varphi(\cdot-k)\}$ is a Riesz basis for V_0.

Definition $\varphi_{j,k} = 2^{j/2}\varphi(2^j t - k)$.

Here 3) means that for any $f \in L^2(\mathbb{R})$, there is a sequence of functions $f_k \in V_k$ such that

$$\|f_k - f\| \longrightarrow 0, \quad \text{as } k \longrightarrow \infty.$$

4) means that if $f \in V_k$, $\forall k \in \mathbb{Z}$, then $f(t) = 0$ for all t.

Properties of scaling function

1) $\int_{-\infty}^{\infty} \varphi(t)\,dt = 1$
2) If $\{\varphi(\cdot-k)\}_{k\in\mathbb{Z}}$ is a basis for V_0, then we must have $\{2^{1/2}\varphi(2\cdot-k)\}$ is a basis for V_1. And since $V_0 \subset V_1$, $\varphi \in V_1$ and hence

$$\varphi(t) = 2\sum_k h_k \varphi(2t-k) \quad \text{for some } \{h_k\}.$$

This is called the **scaling equation**.

Properties of the scaling function

3) Let $H(\omega) = \sum_k h_k e^{-i\omega k}$ and let $\hat{\varphi}(\omega)$ be the Fourier transform of φ.

Then

$$\hat{\varphi}(\omega) = \sum_k h_k \mathcal{F}\left(2\varphi(2\cdot - k)\right)(\omega)$$

$$\overset{(\dagger)}{=} \sum_{k=-\infty}^{\infty} h_k e^{-ik\frac{\omega}{2}} \hat{\varphi}\left(\frac{\omega}{2}\right) = H\left(\frac{\omega}{2}\right)\hat{\varphi}\left(\frac{\omega}{2}\right). \quad (*)$$

Then, by induction $\hat{\varphi}(\omega) = \prod_{j\geq 0} H\left(\frac{\omega}{2^j}\right)$.

From $\hat{\varphi}(0) = 1$ we conclude that $\sum_{k=-\infty}^{\infty} h_k \overset{(\dagger)}{=} 1$

Example

$$\varphi(t) = \operatorname{sinc} t = \frac{\sin \pi t}{\pi t} \Longrightarrow \hat{\varphi}(\omega) = \mathbb{1}_{[-\pi,\pi]}.$$

Wavelets and detail spaces

In a multiresolution $\{V_j\}$ let f be approximated by f_0 and f_1 in V_0 and V_1 respectively.

Hence $f_0 \in V_0$, $f_1 \in V_1$. But also $f_0 \in V_1 \Rightarrow$

$f_1 - f_0 \in V_1$.

For the Haar wavelet we have $\psi(t) = \begin{cases} 1 & 0 \leq t < \frac{1}{2} \\ -1 & \frac{1}{2} \leq t < 0 \\ 0 & \text{else} \end{cases}$

and $\psi(t) = \frac{1}{\sqrt{2}} \underbrace{\varphi_{1,0}}_{f_0} - \frac{1}{\sqrt{2}} \underbrace{\varphi_{1,1}}_{f_1}$

Def For a MRA, ψ is called a wavelet if $W_0 \subset V_1$ is spanned by $\{\psi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $V_1 = W_0 \oplus V_0$, i.e. each $f_1 \in V_1$ can be written (uniquely) as

$$f_1 = f_0 + d_0 \quad \text{with } f_0 \in V_0,\ d_0 \in W_0.$$

W_0 is called a detail space.

Properties of the wavelets

1) $\int \psi(t)\,dt = 0 \quad (\Leftrightarrow \hat{\psi}(0) = 0)$

2) $\psi(t) \in V_1 \iff \psi(t) = 2\sum_k g_k \varphi(2t-k)$

Hence $\hat{\psi}(\omega) = G(\frac{\omega}{2})\hat{\varphi}(\frac{\omega}{2})$ (*)

with $G(\omega) = \sum_k g_k e^{-ik\omega}$,

and because $\hat{\varphi}(0) = 1 \quad (\hat{\varphi}(0) = 0)$

$(*) \Rightarrow G(0) = \sum_k g_k = 0.$

MRA and wavelet decomposition

We have $V_1 = V_0 \oplus W_0$, where

V_0 is spanned by $\{\varphi(\cdot - k)\}$, and

W_0 is spanned by $\{\psi(\cdot - k)\}$, both of which are supposed to be Riesz basis.

Let now $\psi_{j,k} = 2^{j/2}\psi(2^{j/2}t - k)$ and define

W_j = linear span of $\{\psi_{j,k}\}_{k\in\mathbb{Z}}$,

i.e.

$$W_j = \{d_j(t) = \sum_k w_{jk}\psi_{jk}(t), \quad w_{jk} \in \mathbb{C}\}.$$

Let J be an integer corresponding to highest resolution, or in other words, the finest detail of interest.

Take f_J be an approximation of $f \in L^2(\mathbb{R})$. Then,

$$\begin{aligned} f_J \in V_J &= W_{J-1} \oplus V_{J-1} \\ &= W_{J-1} \oplus W_{J-2} \oplus V_{J-2} \\ &= \cdots \\ &= W_{J-1} \oplus W_{J-2} \oplus W_{J-3} \oplus \cdots \oplus W_{j_0} + V_{j_0}. \end{aligned}$$

Then we can write

$$\begin{aligned} f_J(t) &= d_{J-1}(t) + d_{J-2}(t) + \cdots + d_{j_0}(t) + f_{j_0}(t) \\ &= \sum_{j=j_0}^{J-1} \sum_k w_{jk}\psi_{jk}(t) + \sum_k s_{j_0 k}\varphi_{j_0 k}(t). \end{aligned}$$

The last sum converges to zero as $j_0 \to \infty$, because $\bigcap V_{j_0} = \{0\}$.

Also because $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$

one can choose $f_J \to f$ in L^2, $f_J \in V_J$.

So letting $J \to \infty$ we find

$$f(t) = \sum_{j,k} w_{j,k}\, \psi_{j,k}(t),$$

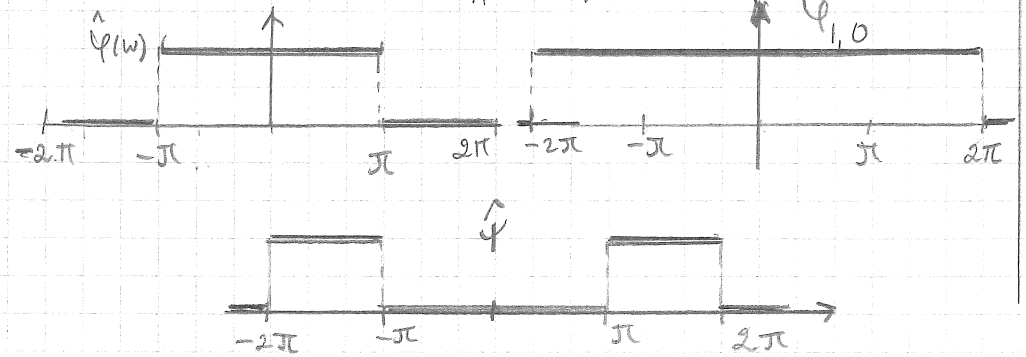
This is the wavelet decomposition of f.

Example $\varphi(t) = \operatorname{sinc} t = \frac{\sin \pi t}{\pi t}$

Corresponding to $\hat\varphi(\omega) = \mathbb{1}_{-\pi<\omega<\pi}$.

Then V_0 is the set of bandlimited functions, with cut off π, and V_j the set of bandlimited functions with cut off $2^j\pi$.

Here $\hat\psi(\omega) = \mathbb{1}_{\pi<|\omega|<2\pi}$



Orthogonal Wavelet decomposition

General properties of scaling function φ and wavelet ψ:

$$\begin{cases} \varphi(t) = \sum_k h_k\, 2\varphi(2t-k), & \int \varphi(t)\,dt = 1 \\ \hat\varphi(\omega) = H\left(\frac{\omega}{2}\right)\hat\varphi\left(\frac{\omega}{2}\right), \quad \text{where } H(\omega) = \sum h_k e^{-ik\omega}, \\ \hat\varphi(0) = 1 \\ H(0) = 1 \iff \sum h_k = 1 \end{cases}$$

$$\begin{cases} \psi(t) = \sum_k g_k\, 2\varphi(2t-k), & \int \psi(t)\,dt = 0 \\ \hat\psi(\omega) = G\left(\frac{\omega}{2}\right)\hat\varphi\left(\frac{\omega}{2}\right), & G(\omega) = \sum g_k e^{-ik\omega} \\ \hat\psi(\omega) = 0 \ \&\ \hat\varphi(0) = 1 \Rightarrow & G(0) = 0. \end{cases}$$

Not The decomposition $V_{j+1} = V_j \oplus W_j$.

It is defined by the filter G. However, there is only one way of writing $V_{j+1} = V_j \oplus W_j$ so that

$\langle v, w\rangle = 0$ for all $v \in V_j$ & $w \in W_j$.

For an orthonormal system we require

$$\int_{-\infty}^{\infty} \varphi(t-k)\,\overline{\varphi(t-n)}\,dt = \begin{cases} 0, & \text{when } n \neq k \\ 1 & \text{when } n = k. \end{cases}$$

In the scaling equation:

$$\int_{-\infty}^{\infty} \varphi(t-n)\,\overline{\varphi(t-m)}\,dt = 4 \sum_{k,k'} h_k h_{k'} \int \varphi(2(t-n)-k)\,\overline{\varphi(2(t-m)-k')}\,dt$$

$$= 2 \sum_{k,k'} h_k h_{k'} \underbrace{\int \varphi(t-2n-k)\,\overline{\varphi(t-2m-k')}\,dt}_{= \begin{cases} 1 & \text{if } 2n+k = 2m+k' \\ 0 & \text{else} \end{cases}}$$

$$= 2 \sum_{2n+k = 2m+k'} h_k h_{k'}.$$

Without loss of generality, we may take $n = 0$ $\Rightarrow$

$$\int \varphi(t)\,\overline{\varphi(t-m)}\,dt = 2 \sum_{k=2m+k'} h_k h_{k'} = 2 \sum_{k'} h_{k'+2m} h_{k'}$$

So $$2 \sum_k h_k h_{k+2m} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & m \neq 0. \end{cases} \qquad (a)$$

[i.e. $\sum_k h_k^2 = \frac{1}{2}$ (obs! $\sum_k h_k = 1$).]

Next we demand that $\{\psi(t-k)\}_{k=-\infty}^{\infty}$ is an ON-basis for W_0.

$$\int_{-\infty}^{\infty} \psi(t-k)\,\overline{\psi(t-m)}\,dt = \begin{cases} 1 & \text{if } k = m \\ 0 & \text{else} \end{cases} \quad (\Rightarrow \ldots$$

$$2 \sum_k g_k g_{k+2m} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & m \neq 0. \end{cases} \qquad (b)$$

[i.e. $\sum_k g_k^2 = \frac{1}{2}$ which $\sum_k g_k = 0$.]

And finally we demand $V_0 \perp W_0$,

which means that

$$\int_{-\infty}^{\infty} \varphi(t-k)\,\overline{\psi(t-n)}\,dt = 0 \quad \text{for all } k, n,$$

which means that

$$\sum_m h_{m+2k}\, g_{m+2n} = 0. \qquad (c)$$

The Haar wavelet is an example showing that such systems exist.

Once constructed such a system can be used to approximate $f \in L^2(\mathbb{R})$:

$$f \approx f_{j_0} + d_{j_0} \quad \text{where} \quad f_{j_0} = P_{j_0} f = \sum_k \langle f, \varphi_{j_0 k} \rangle \varphi_{j_0 k}$$

$$d_{j_0} = Q_{j_0} f = \sum_k \langle f, \psi_{j_0 k} \rangle \psi_{j_0 k}.$$

Proposition The ON-condition implies that:

$$|H(\omega)|^2 + |H(\omega+\pi)|^2 = 1$$

$$|G(\omega)|^2 + |G(\omega+\pi)|^2 = 1$$

$$H(\omega)\overline{G(\omega)} + H(\omega+\pi)\overline{G(\omega+\pi)} = 0.$$

Proof (of the 1st statement)

$$H(\omega)\overline{H(\omega)} + H(\omega+\pi)\overline{H(\omega+\pi)} = \{\text{assuming } h_k \text{ real}\}$$

$$= \left(\sum_k h_k e^{-i\omega k}\right)\left(\sum_{k'} h_{k'} e^{+i\omega k'}\right) + \left(\sum_k h_k (-1)^k e^{-i\omega k}\right)\left(\sum_{k'} h_{k'} (-1)^{k'} e^{i\omega k'}\right)$$

$$= \sum_{k,k'} h_k h_{k'} \left(1 + (-1)^{k+k'}\right) e^{-i\omega(k-k')} = 2 \sum_{k+k'=2m} h_k h_{k'} e^{-i\omega(k-k')}$$

$$= \left\{\begin{matrix} u = k-k' \\ k' = k-u \end{matrix}\right\} = 2 \sum_{2k-u=2m} h_k h_{k-u} e^{-i\omega u} = \{\text{need } u = 2u'\}$$

$$= 2 \sum_{k,k'} h_k h_{k-2u'} e^{-i\omega(2u')} = \sum_{k'} e^{-i\omega 2u'} \cdot \underbrace{2 \sum_k h_k h_{k-2u'}}_{=1, \text{ if } u'=0} = 1.$$

Proposition (conditions on the scaling function and the wavelet.)

1) $\sum |\hat{\varphi}(\omega + 2k\pi)|^2 = 1$

2) $\lim_{j \to -\infty} \hat{\varphi}(2^{-j}\omega) = 1$ (obvious if $\hat{\varphi}$ continuous at $\omega = 0$)

3) $\hat{\varphi}(2\omega) = H(\omega)\hat{\varphi}(\omega)$, H 2π-periodic

4) $\sum_{j'} |\hat{\psi}(2^{j'}\omega)|^2 = 1$

5) $\sum_{j \geq 0} \hat{\psi}(2^j\omega)\overline{\hat{\psi}(2^j(\omega + 2k\pi))} = 0$, when k is odd.

6) $\sum_{j \geq 0} \sum_k |\hat{\psi}(2^j(\omega + 2k\pi))|^2 = 1.$

Proof of 1)

$$\int_{\mathbb{R}} \varphi(t)\overline{\varphi(t-k)}\,dt = \left\{\begin{matrix} 0 & k \neq 0 \\ 1 & k = 0 \end{matrix}\right\} = \delta_k \iff$$

$$\delta_k = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{-ik\omega} \overline{\hat{\varphi}(\omega)}\,d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\varphi}(\omega) e^{i\omega k} \overline{\hat{\varphi}(\omega)}\,d\omega$$

$$= \sum_n \frac{1}{2\pi}\int_0^{2\pi} |\hat{\varphi}(\omega + 2\pi n)|^2 e^{i(\omega + 2\pi n)k}\,d\omega$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{i\omega k} \sum_n |\hat{\varphi}(\omega + 2\pi n)|^2\,d\omega.$$

So δ_k is the Fourier series of the period function

$$r(\omega) = \sum_{n=-\infty}^{\infty} |\hat{\varphi}(\omega + 2\pi n)|^2, \quad \text{and this} \Rightarrow r(\omega) = 1.$$

□

The continuous wavelet transform

We have seen how to write $f \in L^2(\mathbb{R})$ as a "wavelet series" (versus "Fourier series"):

$$f = \sum_{j,k} c_{jk} \psi_{jk}.$$

Here is a different kind of decomposition:

Take $\psi \in L^2(\mathbb{R})$, and assume that

$$C_\psi = \frac{1}{2\pi} \int_{\mathbb{R}} \frac{1}{|\xi|} |\hat{\psi}(\xi)|^2 d\xi < \infty.$$

If $\psi \in L^1(\mathbb{R})$, then $\hat{\psi}(\xi)$ is continuous, then we must have

$$\hat{\psi}(0) = \int_{\mathbb{R}} \psi(x)\, dx \overset{\longleftarrow}{=} 0 \quad \text{(wavelet)}$$

The continuity of $\hat{\psi}$ follows from

$$|\hat{\psi}(\xi+h) - \hat{\psi}(\xi)| = \left| \int_{\mathbb{R}} \left(e^{-i(\xi+h)x} - e^{-i\xi x} \right) f(x)\, dx \right|$$

$$\leq \int_{-\infty}^{\infty} |e^{-ihx} - 1| |f(x)|\, dx \leq \int_{|x| \leq M} |e^{-ihx} - 1| |f(x)|\, dx + \int_{|x|>} 2|f(x)|.$$

Given $\varepsilon > 0$, take M suff. large: $\int_{|x|>M} |f(x)|\, dx < \frac{\varepsilon}{4}$, and then δ sufficiently small that

$$|1 - e^{-ihx}| < \varepsilon \left(2 \int |f(x)|\, dx \right)^{-1} \text{ for } |x| < M \text{ & } |h| < \delta.$$

We then find that $|\hat{\psi}(\xi+h) - \hat{\psi}(\xi)| < \varepsilon$ for $|h| < \delta$.

Definition $\psi^{ab}(x) = |a|^{-1/2} \psi\left(\frac{x-b}{a}\right)$.

This is doubly indexed family of wavelets.

Definition:

$$(T^{wav} f)(a,b) = \langle f, \psi^{ab} \rangle = \int f(x) \overline{\psi\left(\frac{x-b}{a}\right)} |a|^{-1/2} dx$$

Note Because $\|\psi^{ab}\|_{L^2} = 1$, $|(T^{wav} f)(a,b)| \leq \|f\|_{L^2}$.

Proposition (The resolution of the identity).

For all $f, g \in L^2(\mathbb{R})$, the following equality holds:

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (T^{wav} f)(a,b) \overline{(T^{wav} g)(a,b)} \frac{1}{|a|^2} da\, db = C_\psi \langle f, g \rangle$$

This result should be compared with

$$\int f(x) \overline{g(x)}\, dx = \frac{1}{2\pi} \int \hat{f}(\xi) \overline{\hat{g}(\xi)}\, d\xi.$$

We can write the equality as:

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(T^{wav}f)(a,b)\overline{\int_{-\infty}^{\infty} g(x)\,|a|^{-1/2}\psi\left(\frac{x-b}{a}\right)dx}\,\frac{1}{|a|^2}\,da\,db$$

$$=\int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(T^{wav}f)(a,b)\,\psi\left(\frac{x-b}{a}\right)\frac{1}{|a|^{2+1/2}}\,da\,db\right)\overline{g(x)}\,dx$$

$$=\int_{-\infty}^{\infty}\underbrace{\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(T^{wav}f)(a,b)\,\psi^{ab}(x)\,\frac{1}{|a|^2}\,da\,db\right)}_{\text{"in a weak sense" this must be equal to } f.}\overline{g(x)}\,dx$$

Proof

$$(\star)\quad \iint\frac{da\,db}{|a|^2}(T^{wav}f)(a,b)\overline{(T^{wav}g)(a,b)}$$

{scaling + translation}

$$=\iint\frac{da\,db}{|a|^2}\left[\frac{1}{2\pi}\int d\xi\,\hat f(\xi)|a|^{1/2}e^{-ib\xi}\overline{\hat\psi(a\xi)}\right]\times$$

$$\times\left[\frac{1}{2\pi}\int d\xi'\,\overline{\hat g(\xi')}|a|^{1/2}e^{+ib\xi'}\hat\psi(a\xi')\right]$$

i.e. we have expressed $(T^{wav}f)(a,b)$ using Parseval's formula:

$$\left[\overline{\mathcal{F}}=\chi,\ \mathcal{F}\left(a^{-1/2}\chi\left(\frac{\bullet-b}{a}\right)\right)=a^{-1/2}\cdot a\,e^{-ib\xi}\hat\chi(a\xi)=a^{1/2}e^{-ib\xi}\hat\chi(a\xi)\right]$$

$$T^{wav}f(a,b)=\int_{-\infty}^{\infty}f(x)\,|a|^{-1/2}\overline{\psi\left(\frac{x-b}{a}\right)}\,dx$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(\xi)\,\hat{\mathcal{F}}\left(|a|^{-1/2}\overline{\psi\left(\frac{\bullet-b}{a}\right)}\right)(\xi)\,d\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat f(\xi)\,|a|^{1/2}e^{-ib\xi}\overline{\hat\psi}(a\xi)\,d\xi.$$

Next we note that

$$\int_{-\infty}^{\infty}\underbrace{\hat f(\xi)\,|a|^{1/2}\overline{\hat\psi(a\xi)}}_{\equiv F_a(\xi)}e^{-ib\xi}\,d\xi=\hat F_a(b),$$

where ξ and b are treated as dual variables in a Fourier transform.

Similarly: $\int\underbrace{e^{ib\xi'}|a|^{1/2}\overline{\hat g(\xi')}}_{\equiv\overline{G_a(\xi')}}\hat\psi(a\xi')\,d\xi'=\overline{\hat G_a(b)}$.

Therefore we can write

$$(\star)=\frac{1}{4\pi^2}\int\frac{da}{a^2}\int db\,\hat F_a(b)\overline{\hat G_a(b)}=\frac{1}{2\pi}\int\frac{da}{a^2}\int F_a(\xi)\overline{G_a(\xi)}\,d\xi$$

$$=\frac{1}{2\pi}\int\frac{da}{a}\int\hat f(\xi)\overline{\hat g(\xi)}\,|\hat\psi(a\xi)|^2\,d\xi=\frac{1}{2\pi}\int\hat f(\xi)\overline{\hat g(\xi)}\int\frac{|\hat\psi(a\xi)|^2}{|a|}\,da\,d\xi$$

$$:=C_\psi\,\frac{1}{2\pi}\int\hat f(\xi)\overline{\hat g(\xi)}=C_\psi\langle f,g\rangle.$$

[Note that all integrals of the form $\int\frac{F(x,y)}{y}\,dy$ are independent of x, if the integral is convergent.] ☐

The family $\{\psi^{ab}\}_{a,b\in\mathbb{R}}$ is uncountable, but in the discrete wavelet case, we have only countably many ψ_{jk}. How many ψ_{jk}:s do we really need?

Definition A family of functions $(\varphi_j)_{j\in J}$ in a Hilbert space $\mathcal{H}$ is called a frame if there exists $0 < A \le B$ such that for all $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_j |\langle f, \varphi_j\rangle|^2 \le B\|f\|^2.$$

A and B are called the frame bands.

A frame is called tight if $A = B$.

Proposition If $(\varphi_j)_{j\in J}$ is a tight frame with $A=1$ and $\|\varphi_j\| = 1$ for all $j \in J$, then (φ_j) is an ON-basis.

Proof If $\langle f, \varphi_j\rangle = 0$ for all $j \in J$, then $\|f\| = 0$ and therefore $(\varphi_j)_{j\in J}$ spann all of $\mathcal{H}$.

Otherwise, let $\mathcal{H}_0$ be a spann of $(\varphi_j)_{j\in J}$ and let $\mathcal{H}_1$ be the orthogonal complement of $\mathcal{H}_0$ in $\mathcal{H}$; i.e. $\mathcal{H}_0 \perp \mathcal{H}_1$ and $\mathcal{H}_0 \oplus \mathcal{H}_1 = \mathcal{H}$.

Take $g \in \mathcal{H}_1$, $\|g\| = 1$. But then $\langle g, \varphi_j\rangle = 0$ for all $j \in J$, which contradicts the tightness of the frame.

$$\|\varphi_j\|^2 = \sum_j |\langle \varphi_j, \varphi_{j'}\rangle|^2 = \|\varphi_j\|^4 + \sum_{j'\neq j} |\langle \varphi_j, \varphi_{j'}\rangle|^2.$$

But $\|\varphi_j\|^2 = \|\varphi_j\|^4 = 1 \Rightarrow \sum_{j'\neq j} |\langle \varphi_j, \varphi_{j'}\rangle|^2 = 0$

so $\langle \varphi_j, \varphi_{j'}\rangle = 0$ if $j \neq j'$.

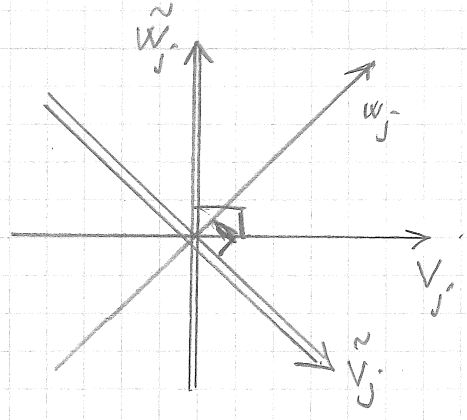
If a frame is tight $\sum_{j\in J} |\langle f, \varphi_j\rangle|^2 = A\|f\|^2$

$$\Rightarrow A\langle f, g\rangle = \sum_{j\in J} \langle f, \varphi_j\rangle\langle \varphi_j, g\rangle \Rightarrow$$

$$f = \frac{1}{A}\sum \langle f, \varphi_j\rangle \varphi_j.$$

For a frame that is not a ON-basis, there may be many ways of writing $f = \sum c_j \varphi_j$. The most "economical" way is by aid of the "dual frame" $\tilde{\varphi}_j$.

Biorthogonal Systems



The figure indicates how V_j and W_j together span $\mathbb{R}^2$, which serves as an image of V_{j+1}.

But they are not orthogonal.

Instead there is a space $\tilde{W}_j$ that is the orthogonal complement to V_j. $\tilde{W}_j$ is the detail space of a "dual MRA": $\{\tilde{V}_j\}_{j=-\infty}^{\infty}$; $\tilde{V}_j \subset \tilde{V}_{j+1} \cdots$.

There is a dual scaling function $\tilde{\varphi}$ that satisfies a

scaling equation:

$$\tilde{\varphi}(t) = 2\sum \tilde{h}_k \tilde{\varphi}(2t-k),$$

and dual mother wavelet that satisfies $\tilde{\psi}(t) = 2\sum \tilde{g}_k \tilde{\varphi}(2t-k)$.

The conditions for biorthogonality are

$$\begin{cases} \langle \varphi_{jk}, \tilde{\varphi}_{jn} \rangle = \delta_{kn} \\ \langle \varphi_{jk}, \tilde{\psi}_{jn} \rangle = 0 \end{cases} \quad \& \quad \begin{cases} \langle \psi_{jk}, \tilde{\psi}_{jn} \rangle = \delta_{kn} \\ \langle \tilde{\varphi}_{jk}, \psi_{jn} \rangle = 0. \end{cases}$$

One may derive orthogonality conditions for H, G, $\tilde{H}$ and $\tilde{G}$:

$$\tilde{H}(\omega)\overline{H(\omega)} + \tilde{H}(\omega+\pi)\overline{H(\omega+\pi)} = 1$$
$$\tilde{G}(\omega)\overline{G(\omega)} + \tilde{G}(\omega+\pi)\overline{G(\omega+\pi)} = 1$$
$$\tilde{G}(\omega)\overline{H(\omega)} + \tilde{G}(\omega+\pi)\overline{H(\omega+\pi)} = 0$$
$$\tilde{H}(\omega)\overline{G(\omega)} + \tilde{H}(\omega+\pi)\overline{G(\omega+\pi)} = 0.$$

$$\Leftrightarrow \quad \overline{M(\omega)}\,\tilde{M}(\omega)^{T} = I \quad \text{where} \quad M(\omega) = \begin{bmatrix} H(\omega) & H(\omega+\pi) \\ G(\omega) & G(\omega+\pi) \end{bmatrix}$$

(^{T}: Transpose)

or $\tilde{M}(\omega)^{T}\,\overline{M(\omega)} = I$ (I: identity matrix).

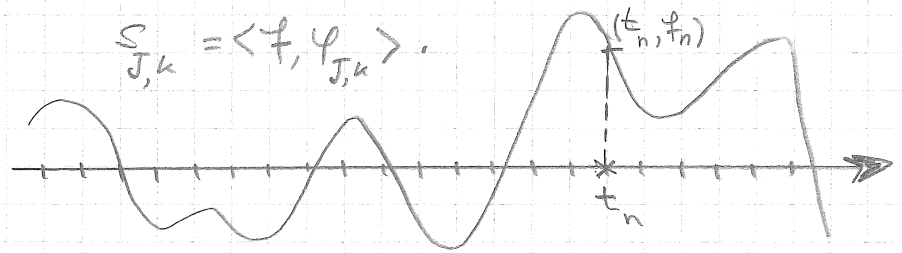
The discrete wavelet transform

Question: How to compute the scaling and wavelet coefficients?

$$f_J(t) = \sum_{j=j_0}^{J-1} \sum_k w_{jk}\psi_{jk}(t) + \sum_k s_{j_0}\varphi_{j_0 k}(t).$$

First: we have to decide a finest scale J_0 and determine

$$s_{J,k} = \langle f, \varphi_{J,k} \rangle.$$



Usually the signal is sampled to give a discrete signal $(f_n)_{n=-\infty}^{\infty}$.

In the case of the Haar wavelet the $s_{J,k}$ are simply the f_n, but for higher order wavelets the calculation is more involved.

Note that the signal must not contain too high frequencies.

Once the $s_{J,k}$ are known, we may proceed recursively:

Assume that $s_{j+1,k} = \langle f, \varphi_{j+1,k} \rangle$ are known.

We may write $f_{j+1} = \sum \langle f, \varphi_{j+1,k} \rangle \varphi_{j+1,k}$ (obs! notation!)

and we know that $f_{j+1} = f_j + d_j$.

Hence

$$f_{j+1}(t) = \sum_k s_{j+1,k} \varphi_{j+1,k}(t) = \sum_k s_{j,k} \varphi_{j,k}(t) + \sum_k w_{j,k} \psi_{j,k} = f_j(t) + d_j(t).$$

We may compute the scalar product with $\varphi_{j,\ell}$, to get

$$\sum_k s_{j+1,k} \langle \varphi_{j+1,k}, \varphi_{j,\ell} \rangle = \sum_k s_{j,k} \underbrace{\langle \varphi_{j,k}, \varphi_{j,\ell} \rangle}_{\delta_{k,\ell}} + \sum_k w_{j,k} \underbrace{\langle \psi_{j,k}, \varphi_{j,\ell} \rangle}_{=0}$$

Now we compute the scalar flux on the LHS:

To this end we use: $\varphi_{j,k} = \sqrt{2} \sum_m h_m \varphi_{j+1,m+2k}$ (*)

We check (*):

$$\varphi_0 = \varphi(t) = \sum_m h_m 2 \varphi(2t-m).$$

Then $\varphi_{0,k} = \varphi(t-k) = \sum_m h_m \sqrt{2}\sqrt{2} \varphi(2t-m-2k)$

$$= \sum_m \sqrt{2} h_m \varphi_{1,2k+m}(t), \quad \text{...}$$

So

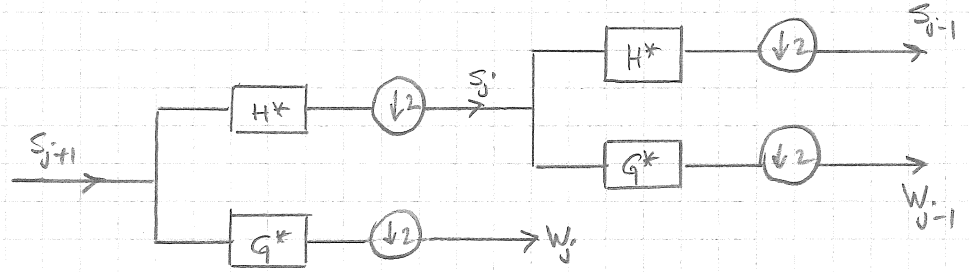
$$\langle \varphi_{j+1,k}, \varphi_{j,\ell} \rangle = \sqrt{2} \sum_m h_m \underbrace{\langle \varphi_{j+1,k}, \varphi_{j+1,m+2\ell} \rangle}_{\delta_{k,m+2\ell}} = \sqrt{2} h_{k-2\ell}.$$

A similar calculation holds for $\langle f_{j+1}, \psi_{j,k} \rangle$.

In summary:

$$\underbrace{s_{j,k}}_{\text{RHS in } J} \underset{\ell=k}{=} \sqrt{2} \sum_\ell h_{\ell-2k} s_{j+1,\ell} \quad \& \quad w_{j,k} = \sqrt{2} \sum_\ell g_{\ell-2k} s_{j+1,\ell}.$$

This may be expressed with a filter bank



The inverse calculation is carried out similarly:

write

$$f_{j+1} = \sum_\ell s_{j,\ell}\,\varphi_{j,\ell} + \sum_\ell w_{j,\ell}\,\psi_{j,\ell}$$

$$= \sqrt{2}\sum_\ell \sum_m s_{j,\ell}\,h_m\,\varphi_{j+1,m+2\ell} + \sqrt{2}\sum_\ell\sum_m w_{j,\ell}\,g_m\,\varphi_{j+1,m+2\ell}$$

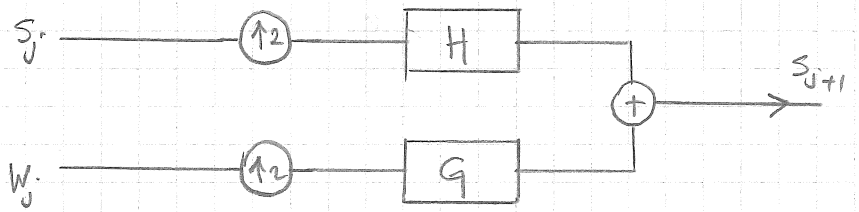
and

$$s_{j+1,k} = \langle f_{j+1}, \varphi_{j+1,k}\rangle = \sqrt{2}\sum_\ell\sum_m \left(s_{j,k}\,h_m + w_{j,k}\,g_m\right)\langle \varphi_{j+1,m+2\ell}, \varphi_{j+1,k}\rangle$$

$$= \sqrt{2}\sum \left(s_{j,\ell}\,h_{k-2\ell} + w_{j,k}\,g_{k-2\ell}\right)$$

This correspond to the following filter:

(The factors $\sqrt{2}$ may be included in the filters).



Biorthogonal systems

Let $\{V_j\}$ be a MRA, and $\{\tilde V_j\}$ a dual MRA.

The scaling function and mother wavelets for $\tilde V_j$ satisfies

$$\tilde\varphi(t) = 2\sum \tilde h_k\,\tilde\varphi(2t-k) \quad \&\quad \tilde\psi(t) = 2\sum \tilde g_k\,\tilde\varphi(2t-k).$$

The biorthogonality conditions are:

$$\langle \varphi_{j,k}, \tilde\varphi_{j,\ell}\rangle = \delta_{k\ell}\,, \quad \langle \psi_{j,k}, \tilde\psi_{j,\ell}\rangle = \delta_{k\ell}$$

$$\langle \varphi_{j,k}, \tilde\psi_{j,\ell}\rangle = 0\,, \quad \langle \tilde\varphi_{j,k}, \psi_{j,\ell}\rangle = 0.$$

At the finest resolution we may then write

$$f_J(t) = \sum_k \langle f, \tilde\varphi_{J,k}\rangle\,\varphi_{J,k}(t)$$

$$= \sum_k s_{J-1,k}\,\varphi_{J-1,k} + \sum_k w_{J-1,k}\,\psi_{J-1,k}\,,$$

and just as in the orthogonal case

$$f(t) = \sum_{j,k} \langle f, \tilde\psi_{j,k}\rangle\,\psi_{j,k}.$$

Note that in this formula we have both $\tilde\psi_{j,k}$ and $\psi_{j,k}$.