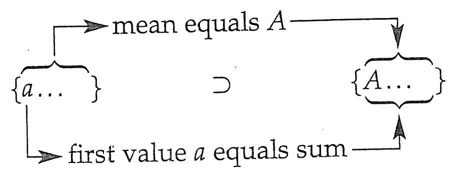
We see that the mean of a sequence equals the first value of its DFT, but conversely the first value of the sequence equals the sum of the DFT.



GENERALIZED PARSEVAL-RAYLEIGH THEOREM

$$\sum_{\tau=0}^{N-1} |f(\tau)|^2 = N \sum_{\nu=0}^{N-1} |F(\nu)|^2.$$

Example:

$$\{1\,1\,0\,0\} \supset \tfrac{1}{4}\{2 \;\; 1-i \;\; 0 \;\; 1+i\}.$$

We see that $\Sigma f^2 = 2$ and that $N\Sigma F^2 = 4 \times 0.5 = 2$.

PACKING THEOREM

The packing operator Pack_K packs a given N-member sequence $f(\tau)$ with trailing zeros so as to increase the number of elements to KN.

$$\text{Pack}_K\{f(\tau)\} = \{g(\tau)\},$$

where

$$g(\tau) = \begin{cases} f(\tau) & 0 \leqslant \tau \leqslant N-1 \\ 0 & N \leqslant \tau \leqslant KN-1. \end{cases}$$

Thus

$$\text{Pack}_2\,\{1\,2\,3\,4\} = \{1\,2\,3\,4\,0\,0\,0\,0\}.$$

This theorem is

$$\text{Pack}_K\,\{f(\tau)\} \supset G(\nu),$$

where $G(\nu) = \dfrac{1}{K} F\left(\dfrac{\nu}{K}\right), \qquad \nu = 0, K, 2K, \ldots, KN-K.$

The intermediate values of $G(\nu)$, not given by this relation, can be determined by sinc-function interpolation between the known values [e.g., by midpoint interpolation (p. 225) when $K = 2$], but for a better method, see Problem 11.8.

SIMILARITY THEOREM

To have an analogy with expansion or contraction of the scale of continuous time, we must supply sufficient zero elements, either at the end, as with packing, so that the sequence may expand, or between elements, so that there is room for contraction. The operation of inserting zeros between elements so as to increase the total number of elements by a factor K will be denoted by the stretch operator Stretch_K.

$$\text{Stretch}_K \{f(\tau)\} = \{g(\tau)\},$$

where

$$g(\tau) = \begin{cases} f(\tau/K) & \tau = 0, K, 2K, \ldots, (N-1)K \\ 0 & \text{otherwise.} \end{cases}$$

Example:

$$\text{Stretch}_2 \{1\,2\,3\,4\} = \{1\,0\,2\,0\,3\,0\,4\,0\}.$$

The theorem is, if $\{g\} \supset \{G\}$,

$$G(\nu) = \begin{cases} \dfrac{1}{K}F(\nu) & \nu = 0, \ldots, N-1 \\ \dfrac{1}{K}F(\nu - N) & \nu = N, \ldots, 2N-1 \\ \cdots\cdots\cdots & \cdots\cdots\cdots\cdots\cdots \\ \dfrac{1}{K}F(\nu - \overline{K-1}N) & \nu = (K-1)N, \ldots, KN-1. \end{cases}$$

Thus, stretching by a factor K in the τ domain results in K-fold repetition of $F(\nu)$ in the ν domain; the frequency scale is not compressed by a factor K.

EXAMPLES USING MATLAB

The DFT is readily evaluated using MATLAB^R, a high-level language that can execute operations on arrays from the keyboard and therefore is particularly adapted to displaying numerical results as the sequence of complex values constituting the DFT of a given sequence. For example, to obtain the DFT of the sequence {1 1 1 1 0 0 0 0} just type

fft([1 1 1 1 0 0 0 0]),

press the return key, and the following is displayed:

4.0000   1.0000−2.4142*i*   0   1.0000−0.4142*i*   0   1.0000+0.4142*i*   0   1.0000+2.4142*i*

4. **Convolution theorem.** Obtain the cyclic convolutions

$$\{0\,1\,0\,0\} * \{0\,0\,1\,0\} \quad \text{and} \quad \{1\,1\,0\,0\} * \{0\,0\,1\,1\}$$

and verify the results by the convolution theorem.

5. **Two-dimensional convolution.** Obtain the two-dimensional cyclic convolution sums

$$\begin{Bmatrix}0 & 0\\1 & 0\end{Bmatrix} ** \begin{Bmatrix}0 & 0\\0 & 1\end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix}1 & 0\\0 & 2\end{Bmatrix} ** \begin{Bmatrix}0 & 1\\0 & 0\end{Bmatrix}$$

and verify that the results are correct by mean of the two-dimensional convolution theorem.

6. **Two-dimensional DFT.** Verify the following four DFT pairs in two dimensions and show that the fifth results from adding the four together.

$$\begin{Bmatrix}0 & 0\\1 & 0\end{Bmatrix} \supset \frac{1}{4}\begin{Bmatrix}1 & 1\\1 & 1\end{Bmatrix}$$

$$\begin{Bmatrix}0 & 0\\0 & 1\end{Bmatrix} \supset \frac{1}{4}\begin{Bmatrix}1 & -1\\1 & -1\end{Bmatrix} \qquad \begin{matrix} & \tau & \\ & \uparrow & \\ \{ & & \} \rightarrow \sigma\end{matrix}$$

$$\begin{Bmatrix}1 & 0\\0 & 0\end{Bmatrix} \supset \frac{1}{4}\begin{Bmatrix}-1 & -1\\1 & 1\end{Bmatrix}$$

$$\begin{Bmatrix}0 & 1\\0 & 0\end{Bmatrix} \supset \frac{1}{4}\begin{Bmatrix}-1 & 1\\1 & -1\end{Bmatrix} \qquad \begin{matrix} & \nu & \\ & \uparrow & \\ \{ & & \} \rightarrow \mu\end{matrix}$$

$$\begin{Bmatrix}1 & 1\\1 & 1\end{Bmatrix} \supset \frac{1}{4}\begin{Bmatrix}0 & 0\\4 & 0\end{Bmatrix}$$

Enunciate two-dimensional sum and first-value theorems. Reconcile the pattern of minuses in the transform with what might be expected from the two-dimensional shift theorem. Verify the two-dimensional Rayleigh-Parseval theorem on the examples.

7. **Midpoint interpolation.** A 16-element sequence whose DFT is $F(\nu)$ is extended to 32 elements by the addition of 16 trailing zeros. The new DFT is $G(\nu)$. By the packing theorem we can obtain half the values of $G(\nu)$ immediately; for example, $G(0) = 0.5F(0)$, $G(2) = 0.5F(1)$, ... $G(30) = 0.5F(15)$. Show that the intermediate values of $G(\nu)$ can be obtained by cyclic convolution of the known values with the midpoint interpolation sequence (p. 225), that is, that

$$G(\nu) = 0.5 \sum_{\kappa=-\infty}^{\infty} F\left(\frac{\nu}{2} - \kappa - \tfrac{1}{2}\right) \operatorname{sinc}(\kappa + \tfrac{1}{2}), \qquad \nu = 1, 3, \ldots, 31.$$

8. **Supplement to packing theorem.** The preceding problem requires summing an infinite number of terms to obtain intermediate values of $G(\nu)$. Show that the sum reduces to a finite number of terms if the interpolating coefficients are modified as follows:

$$G(\nu) = 0.5 \sum_{\kappa=0}^{N-1} F\left(\frac{\nu}{2} - \kappa - \tfrac{1}{2}\right) \frac{\sin\,[\pi(\kappa + \frac{1}{2})]}{\sin\,[\pi N^{-1}(\kappa + \frac{1}{2})]}, \qquad \nu = 1, 3, \ldots, 2N - 1,$$