Mathematics Chalmers & GU

TMA372/MMG800: Partial Differential Equations, 2010–03–08; kl 8.30-13.30.

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Calculators, formula notes and other subject related material are not allowed.
Each problem gives max 7p. Valid bonus poits will be added to the scores.
Breakings: **3**: 20-29p, **4**: 30-39p och **5**: 40p- For GU **G** students :20-35p, **VG**: 36p-
For solutions and gradings information see the couse diary in:
http://www.math.chalmers.se/Math/Grundutb/CTH/tma372/0910/index.html

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1. Let $f \in C^2(a,b)$ and prove the following interpolation error estimate in the L_∞ norm,

$$\|f - \pi_1 f\|_{L_\infty(a,b)} \le (b-a)^2 \|f''\|_{L_\infty(a,b)}.$$

2. Consider the initial value problem: $\dot{u}(t) + au(t) = 0, \quad t > 0, \quad u(0) = 1.$

a) Let $a = 40$, and the time step $k = 0.1$. Draw the graph of $U_n := U(nk)$, $k = 1, 2, \dots$, approximating u using (i) explicit Euler, (ii) implicit Euler, and (iii) Cranck-Nicholson methods.

b) Consider the case $a = i$, $(i^2 = -1)$, having the complex solution $u(t) = e^{-it}$ with $|u(t)| = 1$ for all t. Show that this property is preserved in Cranck-Nicholson approximation, i.e. $|U_n| = 1$, but not in any of the Euler approximations.

3. Let α and β be positive constants. Give the piecewise linear finite element approximation procedure and derive the corresponding stiffness matrix, mass matrix and load vector using the uniform mesh with size $h = 1/4$ for the problem

$$-u''(x) + u = 1, \quad 0 < x < 1; \qquad u(0) = \alpha, \quad u'(1) = \beta.$$

4. Let p be a positive constant. Prove an a priori and an a posteriori error estimate (in the H^1-norm: $||e||^2_{H^1} = ||e'||^2 + ||e||^2$) for a finite element method for problem

$$-u'' + pxu' + (1 + \frac{p}{2})u = f, \quad \text{in } (0,1), \qquad u(0) = u(1) = 0.$$

5. Consider the initial boundary value problem for the heat equation

$$\begin{cases} \dot{u} - \Delta u = 0, & x \in \Omega \subset \mathbb{R}^2, \quad 0 < t \le T, \\ u(x,t) = 0, & x \in \partial\Omega, \quad 0 < t \le T, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Prove the following stability estimates

$$i) \qquad \|u\|^2(t) + 2\int_0^t \|\nabla u\|^2(s)\, ds = \|u_0\|^2,$$

$$ii) \qquad \int_0^t s\|\Delta u\|^2(s)\, ds \le \frac{1}{4}\|u_0\|^2, \qquad \text{and} \quad iii) \qquad \|\nabla u\|(t) \le \frac{1}{\sqrt{2t}}\|u_0\|.$$

6. Consider the convection-diffusion problem

$$-div(\varepsilon \nabla u + \beta u) = f, \text{ in } \Omega \subset \mathbb{R}^2, \quad u = 0, \text{ on } \quad \partial\Omega,$$

where Ω is a bounded convex polygonal domain, $\varepsilon > 0$ is constant, $\beta = (\beta_1(x), \beta_2(x))$ and $f = f(x)$. Determine the conditions in the Lax-Milgram theorem that would guarantee existence of a unique solution for this problem. Prove a stability estimate for u i terms of $||f||_{L_2(\Omega)}$, ε and $diam(\Omega)$, and under the conditions that you derived.

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TMA372/MMG800: Partial Differential Equations, 2010–03–08; kl 8.30-13.30..
Lösningar/Solutions.

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1. See Lecture Notes or the text book, Chapter 5.

2. a) With $a = 40$ and $k = 0.1$ we get the explicit Euler:

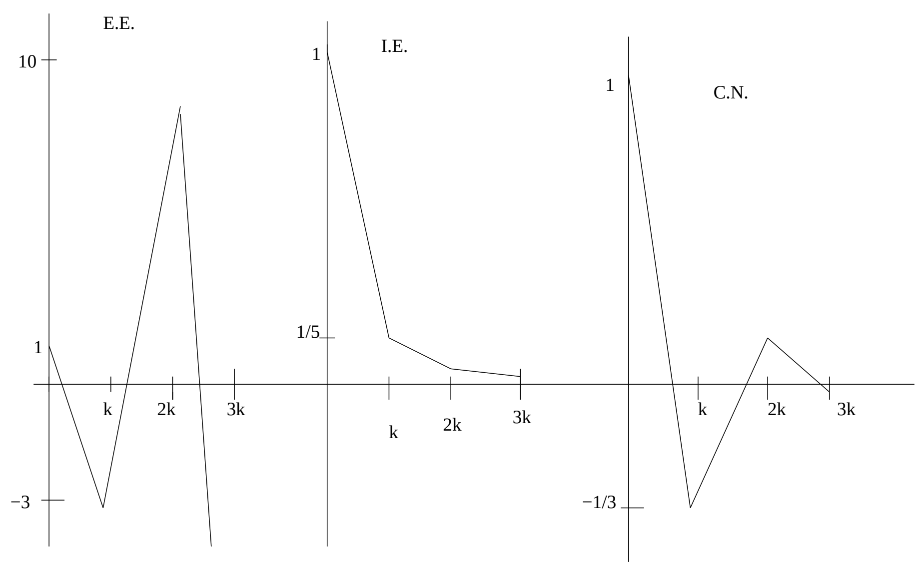
$$\begin{cases} U_n - U_{n-1} + 40 \times (0.1) U_{n-1} = 0, \\ U_0 = 1. \end{cases} \Longrightarrow \begin{cases} U_n = -3U_{n-1}, \quad n = 1, 2, 3, \dots, \\ U_0 = 1. \end{cases}$$

Implicit Euler:

$$\begin{cases} U_n = \frac{1}{1+40\times(0.1)} U_{n-1} = \frac{1}{5} U_{n-1}, \quad n = 1, 2, 3, \dots, \\ U_0 = 1. \end{cases}$$

Cranck-Nicolson:

$$\begin{cases} U_n = \frac{1-\frac{1}{2}\times 40\times(0.1)}{1+\frac{1}{2}\times 40\times(0.1)} U_{n-1} = -\frac{1}{3} U_{n-1}, \quad n = 1, 2, 3, \dots, \\ U_0 = 1. \end{cases}$$



b) With $a = i$ we get

Explicit Euler

$$|U_n| = |1 - (0.1) \times i| |U_{n-1}| = \sqrt{1 + 0.01} |U_{n-1}| \Longrightarrow |U_n| \geq |U_{n-1}|.$$

Implicit Euler

$$|U_n| = \Big| \frac{1}{1 + (0.1) \times i} \Big| |U_{n-1}| = \frac{1}{\sqrt{1 + 0.01}} |U_{n-1}| \leq |U_{n-1}|.$$

Crank-Nicolson

$$|U_n| = \Big| \frac{1 - \frac{1}{2}(0.1) \times i}{1 + \frac{1}{2}(0.1) \times i} \Big| |U_{n-1}| = |U_{n-1}|.$$

3. Multiply the pde by a test function v with $v(0) = 0$, integrate over $x \in (0, 1)$ and use partial integration to get

$$
\begin{aligned}
&-[u'v]_0^1 + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 v\,dx \qquad \Longleftrightarrow \\
(1) \qquad &-u'(1)v(1) + u'(0)v(0) + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 v\,dx \quad \Longleftrightarrow \\
&-\beta v(1) + \int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 v\,dx.
\end{aligned}
$$

The continuous variational formulation is now formulated as follows: Find

$$(VF) \qquad u \in V := \{w : \int_0^1 \Big(w(x)^2 + w'(x)^2\Big)\,dx < \infty, \quad w(0) = \alpha\},$$

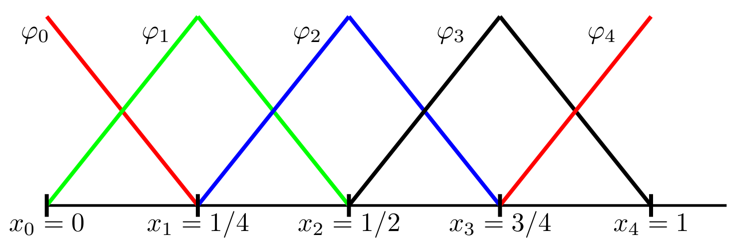
such that

$$\int_0^1 u'v'\,dx + \int_0^1 uv\,dx = \int_0^1 v\,dx + \beta v(1), \quad \forall v \in V^0,$$

where

$$V^0 := \{v : \int_0^1 \Big(v(x)^2 + v'(x)^2\Big)\,dx < \infty, \quad v(0) = 0\}.$$

For the discrete version we let $\mathcal{T}_h$ be a uniform partition: $0 = x_0 < x_1 < \ldots < x_{N+1}$ of $[0, 1]$ into the subintervals $I_n = [x_{n-1}, x_n],\ n = 1, \ldots N + 1$. Here, we have N interior nodes: $x_1, \ldots x_N$, two boundary points: $x_0 = 0$ and $x_{N+1} = 1$ (see Fig. below for $N = 3$, $h = 1/4$, and hence $N + 1 = 4$ intervals).



We shall keep the general framework and let $N = 3$, $h = 1/4$ at the very end. The finite element method (discrete variational formulation) is now formulated as follows: Find

$$(FEM) \qquad u_h \in V_h := \{w_h : w_h \text{ is piecewise linear and continuous on } \mathcal{T}_h,\ w_h(0) = \alpha\},$$

such that

$$(2) \qquad \int_0^1 u_h' v_h'\,dx + \int_0^1 u_h v_h\,dx = \int_0^1 v_h\,dx + \beta v_h(1), \quad \forall v \in V_h^0,$$

where

$$V_h^0 := \{v_h : v_h \text{ is piecewise linear and continuous on } \mathcal{T}_h,\ v_h(0) = 0\}.$$

Using the basis functions $\varphi_j,\ j = 0, \ldots N+1$, where $\varphi_1, \ldots \varphi_N$ are the usual *hat-functions* whereas φ_0 and φ_{N+1} are *semi-hat-functions* viz;

$$(3) \qquad \varphi_j(x) = \begin{cases} 0, & x \notin [x_{j-1}, x_j] \\ \frac{x - x_{j-1}}{h} & x_{j-1} \le x \le x_j \\ \frac{x_{j+1} - x}{h} & x_j \le x \le x_{j+1} \end{cases}, \qquad j = 1, \ldots N.$$

and

$$\varphi_0(x) = \begin{cases} \frac{x_1 - x}{h} & 0 \le x \le x_1 \\ 0, & x_1 \le x \le 1 \end{cases}, \qquad \varphi_{N+1}(x) = \begin{cases} \frac{x - x_N}{h} & x_N \le x \le x_{N+1} \\ 0, & 0 \le x \le x_N. \end{cases}$$

In this way we may write

$$V_h = \alpha\varphi_0 \oplus [\varphi_1, \ldots, \varphi_{N+1}], \quad V_h^0 = [\varphi_1, \ldots, \varphi_{N+1}].$$

Thus every $u_h \in V_h$ can be written as $u_h = \alpha\varphi_0 + v_h$ where $v_h \in V_h^0$, i.e.,

$$u_h = \alpha\varphi_0 + \xi_1\varphi_1 + \dots \xi_{N+1}\varphi_{N+1} = \alpha\varphi_0 + \sum_{j=1}^{M+1} \xi_j\varphi_j \equiv \alpha\varphi_0 + \tilde{u}_h,$$

where $\tilde{u}_h \in V_h^0$. Hence the problem (2) can equivalently be formulated as follows

$$\int_0^1 \Big(\alpha\varphi_0' + \sum_{i=1}^{N+1} \xi_j\varphi_j'\Big)\varphi_i'\,dx + \int_0^1 \Big(\alpha\varphi_0 + \sum_{i=1}^{N+1} \xi_j\varphi_j\Big)\varphi_i\,dx = \int_0^1 \varphi_i\,dx + \beta\varphi_i(1), \quad i = 1,\dots N+1,$$

or, more specifically, as: For $i = 1,\dots N+1$, find ξ_j from the following linear system of equations:

$$\sum_{j=1}^{N+1} \Big(\int_0^1 \varphi_i'\varphi_j'\,dx\Big)\xi_j + \sum_{j=1}^{N+1} \Big(\int_0^1 \varphi_i\varphi_j\,dx\Big)\xi_j + = -\alpha\int_0^1 \varphi_0'\varphi_i'\,dx - \alpha\int_0^1 \varphi_0\varphi_i\,dx + \int_0^1 \varphi_i\,dx + \beta\varphi_i(1),$$

or equivalently $A\xi = \mathbf{b}$ where $A = S + M$ with $S = (s_{ij})$ being the stiffness matrix and $M = (m_{ij})$ the mass matrix. Now, since we have a uniform mesh with $N = 3$; the standard values for entries of these matrices are as follows

$$s_{ii} = 2/h, \quad a_{i,i+1} = a_{i+1,i} = -1/h, \quad i = 1,\dots N, \quad \text{and} \qquad a_{N+1,N+1} = 1/h,$$

and

$$m_{ii} = 2h/3, \quad a_{i,i+1} = a_{i+1,i} = h/6, \quad i = 1,\dots N, \quad \text{and} \qquad a_{N+1,N+1} = h/3.$$

Now we return to our specific basis functions as in the Figure above ($N+1 = 4$, $h = 1/4$), note that φ_4 is a half-hat function. Then

$$A = 4\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix} + \frac{1}{24}\begin{bmatrix} 4 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix},$$

and the unknown $\xi := [\xi_1\xi_2, \xi_3, \xi_4]^t$ is determined by solving $A\xi = \mathbf{b}$, with A as above and the load vector $\mathbf{b}$ given by

$$\mathbf{b} = \begin{bmatrix} -\alpha\int_0^1 \varphi_0'\varphi_1'\,dx - \alpha\int_0^1 \varphi_0\varphi_1\,dx + \int_0^1 \varphi_1\,dx \\ \int_0^1 \varphi_2\,dx \\ \int_0^1 \varphi_3\,dx \\ \int_0^1 \varphi_4\,dx + \beta\varphi_4(1) \end{bmatrix} = \begin{bmatrix} 4\alpha - \alpha/24 + 1/4 \\ 1/4 \\ 1/4 \\ \beta + 1/8 \end{bmatrix}.$$

4. We multiply the differential equation by a test function $v \in H_0^1(I)$, $I = (0,1)$ and integrate over I. Using partial integration and the boundary conditions we get the following *variational problem*: Find $u \in H_0^1(I)$ such that

$$\int_I \Big(u'v' + pxu'v + (1+\frac{p}{2})uv\Big) = \int_I fv, \quad \forall v \in H_0^1(I). \tag{4}$$

A *Finite Element Method* with *cG*(1) reads as follows: Find $U \in V_h^0$ such that

$$\int_I \Big(U'v' + pxU'v + (1+\frac{p}{2})Uv\Big) = \int_I fv, \quad \forall v \in V_h^0 \subset H_0^1(I), \tag{5}$$

where

$$V_h^0 = \{v : v \text{ is piecewise linear and continuous in a partition of } I,\ v(0) = v(1) = 0\}.$$

Now let $e = u - U$, then (1)-(2) gives that

$$\int_I \Big(e'v' + pxe'v + (1+\frac{p}{2})ev\Big) = 0, \quad \forall v \in V_h^0. \tag{6}$$

A posteriori error estimate: We note that using $e(0) = e(1) = 0$, we get

$$\int_I pxe'e = \frac{p}{2}\int_I x\frac{d}{dx}(e^2) = \frac{p}{2}(xe^2)|_0^1 - \frac{p}{2}\int_I e^2 = -\frac{p}{2}\int_I e^2, \tag{7}$$

so that

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I \Big(e'e' + pxe'e + (1+\frac{p}{2})ee\Big) \\
&= \int_I \Big((u-U)'e' + px(u-U)'e + (1+\frac{p}{2})(u-U)e\Big) = \{v = e \text{ in}(1)\} \\
&= \int_I fe - \int_I \Big(U'e' + pxU'e + (1+\frac{p}{2})Ue\Big) = \{v = \pi_h e \text{ in}(2)\} \qquad (8)\\
&= \int_I f(e-\pi_h e) - \int_I \Big(U'(e-\pi_h e)' + pxU'(e-\pi_h e) + (1+\frac{p}{2})U(e-\pi_h e)\Big) \\
&= \{P.I. \text{ on each subinterval}\} = \int_I \mathcal{R}(U)(e-\pi_h e),
\end{aligned}
$$

where $\mathcal{R}(U) := f + U'' - pxU' - (1+\frac{p}{2})U = f - pxU' - (1+\frac{p}{2})U$, (for approximation with piecewise linears, $U \equiv 0$, on each subinterval). Thus (5) implies that

$$
\begin{aligned}
\|e\|_{H^1}^2 &\le \|h\mathcal{R}(U)\|\|h^{-1}(e-\pi_h e)\| \\
&\le C_i\|h\mathcal{R}(U)\|\|e'\| \le C_i\|h\mathcal{R}(U)\|\|e\|_{H^1},
\end{aligned}
$$

where C_i is an interpolation constant, and hence we have with $\|\cdot\| = \|\cdot\|_{L_2(I)}$ that

$$
\|e\|_{H^1} \le C_i\|h\mathcal{R}(U)\|.
$$

A priori error estimate: We use (4) and write

$$
\begin{aligned}
\|e\|_{H^1}^2 &= \int_I (e'e' + ee) = \int_I (e'e' + pxe'e + (1+\frac{p}{2})ee) \\
&= \int_I \Big(e'(u-U)' + pxe'(u-U) + (1+\frac{p}{2})e(u-U)\Big) = \{v = U - \pi_h u \text{ in}(3)\} \\
&= \int_I \Big(e'(u-\pi_h u)' + pxe'(u-\pi_h u) + (1+\frac{p}{2})e(u-\pi_h u)\Big) \\
&\le \|(u-\pi_h u)'\|\|e'\| + p\|u-\pi_h u\|\|e'\| + (1+\frac{p}{2})\|u-\pi_h u\|\|e\| \\
&\le \{\|(u-\pi_h u)'\| + (1+p)\|u-\pi_h u\|\}\|e\|_{H^1} \\
&\le C_i\{\|hu''\| + (1+p)\|h^2u''\|\}\|e\|_{H^1},
\end{aligned}
$$

this gives that

$$
\|e\|_{H^1} \le C_i\{\|hu''\| + (1+p)\|h^2u''\|\},
$$

which is the a priori error estimate.

5. See Lecture Notes or text book chapter 16.

6. Consider

$$
-div(\varepsilon\nabla u + \beta u) = f, \text{ in } \Omega, \qquad u = 0 \text{ on } \Gamma = \partial\Omega. \qquad (9)
$$

a) Multiply the equation (6) by $v \in H_0^1(\Omega)$ and integrate over Ω to obtain the Green's formula

$$
-\int_\Omega div(\varepsilon\nabla u + \beta u)v\,dx = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx = \int_\Omega fv\,dx.
$$

Variational formulation for (6) is as follows: Find $u \in H_0^1(\Omega)$ such that

$$
a(u,v) = L(v), \qquad \forall v \in H_0^1(\Omega), \qquad (10)
$$

where

$$
a(u,v) = \int_\Omega (\varepsilon\nabla u + \beta u)\cdot\nabla v\,dx,
$$

and

$$
L(v) = \int_\Omega fv\,dx.
$$

According to the Lax-Milgram theorem, for a unique solution for (7) we need to verify that the following relations are valid:

i)

$$|a(v,w)| \le \gamma ||u||_{H^1(\Omega)} ||w||_{H^1(\Omega)}, \qquad \forall v,\, w \in H_0^1(\Omega),$$

ii)

$$a(v,v) \ge \alpha ||v||^2_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

iii)

$$|L(v)| \le \Lambda ||v||_{H^1(\Omega)}, \qquad \forall v \in H_0^1(\Omega),$$

for some $\gamma,\ \alpha,\ \Lambda > 0$.

Now since

$$|L(v)| = |\int_\Omega f v\, dx| \le ||f||_{L_2(\Omega)} ||v||_{L_2(\Omega)} \le ||f||_{L_2(\Omega)} ||v||_{H^1(\Omega)},$$

thus iii) follows with $\Lambda = ||f||_{L_2(\Omega)}$.

Further we have that

$$\begin{aligned}
|a(v,w)| &\le \int_\Omega |\varepsilon \nabla v + \beta v| |\nabla w|\, dx \le \int_\Omega (\varepsilon |\nabla v| + |\beta||v|)|\nabla w|\, dx \\
&\le \Big(\int_\Omega (\varepsilon |\nabla v| + |\beta||v|)^2\, dx \Big)^{1/2} \Big(\int_\Omega |\nabla w|^2\, dx \Big)^{1/2} \\
&\le \sqrt{2} \max(\varepsilon, ||\beta||_\infty) \Big(\int_\Omega (|\nabla v|^2 + v^2)\, dx \Big)^{1/2} ||w||_{H^1(\Omega)} \\
&= \gamma ||v||_{H^1(\Omega)} ||w||_{H^1(\Omega)},
\end{aligned}$$

which, with $\gamma = \sqrt{2}\max(\varepsilon, ||\beta||_\infty)$, gives i).

Finally, if $div \beta \le 0$, then

$$\begin{aligned}
a(v,v) &= \int_\Omega \Big(\varepsilon |\nabla v|^2 + (\beta \cdot \nabla v) v \Big)\, dx = \int_\Omega \Big(\varepsilon |\nabla v|^2 + (\beta_1 \frac{\partial v}{\partial x_1} + \beta_2 \frac{\partial v}{\partial x_2}) v \Big)\, dx \\
&= \int_\Omega \Big(\varepsilon |\nabla v|^2 + \frac{1}{2} (\beta_1 \frac{\partial}{\partial x_1} (v)^2 + \beta_2 \frac{\partial}{\partial x_2} (v)^2) \Big)\, dx = \text{Green's formula} \\
&= \int_\Omega \Big(\varepsilon |\nabla v|^2 - \frac{1}{2} (div \beta) v^2 \Big)\, dx \ge \int_\Omega \varepsilon |\nabla v|^2\, dx.
\end{aligned}$$

Now by the Poincare's inequality

$$\int_\Omega |\nabla v|^2\, dx \ge C \int_\Omega (|\nabla v|^2 + v^2)\, dx = C ||v||^2_{H^1(\Omega)},$$

for some constant $C = C(diam(\Omega))$, we have

$$a(v,v) \ge \alpha ||v||^2_{H^1(\Omega)}, \qquad \text{with } \alpha = C\varepsilon,$$

thus ii) is valid under the *condition that* $div\beta \le 0$.

From ii), (7) (with $v = u$) and iii) we get that

$$\alpha ||u||^2_{H^1(\Omega)} \le a(u,u) = L(u) \le \Lambda ||u||_{H^1(\Omega)},$$

which gives the *stability estimate*

$$||u||_{H^1(\Omega)} \le \frac{\Lambda}{\alpha},$$

with $\Lambda = ||f||_{L_2(\Omega)}$ and $\alpha = C\varepsilon$ defined above.

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