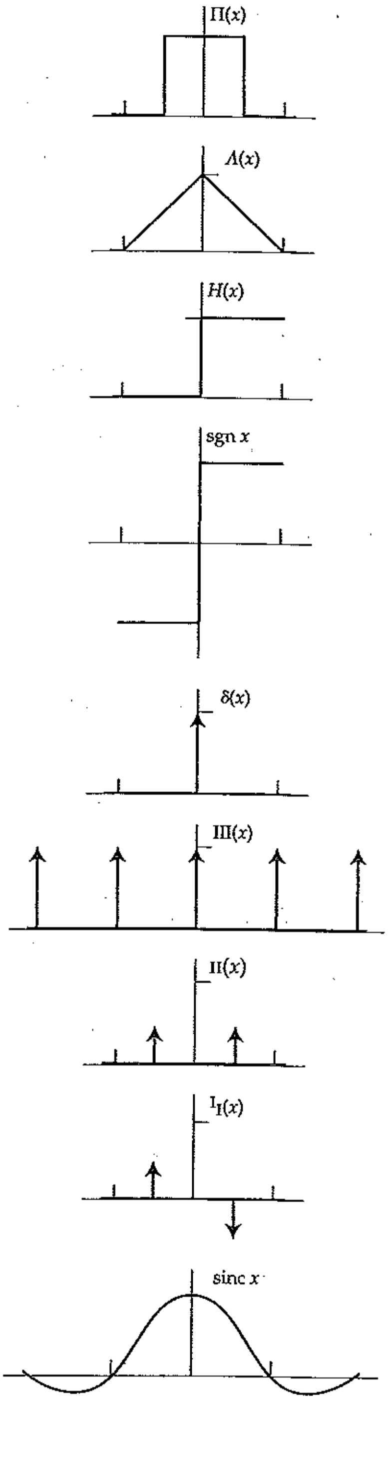
Special symbols

Function	Notation
Rectangle	$\Pi(x) = \begin{cases} 1 & \lvert x\rvert < \frac{1}{2} \\ 0 & \lvert x\rvert > \frac{1}{2} \end{cases}$
Triangle	$\Lambda(x) = \begin{cases} 1 - \lvert x\rvert & \lvert x\rvert < 1 \\ 0 & \lvert x\rvert > 1 \end{cases}$
Heaviside unit step	$H(x) = \begin{cases} 1 & x > 0 \\ 0 & x < 0 \end{cases}$
Sign (signum)	$\operatorname{sgn} x = \begin{cases} 1 & x > 0 \\ -1 & x < 0 \end{cases}$
Impulse symbol	$\delta(x)$
Sampling or replicating symbol	$\mathrm{III}(x) = \sum_{-\infty}^{\infty} \delta(x - n)$
Even impulse pair	$\mathrm{II}(x) = \frac{1}{2}\delta(x + \frac{1}{2}) + \frac{1}{2}\delta(x - \frac{1}{2})$
Odd impulse pair	${}^{\mathrm{I}}{}_{\mathrm{I}}(x) = \frac{1}{2}\delta(x + \frac{1}{2}) - \frac{1}{2}\delta(x - \frac{1}{2})$
Filtering or interpolating	$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$, $\operatorname{jinc} r = \frac{J_1(\pi r)}{2r}$
Asterisk notation for convolution	$f(x) * g(x) \triangleq \int_{-\infty}^{\infty} f(u)g(x - u)\, du$
Asterisk notation for serial products	$\{f_i\} * \{g_i\} \triangleq \left\{ \sum_j f_j g_{i-j} \right\}$
Pentagram notation	$f(x) \star g(x) \triangleq \int_{-\infty}^{\infty} f(u)g(x + u)\, du$
Various two-dimensional functions	${}^2\Pi(x,y) = \Pi(x)\Pi(y)$ ${}^2\delta(x,y) = \delta(x)\,\delta(y)$ ${}^2\mathrm{III}(x,y) = \mathrm{III}(x)\mathrm{III}(y)$ ${}^2\operatorname{sinc}(x,y) = \operatorname{sinc} x \operatorname{sinc} y$

Guide to reference data

Summary of special symbols	70, 71
Oddness and evenness	13, 15
Conjugates	16
Fourier transforms	107, 131, 265, 573
Theorems for Fourier transform	130, 206, 268
Correspondences in the two domains	190, 199
Laplace transforms	385, 388
Two-dimensional Fourier transforms	333–335
Three-dimensional Fourier transforms	342
Hankel transforms	338
Mellin transforms	345–346
z transforms	350–351
Abel transforms	354
Hilbert transforms	365


Π(x)
Λ(x)
H(x)
sgn x
δ(x)
III(x)
II(x)
I_I(x)
sinc x

Filters and Matrices

- $h = (h(0),\ \dots,\ h(N))$ Causal FIR lowpass filter (impulse response) with $\sum h(k) = 1$
- H Toeplitz matrix with $h(k)$ on the kth diagonal: $H_{ij} = h(i-j)$
- $H(\omega) = \sum h(k)\, e^{-ik\omega}$ (frequency response in reduced notation)
  $H(e^{j\omega}) = \sum h(k)\, e^{-jk\omega}$ (signal processing notation)
- $H(z) = \sum h(k) z^{-k}$ Transfer function in the z-domain with $z = e^{j\omega}$
- $y = Hx = h * x$ corresponds to $Y(z) = H(z)X(z)$ ***Convolution Rule***
- S Delay gives $(Sh)(k) = h(k-1)$ ***Time Invariance is $SH = HS$***
- S^{-1} Advance gives $(S^{-1}h)(k) = h(k+1)$ Multiply transform by $z = e^{-j\omega}$
- $(\downarrow 2)$ Downsampling operator $(\downarrow 2)h = h_{\text{even}} = (h(0),\ h(2),\ h(4),\ \dots) =$ even phase of h
- $(\downarrow 2)S^{-1}h = h_{\text{odd}} = (h(1),\ h(3),\ h(5),\ \dots)$ Odd phase of h
- $H_p(z) = [H_{\text{even}}(z)\ \ H_{\text{odd}}(z)] = \left[\sum h(2k)z^{-k}\ \ \sum h(2k+1)z^{-k}\right]$ Polyphase representation
- $(\uparrow 2)$ Upsampling operator $(\uparrow 2)v = [v(0)\ \ 0\ \ v(1)\ \ 0\ \ v(2)\ \ 0\ \ \cdots]$ has z-transform $V(z^2)$
- $(\uparrow 2)(\downarrow 2)x = [x(0)\ \ 0\ \ x(2)\ \ 0\ \ x(4)\ \ 0]$ has z-transform $\frac{1}{2}[X(z) + X(-z)] = X_{\text{even}}(z^2)$
- $H(z^{-1})$ Transpose filter with coefficients $h(-k)$: anticausal matrix H^T
- $z^{-N}H(z^{-1})$ Flip to $h(N-k)$ produces $(h(N),\ \dots,\ h(2),\ h(1),\ h(0))$
- $H(-z)$ Alternating sign $(-1)^k h(k)$ produces $(h(0),\ -h(1),\ h(2),\ -h(3),\ \dots)$
- $z^{-N}H(-z^{-1})$ Alternating flip $(-1)^k h(N-k)$ produces $(h(N),\ \dots,\ (-1)^N h(0))$
- $h(k) = h(N-k)$ Symmetric filter (W symmetry for odd N and H symmetry for even N)

ah

Filter Banks

- $L = (\downarrow 2)C = (\downarrow 2)\sqrt{2}H_0$ Lowpass analysis channel: Filter and downsample
- $B = (\downarrow 2)D = (\downarrow 2)\sqrt{2}H_1$ Highpass analysis channel: Filter has $\sum h_1(k) = 0$
- $\sqrt{2}F_0(\uparrow 2)$ Lowpass synthesis channel: Upsample and filter with $F_0(z) = H_1(-z)$
- $\sqrt{2}F_1(\uparrow 2)$ Highpass synthesis channel: Upsample and filter with $F_1(z) = -H_0(-z)$
- $F_0(z)H_0(z) - F_0(-z)H_0(-z) = P_0(z) - P_0(-z) = 2z^{-\ell}$ No distortion: ℓ delays (*odd* ℓ)
- $H_p(z) = \begin{bmatrix} H_{0,\text{even}}(z) & H_{0,\text{odd}}(z) \\ H_{1,\text{even}}(z) & H_{1,\text{odd}}(z) \end{bmatrix}$ and $F_p(z) = \begin{bmatrix} F_{0,\text{even}}(z) & F_{1,\text{even}}(z) \\ F_{0,\text{odd}}(z) & F_{1,\text{odd}}(z) \end{bmatrix}$ Polyphase matrices
- $H_m(z) = \begin{bmatrix} H_0(z) & H_0(-z) \\ H_1(z) & H_1(-z) \end{bmatrix}$ and $F_m = \begin{bmatrix} F_0(z) & F_1(z) \\ F_0(-z) & F_1(-z) \end{bmatrix}$ Modulation matrices
- $F_p(z)H_p(z) = I$ Perfect Reconstruction with no delays
- $F_m(z)H_m(z) = \begin{bmatrix} 2z^{-\ell} & 0 \\ 0 & -2z^{-\ell} \end{bmatrix}$ Perfect Reconstruction with ℓ delays (*odd* ℓ)
- $H_p^T(z^{-1})H_p(z) = I$ $H_p(z)$ is paraunitary and the filter bank is orthogonal
- $\sum h_0(n)h_0(n+2k) = \delta(k)$ and $H_0(z)H_0(z^{-1}) + H_0(-z)H_0(-z^{-1}) = 2$ Orthogonal lowpass
- $\sum(-1)^k k^m h_0(k) = 0$ for $0 \le m < p$ Condition A_p gives p zeros of $H_0(e^{j\omega})$ at $\omega = \pi$

List of Scaling Functions and Wavelets

- Haar (box function and up-down square wave)
- Daubechies (maxflat orthogonal with $2p$ filter coefficients)
- Splines and biorthogonal wavelets ($F(z) = (1+z^{-1})^p$ gives spline of degree $p-1$)
- Splines and semiorthogonal wavelets (perpendicular to splines, IIR duals)
- Maxflat biorthogonal ($F(z)H(z)$ is Daubechies maxflat product filter)
- Binary biorthogonal (coefficients are integers times 2^{-n}; Section 6.5)
- Shannon (ideal filter, sinc scaling function, box function in ω)
- Meyer (IIR but smooth in ω and t)
- Coifman (zero moments $H^{(k)}(0)$ and $\int t^k\phi(t)dt = 0$ for $0 < k < p$)
- Morlet ($e^{-iat}e^{-t^2/2}$ with $a = \pi\sqrt{2/ln2}$)

Multiresolution and Wavelets

- $\phi(t)$ from the dilation equation (refinement equation) $\phi(t) = \sum 2h(k)\phi(2t - k)$
- $w(t)$ from the wavelet equation $w(t) = \sum 2h_1(k)\phi(2t - k)$
- $w_{jk}(t) = 2^{j/2}w(2^j t - k)$: Normalized wavelet on $[k\Delta t, (k+N)\Delta t]$
- Scale parameter j for stepsize $\Delta t = 2^{-j}$ ($\Delta t = 2^j$ in [D] and MATLAB Toolbox)
- Shift-invariant subspaces $V_j \subset V_{j+1}$ with $V_j \oplus W_j = V_{j+1} : f(t) \in V_0 \leftrightarrow f(2^j t) \in V_j$
- $\{\phi(t-k)\}$ An orthonormal basis (or only a Riesz basis) for V_0
- $\{w(t-k)\}$ An orthonormal basis (or only a Riesz basis) for W_0
- $\{2^{j/2}\phi(2^j t - k)\}$ and $\{2^{j/2}w(2^j t - k)\}$ Bases for V_j and W_j. Joint basis for V_{j+1}
- Orthogonal multiresolution Orthonormal bases with V_j perpendicular to W_j
- Semiorthogonal multiresolution Riesz bases with V_j perpendicular to W_j
- Biorthogonal multiresolution Biorthogonal bases with $V_j \perp \widetilde{W}_j$ and $W_j \perp \widetilde{V}_j$
- $\widetilde{\phi}(t)$ and $\widetilde{w}(t)$ from the analysis filters h_0 and h_1 generate $\widetilde{V}_0 \oplus \widetilde{W}_0 = \widetilde{V}_1$
- $\phi(t)$ and $w(t)$ from the synthesis filters f_0 and f_1 generate $V_0 \oplus W_0 = V_1$
- $\langle \phi(t-k), \widetilde{\phi}(t-\ell) \rangle = \delta(k-l)$ Biorthogonal (dual) bases for V_0 and $\widetilde{V}_0$
- $\langle w(t-k), \widetilde{w}(t-\ell) \rangle = \delta(k-l)$ Biorthogonal (dual) bases for W_0 and $\widetilde{W}_0$
- $a_{jk} = \langle f(t), \widetilde{\phi}_{jk}(t) \rangle$ Coefficients in $f_j(t) = \sum_k a_{jk}\phi_{jk}(t)$ = projection of $f(t)$ onto V_j
- $b_{jk} = \langle f(t), \widetilde{w}_{jk}(t) \rangle$ Wavelet coefficients in $f(t) = \sum\sum b_{jk}w_{jk}(t)$
- $a_{jk} = \sum h_0(\ell - 2k)\, a_{j+1,\ell}$ and $b_{jk} = \sum h_1(\ell - 2k)\, a_{j+1,\ell}$ Mallat Fast Wavelet Transform
- $a_{j+1,\ell} = \sum f_0(\ell - 2k)\, a_{jk} + \sum f_1(\ell - 2k)\, b_{jk}$ Fast Inverse Wavelet Transform

Dilation Equation — Solution and Smoothness

- $\widehat{\phi}(\omega) = H(\frac{\omega}{2})\,\widehat{\phi}(\frac{\omega}{2})$ Fourier transform of the dilation equation
- $\widehat{\phi}(\omega) = \prod_{j=1}^{\infty} H(\frac{\omega}{2^j})$ Fourier transform of the scaling function $\phi(t)$
- $\phi^{(i+1)}(t) = \sum 2h(k)\phi^{(i)}(2t-k)$ Cascade algorithm with $\phi^{(0)}(t) =$ box function
- $M = (\downarrow 2)\; 2\,H$ Cascade matrix with double-shifted rows: $M = \sqrt{2}L$
- $m(0)$ and $m(1)$ N by N submatrices of M Entries $2h(2i-j)$ and $2h(2i-j+1)$
- $m(0)\Phi = \Phi$ Eigenvector with $\lambda = 1$ gives $\Phi = \big(\phi(0),\ \phi(1),\ \ldots\big)$ at the integers
- $\Phi(.\,t_1\,t_2\,\cdots) = m(t_1)\Phi\,(.\,t_2\,t_3\,\cdots)$ Recursion for $\Phi(t) = \big(\phi(t),\ \phi(t+1),\ \ldots\big)$ at dyadic t
- $m(t_1)\,m(t_2)\,\cdots\,m(t_k)$ All products uniformly bounded $\leftrightarrow$ Bounded recursion for $\phi(t)$
- $T = (\downarrow 2)2HH^T$ Transition matrix from autocorrelation $h * h^T$ corresponding to $|H(\omega)|^2$
- $Ta = a$ Eigenvector of T gives the inner products $a(k) = \langle\phi(t),\ \phi(t+k)\rangle$
- $A(z) = \sum a(k)z^{-k}$ and $A(\omega) = \sum|\widehat{\phi}(\omega + 2\pi k)|^2$ Euler-Frobenius polynomial from $h(k)$
- $A(\omega) \equiv 1$ and $A(z) \equiv 1$ and $a = \delta$ Orthonormal basis $\{\phi(t-k)\}$
- $0 < A \leq A(\omega) \leq B$ and $A\sum a_k^2 \leq \|\sum a_k\phi(t-k)\|^2 \leq B\sum a_k^2 \leftrightarrow$ Riesz basis $\{\phi(t-k)\}$
- Condition **E** for Riesz bases $\{\phi(t-k)\}$ and $\{w_{jk}(t)\}$ and L^2 convergence of $\phi^{(i)}(t)$ to $\phi(t)$:

  $\lambda = 1$ is a simple eigenvalue of T and all other $|\lambda(T)| < 1$
- Special eigenvalues $\lambda = 1,\ \ldots,\ (\frac{1}{2})^{p-1}$ of M and $\lambda = 1,\ \ldots,\ (\frac{1}{2})^{2p-1}$ of T from p zeros
- $s_{\max} = -\frac{\log\rho}{\log 4}$ Smoothness of $\phi(t)$ with $\rho = |\lambda_{\max}(T)|$ excluding $\lambda = 1,\ \frac{1}{2},\ \ldots,\ (\frac{1}{2})^{2p-1}$
- $s_{\max} = p - \frac{1}{2}$ Smoothness in L^2 of splines of degree $p-1$ from $H(z) = \left(\frac{1+z^{-1}}{2}\right)^p$
- $s_{\max} \leq p - \frac{1}{2}$ Bound on derivatives of $\phi(t)$ when the filter has p zeros at π