Poisson's summation formula (An approach by Folland)

$$f(x) = \sum_{k=-\infty}^{\infty} c_k e^{2\pi i k x} \Rightarrow \hat{f}(\xi) = \sum_{k=-\infty}^{\infty} c_k \delta(\xi - k)$$

$$\Updownarrow$$

$$\mathcal{F}\Big(\sum_k \delta_k\Big) = \sum_k \delta_k$$

$\Uparrow$) o.k. (have proved)

$\Downarrow$) claim: $f_k \xrightarrow{\text{in } S'} f \Rightarrow \begin{cases} D^k f_k \to D^k f & \text{in } S' \\ \hat{f}_k \to \hat{f} & \text{in } S' \end{cases}$

another approach

Pf $\forall \varphi \in S$ $(\hat{\varphi} \in S)$; Plancherel $\Rightarrow$

$$\langle \hat{f}_k, \varphi \rangle = \langle f_k, \hat{\varphi} \rangle \longrightarrow \langle f, \hat{\varphi} \rangle = \langle \hat{f}, \varphi \rangle \quad ☒$$

Likewise for $D^\alpha f_k$.

$$\tau_a f \overset{\mathcal{F}}{\supset} e^{-2\pi i a \xi} \hat{f} \underset{\delta = f}{\overset{\hat{\delta}=1}{\Longrightarrow}} \hat{\delta}(\xi - a) = e^{-2\pi i a \xi} \quad (*)$$

$$\left.\begin{array}{l} D^\alpha f \overset{\mathcal{F}}{\supset} (2\pi i \xi)^\alpha \hat{f} \\ (2\pi i x)^\alpha f \supset D^\alpha f \end{array}\right\} \underset{f=\delta(x-a)}{\Longrightarrow} \begin{cases} \mathcal{F}(D^\alpha \delta(x-a)) = (2\pi i \xi)^\alpha \mathcal{F}(\delta(x-a))(\xi) \\ (*) = (2\pi i \xi)^\alpha e^{-2\pi i a \xi} \end{cases}$$

$$\overset{\mathcal{F}}{\Longrightarrow} \mathcal{F}\mathcal{F}(D^\alpha \delta(x-a))(\xi) = \mathcal{F}\big((2\pi i \xi)^\alpha e^{-2\pi i a \xi}\big)$$

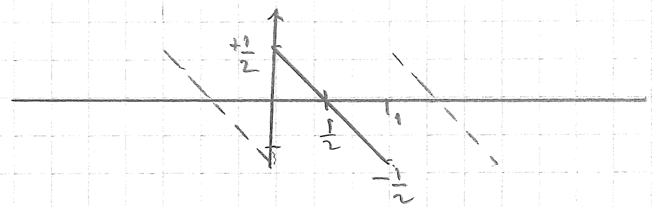
$$\Downarrow \boxed{\mathcal{F}\mathcal{F}\varphi(x) = \varphi(-x)}$$

$$D^\alpha \delta(x+a)(\xi) = \mathcal{F}\big((2\pi i \xi)^\alpha e^{-2\pi i a \xi}\big)$$

Let $\alpha = 0$, $a = -k$ & $\xi = x$ $\Longrightarrow e^{2\pi i k x} \overset{\mathcal{F}}{\supset} \delta(x-k)$

Then $\sum c_k \ldots$ $\supset$ $\sum c_k \ldots$

$\Rightarrow$ Let now $f(x) = \frac{1}{2} - x$, $0 < x < 1$, $f(x+1) = f(x)$



Define $\delta_{per} = \sum_k \delta(x-k)$ does not have mean value equal "0".

Its integral over any interval of length 1 is 1. So it is not the derivative of any periodic function. But "$\delta_{per} - 1$" has mean value 0: $c_k = \int_{-1/2}^{1/2} \delta_{per}(x) e^{-2\pi i k x} dx = \int \delta(x) e^{-2\pi i k x} dx = 1$.

$\Rightarrow \delta_{per} - 1$ is the derivative of sawtooth wave $f(x)$.

f odd $\Rightarrow a_n = 0$ & $b_n = \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx$ $\{2L = 1\}$

$$= 4\int_0^{1/2} \Big(\tfrac{1}{2} - x\Big) \sin(2n\pi x)\, dx = 4\Big(\tfrac{1}{2} - x\Big)\frac{-\cos(2n\pi x)}{2n\pi}\Big|_0^{1/2} - \cancel{4\int (-1)\frac{-\cos(2n\pi x)}{2n\pi} dx} \to 0$$

$$= 0 - 4\Big(\tfrac{1}{2}\Big)\frac{-\cos 0}{2n\pi} = \frac{1}{n\pi}$$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin(2n\pi x) \Rightarrow f'(x) = 2\sum_{n=1}^{\infty} \cos(2n\pi x)$$

$$\Rightarrow \delta_{per} - 1 = 2\sum_{n=1}^{\infty} \cos(2n\pi x) = \sum_{n=1}^{\infty}\big(e^{2\pi n i x} + e^{-2\pi n i x}\big)$$

$$\Rightarrow \delta_{per} = \sum_{n=-\infty}^{\infty} e^{2n\pi i x} = \sum_{n=-\infty}^{\infty} \delta(x-k)$$

Let now in (PSF) $c_k = 1$ & $f = \sum \delta(x-k) = \sum e^{2\pi i n x}$

$$\Rightarrow \hat{f} = \sum_{-\infty}^{\infty} \delta(\xi - k) \quad \therefore f = \sum_k \delta_k \overset{\mathcal{F}}{\supset} \sum_k \delta_k \quad ☒$$